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CONVERGENCE IN COMPARABLE ALMOST PERIODIC REACTION-DIFFUSION SYSTEMS WITH DIRICHLET BOUNDARY CONDITIONS

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Abstract. In this article, we study the asymptotic dynamics in nonmonotone comparable almost periodic reaction-diffusion systems with Dirichlet boundary condition, which are comparable with uniformly stable strongly order-preserving system. By appealing to the theory of skew-product semiflows, we obtain the asymptotic almost periodicity of uniformly stable solutions to the comparable reaction-diffusion system.

1. Introduction

In the previous 50 years or so, many concepts from dynamical systems have been applied to the study of partial differential equations (see [4, 5, 6, 7, 8, 11, 12, 19, 20], etc.). In this paper, we shall study the long-term behaviour of the solutions of some non-autonomous comparable reaction-diffusion equations.

We consider the almost periodic reaction-diffusion system with Dirichlet boundary condition:

$$\begin{gathered}\frac{\partial v_i}{\partial t} = d_i(t)\Delta v_i + F_i(t, v_1, \dots, v_n), \quad x \in \Omega,\ t > 0,\\ v_i(t,x) = 0, \quad x \in \partial\Omega,\ t > 0,\\ v_i(0,x) = v_{0,i}(x), \quad x \in \bar{\Omega},\ 1 \le i \le n,\end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary. $d = (d_1(\cdot), \dots, d_n(\cdot)) \in C(\mathbb{R}, \mathbb{R}^n)$ is assumed to be an almost periodic vector-valued function bounded below by a positive real vector. The nonlinearity $F = (F_1, \dots, F_n) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1-admissible and uniformly almost periodic in t, and F points into $\mathbb{R}^n_+$ along the boundary of $\mathbb{R}^n_+$: $F_i(t, v) \ge 0$ whenever $v \in \mathbb{R}^n_+$ with $v_i = 0$ and $t \in \mathbb{R}^+$. However, F has no monotonicity properties.

To study the properties of the solutions of such a non-monotone equation, an effective approach is to exhibit and utilize certain comparison techniques (see [9, 1, 2, 22]). As pointed out in [21, Section 4], the comparison technique involves monotone systems in a natural way: the original non-monotone systems are comparable with certain monotone ones. Thus, we assume that there exists a function

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$f : \mathbb{R} \times \mathbb{R}^n_+ \to \mathbb{R}^n$ with $f(t, v) \geq F(t, v)$ (or $f(t, v) \leq F(t, v)$), $\forall (t, v) \in \mathbb{R} \times \mathbb{R}^n_+$. Also, we assume that f satisfies (H1)–(H4) in section 2. Then we get a strongly order-preserving system (see section 2 for details):

$$\begin{gathered} \frac{\partial u_i}{\partial t} = d_i(t)\Delta u_i + f_i(t, u_1, \dots, u_n), \quad x \in \Omega,\ t > 0, \\ u_i(t, x) = 0, \quad x \in \partial\Omega,\ t > 0, \\ u_i(0, x) = u_{0,i}(x), \quad x \in \bar{\Omega},\ 1 \leq i \leq n. \end{gathered} \tag{1.2}$$

We want to know whether such a non-monotone system (1.1) inherits certain asymptotic behaviour from its strongly order-preserving partner (1.2). Note that a unified framework to study nonautonomous equations is based on the so-called skew-product semiflows (see [18, 19]). Since even the strongly monotone (which is a stronger notion than strongly order-preserving) skew-product semiflows can possess very complicated chaotic attractors (see [19]), we hence assume that the strongly order-preserving partner is 'uniformly stable', and to establish the asymptotic 1-cover property of the corresponding strongly order-preserving skew-product semiflow.

As far as we know, there are only a few works on the related topics. Jiang [14] proved the global convergence of the comparable discrete-time or continuous-time system provided that all the equilibria of its monotone partner form a totally ordered curve. Recently, Cao, Gyllenberg and Wang[3] established the asymptotic 1-cover property of the comparable skew-product semiflows, whose partner systems are eventually strongly monotone and uniformly stable. Here we emphasize that for reaction-diffusion system with Dirichlet boundary condition, the cone X_+ has empty interior in the state space $X = \Pi_1^n C_0(\bar{\Omega})$ (see section 2 for details). Thus, the skew-product semiflow generated by its partner is only strongly order-preserving, but not eventually strongly monotone (see [13, Chapter 6]). So we have to find another way to get the corresponding asymptotic dynamics for Dirichlet problem.

Motivated by [15], to obtain the asymptotic behavior of solutions to comparable almost periodic reaction-diffusion system (1.1), we first prove that every precompact trajectory of the strongly order-preserving system (1.2) is asymptotic to a 1-cover of the base flow (see Proposition 3.3). Based on this, for the uniformly stable and strongly order-preserving skew-product semiflow generated by (1.2), we can get the topological structure of the set of the union of all 1-covers similarly as [3] (see Lemma 3.4). With such tools, we are able to establish the 1-covering property of uniformly stable omega-limit sets of comparable skew-product semiflow (see Proposition 3.5), and thus obtain the asymptotic almost periodicity of uniformly stable solutions to system (1.1).

This article is organized as follows. In section 2, we present some basic definitions and our main result. In Section 3 we prove the main result.

2. Preliminaries and statement of the main result

A subset S of $\mathbb{R}$ is said to be *relatively dense* if there exists $l > 0$ such that every interval of length l intersects S. A function f, defined and continuous on $\mathbb{R}$, is *almost periodic* if, for any $\varepsilon > 0$, the set $T(f, \varepsilon) = \{s \in \mathbb{R} : |f(t + s) - f(t)| < \varepsilon, \forall t \in \mathbb{R}\}$ is relatively dense. A continuous function $f : \mathbb{R} \times \mathbb{R}^m \mapsto \mathbb{R}^n$ is said to be *admissible* if, for every compact subset $K \subset \mathbb{R}^m$, f is bounded and uniformly continuous on $\mathbb{R} \times K$. Besides, if f is of class $C^r (r \geq 1)$ in $x \in \mathbb{R}^m$, and f and all its partial derivatives with respect to x up to order r are admissible, then we say that f is C^r-*admissible*. A function $f \in C(\mathbb{R} \times \mathbb{R}^m, \mathbb{R}^n)$ is *uniformly almost periodic in* t, if f is both admissible and almost periodic in $t \in \mathbb{R}$.

Let $f \in C(\mathbb{R} \times \mathbb{R}^m, \mathbb{R}^n)$ be uniformly almost periodic, one can define the Fourier series of f (see [19, 23]), and the *frequency module* $\mathcal{M}(f)$ of f as the smallest Abelian group containing a Fourier spectrum. Let $f, g \in C(\mathbb{R} \times \mathbb{R}^m, \mathbb{R}^n)$ be two uniformly almost periodic functions in t. One has $\mathcal{M}(f) = \mathcal{M}(g)$ if and only if the flow $(H(g), \mathbb{R})$ is isomorphic to the flow $(H(f), \mathbb{R})$ (see, [10] or [19, Section 1.3.4]). Here $H(f) = \mathrm{cl}\{f \cdot \tau : \tau \in \mathbb{R}\}$ is called the *hull of* f, where $f \cdot \tau(t, \cdot) = f(t + \tau, \cdot)$ and the closure is taken under the compact open topology.

Let (Y, d_Y) be a compact metric space with metric d_Y. A *continuous flow* $\sigma : \mathbb{R} \times Y \to Y$, $(t, y) \to \sigma(t, y) = \sigma_t(y) = y \cdot t$ is called *minimal* if Y has no other nonempty compact invariant subset but itself. Here a subset $Y_1 \subset Y$ is *invariant* if $\sigma_t(Y_1) = Y_1$ for every $t \in \mathbb{R}$.

Consider the almost periodic reaction-diffusion system with Dirichlet boundary condition

$$\begin{gathered}
\frac{\partial v_i}{\partial t} = d_i(t)\Delta v_i + F_i(t, v_1, \dots, v_n), \quad x \in \Omega,\ t > 0, \\
v_i(t, x) = 0, \quad x \in \partial\Omega,\ t > 0, \\
v_i(0, x) = v_{0,i}(x), \quad x \in \bar{\Omega},\ 1 \le i \le n,
\end{gathered} \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary. Δ is the Laplacian operator on $\mathbb{R}^n$.

Let $d = (d_1(\cdot), \dots, d_n(\cdot)) \in C(\mathbb{R}, \mathbb{R}^n)$ be an almost periodic vector-valued function and for some $d_0 > 0$, $d_i(t) \ge d_0$, for all $t \in \mathbb{R}$, $1 \le i \le n$. The nonlinearity $F = (F_1, \dots, F_n) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1-admissible and uniformly almost periodic in t. Let $v = (v_1, \dots, v_n)$, we also assume that

(I1) $F_i(t, v) \ge 0$ whenever $v \in \mathbb{R}^n_+$ with $v_i = 0$ and $t \in \mathbb{R}^+$.

Denote $X = \Pi_1^n C_0(\bar{\Omega})$ $(C_0(\bar{\Omega}) := \{\phi \in C(\bar{\Omega}, \mathbb{R}) : \phi|_{\partial\Omega} = 0\})$ and the standard cone $X_+ = \{u \in X : u(x) \in \mathbb{R}^n_+, x \in \bar{\Omega}\}$. Then the cone X_+ induces an *ordering* on X via $x_1 \le x_2$ if $x_2 - x_1 \in X_+$. We write $x_1 < x_2$ if $x_2 - x_1 \in X_+ \setminus \{0\}$. Let $x \in X$ and a subset $U \subset X$. We write $x <_r U$ if $x <_r u$ for all $u \in U$. Given two subsets A, $B \subset X$, we write $A <_r B$ if $a <_r b$ holds for each choice of $a \in A$, $b \in B$. Here $<_r$ represents $\le$ or $<$. $x >_r U$ is similarly defined. Obviously, every compact subset in X has both a greatest lower bound and a least upper bound.

Let $H(d, F)$ be the hull of the function (d, F). Then the time translation $(\mu, G) \cdot t$ of $(\mu, G) \in H(d, F)$ induces a compact and minimal flow on $H(d, F)$ (see [18] or [19]). By the standard theory of reaction-diffusion systems (see [13, Chapter 6]), it follows that for every $v_0 \in X_+$ and $(\mu, G) \in H(d, F)$, the system

$$\begin{gathered}
\frac{\partial v_i}{\partial t} = \mu_i(t)\Delta v_i + G_i(t, v), \quad x \in \Omega,\ t > 0, \\
v_i(t, x) = 0, \quad x \in \partial\Omega,\ t > 0, \\
v(0, x) = v_0(x), \quad x \in \bar{\Omega},\ 1 \le i \le n
\end{gathered} \tag{2.2}$$

admits a (locally) unique regular solution $v(t, \cdot, v_0; \mu, G)$ in X_+. This solution also continuously depends on $(\mu, G) \in H(d, F)$ and $v_0 \in X_+$ (see [12]). Thus, (2.2) induces a (local) skew-product semiflow Γ on $X_+ \times H(d, F)$ with

$$\Gamma_t(v_0, (\mu, G)) = (v(t, \cdot, v_0; \mu, G), (\mu, G) \cdot t), \quad \forall (v_0, (\mu, G)) \in X_+ \times H(d, F),\ t \ge 0.$$

Now we assume that there exists a function $f \in C^1(\mathbb{R} \times \mathbb{R}^n_+, \mathbb{R}^n)$, which is C^1-admissible and uniformly almost periodic in t, satisfying

- (H1) $f(t,v) \geq F(t,v)$ for all $(t,v) \in \mathbb{R} \times \mathbb{R}^n_+$. with its frequency module $\mathcal{M}(f) = \mathcal{M}(F)$ (thus $H(d,f) \cong H(d,F)$);
- (H2) $f_i(t,0) = 0\,(1 \leq i \leq n)$;
- (H3) $\frac{\partial f_i}{\partial x_j}(t,x) \geq 0$ for all $1 \leq i \neq j \leq n$, and there is a $\delta > 0$ such that if two nonempty subsets I, J of $\{1,2,\dots,n\}$ form a partition of $\{1,2,\dots,n\}$, then for any $(t,x) \in \mathbb{R} \times \mathbb{R}^n_+$, there exist $i \in I$, $j \in J$ such that $|\frac{\partial f_i}{\partial x_j}(t,x)| \geq \delta > 0$;
- (H4) Every nonnegative solution of ordinary differential system $\dot{u} = g(t,u), g \in H(f)$, is bounded.

It is easy to see that, for any $(\mu, G) \in H(d,F)$, there exists a $(\mu, g) \in H(d,f)$ such that

$$g(t,v) \geq G(t,v) \text{ for all } (t,v) \in \mathbb{R} \times \mathbb{R}^n_+.$$

Denote $Y = H(d,f)$. Then we can consider the reaction-diffusion system

$$\begin{gathered}\frac{\partial u_i}{\partial t} = \mu_i(t)\Delta u_i + g_i(t,u), \quad x \in \Omega,\ t > 0,\\ u_i(t,x) = 0, \quad x \in \partial\Omega,\ t > 0,\\ u(0,x) = u_0(x) \in X_+, \quad x \in \bar{\Omega},\ 1 \leq i \leq n,\end{gathered} \tag{2.3}$$

which induces the global skew-product semiflow

$$\Pi_t : X_+ \times Y \to X_+ \times Y; \quad (u_0, y = (\mu, g)) \mapsto (u(t,\cdot,u_0,y), y \cdot t),\ t \in \mathbb{R}^+, \tag{2.4}$$

where $u(t,\cdot,u_0,y)$ is the unique regular global solution of (2.3) in X_+. Without any confusion, we also write $u(t,\cdot,u_0,y)$ as $u(t,u_0,y)$.

Clearly, by the comparison principle and (H4), the forward orbit $O^+(x,y) = \{\Pi_t(x,y) : t \geq 0\}$ of any $(x,y) \in X_+ \times Y$ is precompact. Thus the omega-limit set of (x,y), defined by $\omega(x,y) = \{(\hat{x},\hat{y}) \in X_+ \times Y : \Pi_{t_n}(x,y) \to (\hat{x},\hat{y}) (n \to \infty)$ for some sequence $t_n \to \infty\}$, is a nonempty, compact and invariant subset in $X_+ \times Y$. A forward orbit $O^+(x_0,y_0)$ of Π_t is said to be *uniformly stable* if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$, called the *modulus of uniform stability*, such that for every $x \in X_+$, if $s \geq 0$ and $\|u(s,x_0,y_0) - u(s,x,y_0)\| \leq \delta(\varepsilon)$ then

$$\|u(t+s,x_0,y_0) - u(t+s,x,y_0)\| < \varepsilon \text{ for each } t \geq 0.$$

Here we assume that every forward orbit of Π_t in (2.4) is uniformly stable, which can be guaranteed by the existence of invariant functional.

Let $P : X_+ \times Y \to Y$ be the natural projection. A compact positively invariant set $K \subset X_+ \times Y$ is called a 1-*cover* of Y if $P^{-1}(y) \cap K$ contains a unique element for every $y \in Y$. If we write the 1-cover $K = \{(c(y),y) : y \in Y\}$, then $c : Y \to X$ is continuous with $\Pi_t(c(y),y) = (c(y \cdot t), y \cdot t)$, $\forall t \geq 0$. For the sake of brevity, we hereafter also write $c(\cdot)$ as a 1-*cover* of Y.

For skew-product semiflows, we always use the order relation on each fiber $P^{-1}(y)$, and write $(x_1,y) \leq (<)\,(x_2,y)$ if $x_1 \leq x_2$ $(x_1 < x_2)$. Recall that the skew-product semiflow Π_t is called *monotone* if

$$\Pi_t(x_1,y) \leq \Pi_t(x_2,y)$$

whenever $(x_1,y) \leq (x_2,y)$ and $t \geq 0$. Moreover, Π_t is *strongly order-preserving* if it is monotone and there is a $t_0 > 0$ such that, whenever $(x_1,y) < (x_2,y)$ there exist open subsets U, V of X_+ with $x_1 \in U$, $x_2 \in V$ satisfying

$$\Pi_t(U, y) < \Pi_t(V, y) \quad \text{for all } t \geq t_0.$$

Π_t is called *fiber-compact* if there exists a $\bar{t} > 0$ such that, for any $y \in Y$ and bounded subset $B \subset X$, $\Pi_t(B, y)$ has compact closure in $P^{-1}(y \cdot t)$ for every $t > \bar{t}$. Then according to (H3), [13, Chapter 6] and [15, Section 6], one can obtain that Π_t in (2.4) is strongly order-preserving and fibre-compact.

By (H1), similarly as the proof of Lemma 5.2 in [3], we can get that Γ_t is upper-comparable with respect to Π_t in the sense that if $\Gamma_t(x_1, y) \leq \Pi_t(x_2, y)$ whenever $(x_1, y), (x_2, y) \in X_+ \times Y$ with $(x_1, y) \leq (x_2, y)$.

Now we are in a position to state our main result.

Theorem 2.1. *Any uniformly stable L^∞-bounded solution of* (2.1) *is asymptotic to an almost periodic solution.*

Remark 2.2. We note that for reaction-diffusion system with Dirichlet boundary condition (2.1), the cone X_+ has empty interior in the state space $X = \Pi_1^n C_0(\bar{\Omega})$. Thus, the skew-product semiflow generated by its monotone partner (2.3) is only strongly order-preserving, but not eventually strongly monotone. Consequently, the results in [3] can't be used to study the asymptotic behavior of the solutions to system (2.1).

3. Proof of Theorem 2.1

To obtain the asymptotic almost periodicity of solutions to system (2.1), we first investigate the asymptotic behavior of its strongly order-preserving partner. Motivated by [15], we establish the 1-cover property of omega limit sets for the strongly order-preserving and uniformly stable skew-product semiflows Π_t.

The following result is adopted from [17, P. 19] or [19, P. 29], see also [16, P. 634].

Theorem 3.1. *Let Θ_t be a skew-product semiflow on $X_+ \times Y$. If a forward orbit $O^+_\Theta(x_0, y_0)$ of Θ_t is precompact and uniformly stable, then its omega-limit set $\omega_\Theta(x_0, y_0)$ admits a flow extension which is minimal.*

Now fix $(x_0, y_0) \in X_+ \times Y$ and let $K = \omega(x_0, y_0)$ be its omega-limit set with respect to Π_t. For any given $y \in Y$, we define

$$(p(y), y) = \text{g.l.b. of } K \cap P^{-1}(y).$$

Then from [15, Proposition 3.1], it follows that $\omega(p(y), y)$ is 1-cover of Y. Denote $\{(p_*(y), y)\} = \omega(p(y), y) \cap P^{-1}(y)$, by [15, Proposition 3.2] one has

$$u(t, p_*(y), y) = p_*(y \cdot t) \quad \text{for any } y \in Y \text{ and } t \in \mathbb{R}. \tag{3.1}$$

So we can denote the 1-cover $\omega(p(y), y)$ by $p_*(\cdot)$.

Lemma 3.2. *Assume that there exists a point $(z, y) \in K$ such that $p_*(y) < z$. Then for any $t \in \mathbb{R}$, there exist a neighborhood U of $p_*(y)$ and a neighborhood V of z such that*

$$u(t, U, y) < u(t, V, y).$$

Proof. By the minimality of K, for any $t \in \mathbb{R}$, there is $\tau_n \to +\infty$ such that $\tau_n + t \geq 0$ and

$$\Pi_{\tau_n} \circ \Pi_t(z, y) \to \Pi_t(z, y), \quad \text{as } n \to \infty.$$

Note that the monotonicity implies that

$$\Pi_{\tau_n} \circ \Pi_t(p_*(y), y) \le \Pi_{\tau_n} \circ \Pi_t(z, y).$$

Letting $n \to \infty$, we then get $\Pi_t(p_*(y), y) \le \Pi_t(z, y)$, thus,

$$u(t, p_*(y), y) \le u(t, z, y), \quad \forall t \in \mathbb{R}. \tag{3.2}$$

Suppose that the conclusion of the lemma does not hold. Then we claim that there exists $r_0 \in \mathbb{R}$ such that

$$u(t, p_*(y), y) = u(t, z, y), \quad \forall t \le r_0. \tag{3.3}$$

Otherwise. By (3.2), one has that for any $r \in \mathbb{R}$, there exists some $\bar{t} \le r$ such that

$$u(\bar{t}, p_*(y), y) < u(\bar{t}, z, y).$$

Since Π_t is strongly order-preserving, it follows that there exist a neighborhood $\bar{U}$ of $u(\bar{t}, p_*(y), y)$ and a neighborhood $\bar{V}$ of $u(\bar{t}, z, y)$ such that

$$u(r - \bar{t} + t_0, \bar{U}, y \cdot \bar{t}) < u(r - \bar{t} + t_0, \bar{V}, y \cdot \bar{t}).$$

Note that by the continuity of Π_t, there exist a neighborhood $\hat{U}$ of $p_*(y)$ with $u(\bar{t}, \hat{U}, y) \subset \bar{U}$, and a neighborhood $\hat{V}$ of z with $u(\bar{t}, \hat{V}, y) \subset \bar{V}$. So we have

$$u(r - \bar{t} + t_0, u(\bar{t}, \hat{U}, y), y \cdot \bar{t}) < u(r - \bar{t} + t_0, u(\bar{t}, \hat{V}, y), y \cdot \bar{t}).$$

Thus,

$$u(r + t_0, \hat{U}, y) < u(r + t_0, \hat{V}, y).$$

Since r is arbitrary, the conclusion of the lemma holds. A contradiction. So we proved the claim.

By the minimality of K, we obtain that $\alpha(z, y) = K$. Hence, $(z, y) \in \alpha(z, y)$. Then it follows that there exists a sequence $\tau_n \to -\infty$ such that $\tau_n \le r_0$ and $\Pi_{\tau_n}(z, y) \to (z, y)$. Thus the 1-cover property of $\omega(p_*(y), y)$ and (3.1) imply that $\Pi_{\tau_n}(p_*(y), y) \to (p_*(y), y)$. By (3.3), one has

$$u(\tau_n, p_*(y), y) = u(\tau_n, z, y).$$

By letting $n \to +\infty$, we get

$$(p_*(y), y) = (z, y).$$

A contradiction to the assumption. This completes the proof. $\square$

The following Proposition shows the 1-cover property of omega limit sets for Π_t.

Proposition 3.3. *For any $(x_0, y_0) \in X_+ \times Y$, $\omega(x_0, y_0)$ is a 1-cover of Y.*

Proof. Now fix $(x_0, y_0) \in X_+ \times Y$ and set $K = \omega(x_0, y_0)$. For any $y \in Y$, by [15, Proposition 3.1], we have $(p_*(y), y) \le K \cap P^{-1}(y)$.

We claim that $\{(p_*(y), y)\} = K \cap P^{-1}(y)$ for all $y \in Y$. Suppose not. Then there exist some $y \in Y$ and a point $(\hat{z}, y) \in K$ such that $p_*(y) < \hat{z}$. By the minimality of K, we get that

$$p_*(y) < z, \quad \forall (z, y) \in K \cap P^{-1}(y).$$

Then it follows from Lemma 3.2 that there exist a neighborhood U_z of $p_*(y)$ and a neighborhood V_z of z such that

$$U_z < V_z. \tag{3.4}$$

Since $\{V_z : (z,y) \in K \cap P^{-1}(y)\}$ is an open cover of $K \cap P^{-1}(y)$, we can find a finite subcover, denoted by $\{V_1, V_2, \dots, V_n\}$. Note that by (3.4) there exist neighborhoods $U_i,\ i = 1, 2, \dots, n$ of $p_*(y)$ such that

$$U_1 < V_1, \quad U_2 < V_2, \ \dots, \ U_n < V_n.$$

Therefore, $\cap_{i=1}^n U_i < \cup_{i=1}^n V_i$. Since $K \cap P^{-1}(y) \subset \cup_{i=1}^n V_i$, we have

$$\cap_{i=1}^n U_i < K \cap P^{-1}(y).$$

So we can take an $\epsilon_0 > 0$ such that

$$B^+(p_*(y), \epsilon_0) < K \cap P^{-1}(y), \tag{3.5}$$

where $B^+(p_*(y), \epsilon_0) = \{x \in X_+ : x \geq p_*(y),\ \|x - p_*(y)\| \leq \epsilon_0\}$. By the uniform stability of $\Pi_t(p_*(y), y)$, there exists $\delta_0 = \delta_0(\epsilon_0) \leq \epsilon_0$ such that

$$\|u - p_*(y)\| \leq \epsilon_0,\ \forall (u, y) \in \omega(x, y) \cap P^{-1}(y)$$

whenever $\|x - p_*(y)\| \leq \delta_0$. Combing with (3.5), we get

$$(p_*(y), y) \leq \omega(x, y) \cap P^{-1}(y) < K \cap P^{-1}(y)$$

for any $x \in B^+(p_*(y), \delta_0)$. Since $\omega(x, y)$ is minimal, using [15, Proposition 3.1(3)], we obtain

$$\omega(x, y) = \omega(p(y), y) = p_*(\cdot), \quad \forall x \in B^+(p_*(y), \delta_0). \tag{3.6}$$

Set

$$L = \{\tau \in [0,1] : x_\tau = p_*(y) + \tau(\hat{z} - p_*(y)), \omega(x_\tau, y) = p_*(\cdot)\}.$$

By (3.6), there exists a $\bar{\tau} > 0$ such that $[0, \bar{\tau}] \subset L$. It is easy to see that L is an interval. Now we show that L is closed, that is, $L = [0, \tau_0]$ with $0 < \tau_0 = \sup\{\tau : \tau \in L\} < 1$. Note that $\Pi_t(x_{\tau_0}, y)$ is uniformly stable. Let $\delta(\epsilon)$ be the modulus of uniform stability for $\epsilon > 0$. Thus, we take $\tau \in [0, \tau_0)$ with $\|x_\tau - x_{\tau_0}\| < \delta(\epsilon)$ and we get

$$\|u(t, x_\tau, y) - u(t, x_{\tau_0}, y)\| < \epsilon, \quad \forall t \geq 0.$$

Since $\omega(x_\tau, y) = p_*(\cdot)$, there is a $\hat{t}$ such that

$$\|u(t, x_\tau, y) - p_*(y \cdot t)\| < \epsilon, \quad \forall t \geq \hat{t}.$$

Then, we deduce that

$$\|u(t, x_{\tau_0}, y) - p_*(y \cdot t)\| < 2\epsilon,\ \forall t \geq \hat{t},$$

and hence $\omega(x_{\tau_0}, y) = p_*(\cdot)$. So L is closed.

Then by a similar argument in the proof of [15, Theorem 4.1], we can get a contradiction. Indeed, since $L = [0, \tau_0]$ with $0 < \tau_0 < 1$, for any $\tau \in (\tau_0, 1)$ we have $(p_*(y), y) \notin \omega(x_\tau, y)$. For ϵ_0 defined in (3.5), by the uniform stability of the orbit, we get

$$\|u(t, x_\tau, y) - u(t, x_{\tau_0}, y)\| < \epsilon_0, \quad \forall t \geq 0 \tag{3.7}$$

whenever $0 < \tau - \tau_0 \ll 1$. Let $\{t_n\}$ be such that $\Pi_{t_n}(x_{\tau_0}, y) \to (p_*(y), y)$. Choosing a subsequence if necessary, we may assume that $\Pi_{t_n}(x_\tau, y) \to (\tilde{x}, y)$ for $0 < \tau - \tau_0 \ll 1$. By (3.7), we obtain $\|\tilde{x} - p_*(y)\| \leq \epsilon_0$. Thus, from the monotonicity, $\tilde{x} \in B^+(p_*(y), \epsilon_0)$. So by (3.5), $\tilde{x} < K \cap P^{-1}(y)$. Using [15, Proposition 3.1 (3)] again, we get $\omega(\tilde{x}, y) = \omega(p(y), y) = p_*(\cdot)$. Then the minimality of $\omega(x_\tau, y)$ implies that $\omega(x_\tau, y) = \omega(\tilde{x}, y) = p_*(\cdot)$, which is a contradiction to the definition of τ_0. Thus, $K \cap P^{-1}(y) = \{(p_*(y), y)\}$ for all $y \in Y$. The minimality deduces that K is a 1-cover of Y. □

Denote

$$A = \cup_{c(\cdot) \text{ is a 1-cover for } \Pi_t} c(\cdot)$$

of all 1-covers of Y for Π_t. For each $y \in Y$, set $A(y) = A \cap P^{-1}(y)$. Based on Proposition 3.3, we obtain the following result.

Lemma 3.4. *A is totally ordered with respect to '$<$', and for each $y \in Y$, $A(y)$ is homeomorphic to a closed interval in $\mathbb{R}$.*

The proof of the above lemma is similar to that of [3, Theorem 3.1], therefore it is omitted.

For any $(x_0, y_0) \in X_+ \times Y$, denote the forward orbit and the omega-limit set for Γ_t by $O^+_\Gamma(x_0, y_0)$ and $\omega_\Gamma(x_0, y_0)$, respectively. Now we will prove the 1-cover property for the uniformly stable ω-limit sets of the comparable skew-product semiflow Γ_t.

Proposition 3.5. *Assume that for point $(x_0, y_0) \in X_+ \times Y$, $O^+_\Gamma(x_0, y_0)$ is uniformly stable. Let $\hat{K} = \omega_\Gamma(x_0, y_0)$. For any $y \in Y$, if there exists some $(b(y), y) \in A(y)$ such that $\hat{K} \cap P^{-1}(y) \geq (b(y), y)$, then $\hat{K}$ is a 1-cover of Y for Γ_t.*

Proof. Let $C_\Pi = \{c(\cdot) : c(\cdot) \text{ is a 1-cover for } \Pi_t\}$. Then by a similar argument in the proof of [3, Theorem 4.3], using Lemma 3.4 we can define a nonempty totally ordered set $\mathcal{C} \subset C_\Pi$, for which

$$\mathcal{C} = \{c(\cdot) \in C_\Pi : (c(y), y) \geq \hat{K} \cap P^{-1}(y) \text{ for all } y \in Y\},$$

and the greatest lower bound $\inf \mathcal{C} \in \mathcal{C}$ exists.

Denote $q(\cdot) = \inf \mathcal{C}$. Now we assert that $\hat{K}$ is a 1-cover of Y for Γ_t, satisfying

$$\hat{K} \cap P^{-1}(y) = (q(y), y), \quad \forall y \in Y.$$

Otherwise, there exist a $y_1 \in Y$ and some $(c, y_1) \in \hat{K} \cap P^{-1}(y_1)$ such that

$$(q(y_1), y_1) > (c, y_1).$$

According to our assumption, we have

$$(q(y_1), y_1) > (c, y_1) \geq (b(y_1), y_1).$$

Then by [3, Lemma 3.4], there is a strictly order-preserving continuous path

$$J : [0, 1] \to A(y_1) \quad \text{with } J(0) = (b(y_1), y_1) \text{ and } J(1) = (q(y_1), y_1). \tag{3.8}$$

Since $(q(y_1), y_1) > (c, y_1)$, by the strongly order-preserving property of Π_t and the comparability of Γ_t with respect to Π_t, we have that there exists a neighborhood U of $q(y_1)$ such that

$$\Pi_{t_1}(U, y_1) > \Pi_{t_1}(c, y_1) \geq \Gamma_{t_1}(c, y_1) = (v(t_1, c, y_1), y_1 \cdot t_1)$$

for some $t_1 > t_0$. Denote $\bar{c} = v(t_1, c, y_1)$ and $y_2 = y_1 \cdot t_1$. Then $(\bar{c}, y_2) \in \hat{K}$ and

$$(u(t_1, U, y_1), y_2) > (\bar{c}, y_2). \tag{3.9}$$

Note that U is a neighborhood of $q(y_1)$. Then due to (3.8) we can find a point $q_1(y_1) \in U \cap A(y_1)$ with $q_1(y_1) < q(y_1)$. Thus, by (3.9) we obtain

$$(q(y_2), y_2) > (q_1(y_2), y_2) > (\bar{c}, y_2).$$

Since $O^+_\Gamma(x_0, y_0)$ is uniformly stable, by Theorem 3.1 $\hat{K}$ admits a flow extension which is minimal. Thus for any $t \in \mathbb{R}$, there is $t_n \to +\infty$ such that $t_n + t \geq 0$ and

$$\Gamma_{t_n} \circ \Gamma_t(\bar{c}, y_2) \to \Gamma_t(\bar{c}, y_2), \ n \to \infty.$$

Then the monotonicity and the comparability of Γ_t with respect to Π_t imply that

$$\Pi_{t_n} \circ \Pi_t(q_1(y_2), y_2) \geq \Pi_{t_n} \circ \Pi_t(\bar{c}, y_2) \geq \Gamma_{t_n} \circ \Gamma_t(\bar{c}, y_2).$$

By letting $n \to \infty$ in the above, we get $\Pi_t(q_1(y_2), y_2) \geq \Gamma_t(\bar{c}, y_2)$, thus,

$$u(t, q_1(y_2), y_2) \geq v(t, \bar{c}, y_2), \ \forall t \in \mathbb{R}. \tag{3.10}$$

Note that $O^+_\Pi(q_1(y_2), y_2)$ is uniformly stable, by Theorem 3.1 we obtain

$$u(t, q_1(y), y) = q_1(y \cdot t) \quad \text{for any } y \in Y \text{ and } t \in \mathbb{R}. \tag{3.11}$$

So combining (3.10), (3.11) and the comparability of Γ_t with respect to Π_t, similarly as the proof of Lemma 3.2, we can get that for any $t \in \mathbb{R}$, there exist a neighborhood U_t of $q_1(y_2)$ and a neighborhood V_t of $\bar{c}$ such that

$$u(t, U_t, y_2) > v(t, V_t, y_2).$$

In particular, for $t = 0$, there exist a neighborhood U_0 of $q_1(y_2)$ and a neighborhood V_0 of $\bar{c}$ such that

$$(U_0, y_2) > (V_0, y_2). \tag{3.12}$$

Recall that $\hat{K}$ is the omega-limit set of (x_0, y_0) for Γ_t, there exists some sequence $t_n \to +\infty$ such that $\Gamma_{t_n}(x_0, y_0) \to (\bar{c}, y_2) \in \hat{K}$, as $n \to \infty$. Also, since $q_1(\cdot)$ is a 1-cover for Π_t, we get $\Pi_{t_n}(q_1(y_0), y_0) \to (q_1(y_2), y_2)$, as $n \to \infty$. So by (3.12) there exists $N > 1$ such that

$$\Pi_{t_N}(q_1(y_0), y_0) > \Gamma_{t_N}(x_0, y_0). \tag{3.13}$$

Then by a similar argument in the proof of [3, Theorem 4.3], we can get that

$$(q_1(y), y) \geq \hat{K} \cap P^{-1}(y) \quad \text{for all } y \in Y.$$

For the sake of completeness, we include a detailed proof here. As a matter of fact, by the monotonicity of Π_t and the comparability of Γ_t with respect to Π_t, it follows from (3.13) that

$$\Pi_{t+t_N}(q_1(y_0), y_0) \geq \Pi_t \Gamma_{t_N}(x_0, y_0) \geq \Gamma_{t+t_N}(x_0, y_0), \quad \forall t \geq 0. \tag{3.14}$$

For any $(x, y) \in \hat{K}$, there exists $s_n \to +\infty$ such that $\Gamma_{s_n}(x_0, y_0) \to (x, y)$, as $n \to \infty$. Let $t = s_n - t_N$ in (3.14) for all n sufficiently large. Then we get $\Pi_{s_n}(q_1(y_0), y_0) \geq \Gamma_{s_n}(x_0, y_0)$. Letting $n \to +\infty$, one has $(q_1(y), y) \geq (x, y)$. By the arbitrariness of $(x, y) \in \hat{K}$, we get $(q_1(y), y) \geq \hat{K} \cap P^{-1}(y)$ for all $y \in Y$. This contradicts the definition of $q(\cdot)$. So we have proved the assertion, and $\hat{K}$ is a 1-cover of Y for Γ_t. □

Proof of Theorem 2.1. Let $v(t, \cdot, v_0; d, F)$ be an L^∞-bounded solution of (2.1) in X_+. Then from the study in [12] and a priori estimates for parabolic equations, it follows that v is a globally defined classical solution in X_+, and $\{v(t, \cdot, v_0; d, F) : t \geq \tau\}$ is precompact in X_+ for some $\tau > 0$. So $\hat{K} := \omega_\Gamma(v_0, (d, F))$ is a nonempty compact set in $X_+ \times H(d, F)$. Since $0(\cdot) \in C_\Pi$ by (H2),

$$\hat{K} \cap P^{-1}(y) \geq (0, y) \in A(y), \quad \forall y \in Y.$$

If $v(t, \cdot, v_0; d, F)$ is uniformly stable, then by Proposition 3.5 we get that $\hat{K}$ is a 1-cover of Ω for Γ_t, and thus the uniformly stable L^∞-bounded solution $v(t, \cdot, v_0; d, F)$ is asymptotic to an almost periodic solution. □

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