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EXISTENCE AND MULTIPLICITY OF POSITIVE PERIODIC SOLUTIONS FOR SECOND-ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. In this article, the existence and multiplicity results of positive periodic solutions are obtained for the second-order functional differential equation with infinite delay

$$u''(t) + b(t)u'(t) + a(t)u(t) = c(t)f(t, u_t), \quad t \in \mathbb{R}$$

where a, b, c are continuous ω-periodic functions, $u_t \in C_B$ is defined by $u_t(s) = u(t+s)$ for $s \in (-\infty, 0]$, C_B denotes the Banach space of bounded continuous function $\phi : (-\infty, 0] \to \mathbb{R}$ with the norm $\|\phi\|_B = \sup_{s\in(-\infty,0]} |\phi(s)|$, and $f : \mathbb{R} \times C_B \to [0, \infty)$ is a nonnegative continuous functional. The existence conditions concern with the first eigenvalue of the associated linear periodic boundary problem. Our discussion is based on the fixed point index theory in cones.

1. Introduction

Let C_B be the Banach space of bounded continuous function defined on $(-\infty, 0]$ with the norm $\|\phi\|_B = \sup_{s\in(-\infty,0]} |\phi(s)|$ and $f : \mathbb{R} \times C_B \to [0, \infty)$ is a nonnegative continuous functional acting on $\mathbb{R} \times C_B$. If u is a continuous ω-periodic function, then $u_t \in C_B$ for every $t \in \mathbb{R}$, where u_t is defined by $u_t(s) = u(t + s)$ for every $s \in (-\infty, 0]$.

In this article, we sutdy the existence and multiplicity of positive periodic solutions of the second-order functional differential equation with infinite delay

$$u''(t) + b(t)u'(t) + a(t)u(t) = c(t)f(t, u_t), \quad t \in \mathbb{R}, \tag{1.1}$$

where $a(t), b(t), c(t)$ are continuous ω-periodic functions on $\mathbb{R}$.

In recent years, the existence of periodic solutions for some second-order functional differential equations has been researched by some authors, and many results have been obtained by applying monotone iterative technique, fixed point theorem in cones, Leray-Schauder continuation theorem, coincidence degree theory and so on, see [2, 4, 8, 9, 10, 11, 16, 6, 19, 20, 21] and the references therein.

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Jiang [8, 9] and others considered the periodic problem of the second-order delay differential equation

$$-u''(t) = g(t, u(t), u(t - \tau(t))), \quad t \in \mathbb{R},$$

where $g \in C(\mathbb{R}^3, \mathbb{R})$ and $\tau \in C(\mathbb{R}, [0, \infty))$. Using monotone iterative technique, they obtained the existence results of non-constant ω-periodic solutions.

Guo and Guo [4] studied the second-order delay differential equation in $\mathbb{R}^n$,

$$-u''(t) = g(u(t - \tau)), \quad t \in \mathbb{R},$$

where $g \in C(\mathbb{R}^n, \mathbb{R}^n)$ and $\tau > 0$ is a given constant. By using critical point theory and S^1-index theory, they obtained the existence and multiplicity of non-constant periodic solutions.

However, in some practice models, only positive periodic solutions are significant. In [2, 11, 20, 21], the authors obtained the existence of positive periodic solutions of some second-order functional differential equations by using fixed-point theorems of cone mapping. Wu [21] considered the second-order functional differential equation

$$u''(t) + a(t)u(t) = \lambda\, g(t, u(t - \tau_1(t)), \dots, u(t - \tau_n(t))), \quad t \in \mathbb{R}, \tag{1.2}$$

where $g \in C(\mathbb{R} \times [0, \infty)^n, [0, \infty))$, $\tau_1, \dots, \tau_n \in C(\mathbb{R}, [0, \infty))$. He obtained the existence result of positive periodic solution by using the Krasnoselskii fixed-point theorem of cone mapping when the coefficient $a(t)$ satisfies the condition that $0 < a(t) < \frac{\pi^2}{\omega^2}$ for every $t \in \mathbb{R}$. Li [11] obtained the existence results of positive ω-periodic solutions for the second-order functional differential equation with constant delays

$$-u''(t) + a(t)u(t) = g(t, u(t - \tau_1), \dots, u(t - \tau_n)), \quad t \in \mathbb{R} \tag{1.3}$$

by employing the fixed point index theory in cones.

Recently, Kang and Cheng [6] discussed the second-order functional differential equation with damped term

$$u''(t) + b(t)u'(t) + a(t)u(t) = \lambda c(t)\, g(t, u(t - \tau(t))), \quad t \in \mathbb{R} \tag{1.4}$$

and obtained the existence and multiplicity of positive periodic solutions by using the Krasnoselskii fixed point theorem of cone mapping when the coefficients $a(t), b(t)$ are nonnegative continuous functions and $g \in C(\mathbb{R} \times [0, \infty), [0, +\infty))$ is nondecreasing in the second variable. For the second-order differential equation without delay, the existence of positive periodic solutions has been discussed by more authors, see [1, 7, 12, 13, 14, 15, 17, 18].

Motivated by the papers mentioned above, we research the existence and multiplicity of positive periodic solutions of the more general functional differential equation (1.1) with infinite delay, in which the coefficients $a(t), b(t)$ may be sign-changing.

Throughout this paper we make the following assumptions:

(H1) $a, b \in C(\mathbb{R}, \mathbb{R})$ are ω-periodic functions, $a(t) \not\equiv 0$ and one of the following two conditions is satisfied:

(i) the following two inequalities hold

$$\int_0^\omega a(s)\, \Phi(b)(s)\, \Psi(-b)(s)\, ds \ge 0, \tag{1.5}$$

$$\sup_{0 \le t \le \omega} \Big\{ \int_t^{t+\omega} \Phi(-b)(s) ds \int_t^{t+\omega} a^+(s)\, \Phi(b)(s) ds \Big\} \le 4, \tag{1.6}$$

(ii) $\int_0^\omega b(s)ds = 0$, $\int_0^\omega a(s)\Phi(b)(s)ds > 0$ and there exists a constant $1 \leq p \leq +\infty$ such that

$$\|\Phi(-b)\|_1{}^{2-1/p} \cdot \|\Phi^{2-1/p}(b)\, a^+\|_p < K(2p^*), \tag{1.7}$$

where

$$\Phi(b)(t) = \exp\Big(\int_0^t b(s)ds\Big), \quad t \in \mathbb{R}, \tag{1.8}$$

$$\Psi(b)(t) = \Phi(b)(\omega)\int_0^t \Phi(b)(s)ds + \int_t^\omega \Phi(b)(s)ds, \quad t \in \mathbb{R} \tag{1.9}$$

and $a^+(s) = \max\{a(s), 0\}$, $\|a\|_p$ is the p-norm of a in $L^p[0,\omega]$, p^* is the conjugate exponent of p defined by $\frac{1}{p} + \frac{1}{p^*} = 1$, and the function $K(q)$ is defined by

$$K(q) = \begin{cases} \frac{2\pi}{q}\big(\frac{2}{2+q}\big)^{1-2/q}\big(\frac{\Gamma(1/q)}{\Gamma(1/2+1/q)}\big)^2, & \text{if } 1 \leq q < +\infty,\\ 4\,, & \text{if } q = +\infty \end{cases} \tag{1.10}$$

in which Γ is the Gamma function.

(H2) $c \in C(\mathbb{R}, [0,\infty))$ is an ω-periodic function and $c \not\equiv 0$.

(H3) $f : \mathbb{R} \times C_B \to [0,\infty)$ is continuous and it maps every bounded set of $\mathbb{R} \times C_B$ into a bounded set of $[0,+\infty)$, $f(t, \phi)$ is ω-periodic in t.

We aim to discuss the existence and multiplicity of positive ω-periodic solution of (1.1) under Assumptions (H1)–(H3). Condition (H1) is taken from [1, 7]. In our discussion, the maximum principles built by Cabada and Cid in [1] and Hakl and Torres in [7] for the periodic problem of the corresponding linear second-order different equation

$$u'' + b(t)u'(t) + a(t)u(t) = h(t), \quad t \in \mathbb{R} \tag{1.11}$$

plays an important role. According to these maximum principles, we obtain some new existence and multiplicity results by constructing a special cone in $C_\omega(\mathbb{R})$ and applying the fixed-point index theory in cones. Our result improve and extend the results in [6, 21] and other existing results.

The techniques used in this paper are different from those in [6, 21]. Our results are more general than thiers in three aspects. Firstly, equation (1.1) is infinitely delayed, and equations (1.2) and (1.4) discussed in [6, 21] are finitely delayed. Secondly, we relax the conditions of the coefficient $a(t)$ appeared in (1.2) in [21] and the coefficients $b(t)$ appeared in (1.4) in [6], and expand the range of their values, and we do not require that f to be monotonic in the second variable. Thirdly, by constructing a special cone and applying the theory of the fixed-point index in cones, we obtain the essential conditions on the existence of positive periodic solutions of Equations (1.1). The conditions concern the first eigenvalue of the associated linear periodic boundary problem, which improve the existence results in [6, 21]. To our knowledge, there are very few works on the existence of positive periodic solutions for the above functional differential equation under the conditions concerning the first eigenvalue of the corresponding linear differential equation.

Our main results are presented and proved in Section 3. Some preliminaries to discuss Equation (1.1) are presented in Section 2.

2. Preliminaries

To study (1.1), we consider the periodic problem of the corresponding linear ordinary differential equation

$$u'' + b(t)u'(t) + a(t)u(t) = c(t)\,h(t), \quad t \in \mathbb{R}, \tag{2.1}$$

where $h \in C(\mathbb{R})$ is a ω-periodic function. For this we consider the linear periodic boundary value problem

$$\begin{gathered} u'' + b(t)u'(t) + a(t)u(t) = h(t), \quad t \in [0,\omega], \\ u(0) = u(\omega),\ u'(0) = u'(\omega). \end{gathered} \tag{2.2}$$

By the maximum principle in Cabada, Cid and Hakl et al [1, Theorem 5.1] and [7, Theorem 2.2], we have the following Lemma.

Lemma 2.1. *Assume that* (H1) *holds. Then the periodic boundary value problem* (2.2) *has a positive Green's function* $G \in C([0,\omega]^2,(0,\infty))$*, and for every* $h \in C[0,\omega]$*, the equation* (2.2) *has a unique solution expressed by*

$$u(t) = \int_0^\omega G(t,s)\,h(s)\,ds, \quad t \in [0,\omega]. \tag{2.3}$$

Let $C_\omega(\mathbb{R})$ denote the Banach space of all continuous ω-periodic function $u(t)$ with norm $\|u\|_C = \max_{0\le t\le\omega}|u(t)|$. Let $C_\omega^+(\mathbb{R})$ be the nonnegative function cone in $C_\omega(\mathbb{R})$. Generally, for $n \in \mathbb{N}$ we use $C_\omega^n(\mathbb{R})$ to denote the space of all nth-order continuous differentiable ω-periodic functions.

Clearly, if $u \in C_\omega(\mathbb{R})$, the restriction of u on $(-\infty,0]$ belongs to C_B, $u_t \in C_B$ for every $t \in \mathbb{R}$, and

$$\|u\|_B = \|u\|_C; \quad \|u_t\|_B = \|u\|_C, \quad t \in \mathbb{R}. \tag{2.4}$$

Hence, we think that $C_\omega(\mathbb{R}) \subset C_B$.

Assume that (H1) holds and $G(t,s)$ is the positive Green's function of the periodic boundary value problem (2.2). Let

$$\underline{G} = \min_{0\le t,s\le\omega} G(t,s), \quad \overline{G} = \max_{0\le t,s\le\omega} G(t,s), \quad \sigma = \underline{G}/\overline{G} \tag{2.5}$$

and define a cone K in $C_\omega(\mathbb{R})$ by

$$K = \{\, u \in C_\omega(\mathbb{R}) : u(t) \ge \sigma\|u\|_C,\ t \in \mathbb{R}\,\}. \tag{2.6}$$

Lemma 2.2. *Assume that* (H1) *and* (H2) *hold. Then for every* $h \in C_\omega(\mathbb{R})$*, Equation* (2.1) *has a unique* ω*-periodic solution* $u := T\,h \in C_\omega^2(\mathbb{R})$*. Moreover, the periodic solution operator* $T : C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$ *is a completely continuous linear operator and* $T(C_\omega^+(\mathbb{R})) \subset K$*.*

Proof. For $h \in C_\omega(\mathbb{R})$, by Lemma 2.1 the following linear periodic boundary problem with the weighting function c,

$$\begin{gathered} u'' + b(t)u'(t) + a(t)u(t) = c(t)\,h(t), \quad t \in [0,\omega], \\ u(0) = u(\omega), \quad u'(0) = u'(\omega) \end{gathered} \tag{2.7}$$

has a unique solution $u \in C^2[0,\omega]$. Extend u to an ω-periodic function which is still denoted by u, then $u \in C_\omega^2(\mathbb{R})$ is a unique ω-periodic solution of Equation (2.7), we denote it by Th. Thus we obtain the ω-periodic solution operator $T : C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$ of Equation (2.1). By Lemma 2.1, Th is expressed by

$$Th(t) = \int_0^\omega G(t,s)c(s)\,h(s)\,ds, \quad t \in [0,\omega]. \tag{2.8}$$

Form this, we easily see that $T : C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$ is a completely continuous linear operator.

Let $h \in C_\omega^+(\mathbb{R})$. For every $t \in [0,\omega]$, from (2.8) it follows that

$$0 \le Th(t) = \int_0^\omega G(t,s)c(s)\,h(s)\,ds \le \overline{G}\int_0^\omega c(s)\,h(s)\,ds,$$

and therefore

$$\|Th\|_C \le \overline{G}\int_0^\omega c(s)\,h(s)\,ds.$$

By (2.8) and this inequality, we have

$$\begin{aligned} Th(t) &= \int_0^\omega G(t,s)c(t)\,h(s)\,ds \ge \underline{G}\int_0^\omega c(t)\,h(s)\,ds \\ &= \big(\underline{G}/\overline{G}\big)\cdot \overline{G}\int_0^\omega c(t)\,h(s)\,ds \\ &\ge \sigma|Th\|. \end{aligned}$$

Combining this with the periodicity of u, we show that $u \in K$. Hence $T(C_\omega^+(\mathbb{R})) \subset K$. □

Hereafter, we use $r(T)$ to denote the spectral radius of the operator $T : C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$.

Lemma 2.3. *Assume that* (H1) *and* (H2) *hold. Then* $r(T) > 0$.

Proof. Choose $h_0 \equiv 1$. Then by (2.8) and the positivity of $G(t,s)$ we have

$$\begin{gathered} Th_0(t) = \int_0^\omega G(t,s)c(s)\,ds \ge \underline{G}\int_0^\omega c(s)\,ds := m > 0, \quad t \in [0,\omega], \\ T^2h_0(t) = \int_0^\omega G(t,s)c(s)Th_0(s)\,ds \ge m\,\underline{G}\int_0^\omega c(s)ds = m^2, \quad t \in [0,\omega], \\ \dots \end{gathered}$$

Inductively, we obtain that

$$T^k h_0(t) \ge m^k, \quad t \in [0,\omega], \quad k = 1,2,\dots.$$

Consequently,

$$\|T^k\| \ge \|T^k h_0\|_C \ge m^k, \quad k = 1,2,\dots.$$

By this and the Gelfand's formula of spectral radius we have

$$r(T) = \lim_{k\to\infty} \|T^k\|^{1/k} \ge m > 0. \tag{2.9}$$

The proof of Lemma 2.4 is complete. □

Thus by the well-known Krein-Rutman theorem, $r(T)$ is the maximum positive real eigenvalue of the operator T. So we have

Lemma 2.4. *Assume that* (H1) *and* (H2) *hold. Then there exists a eigenfunction* $\phi_1 \in K \setminus \{\theta\}$ *such that*

$$T\,\phi_1 = r(T)\,\phi_1. \tag{2.10}$$

Set $\lambda_1 = 1/r(T)$, then $\phi_1 = T(\lambda_1\phi_1)$. By Lemma 2.2 and the definition of T, $\phi_1 \in C^2_\omega(\mathbb{R})$ satisfies the differential equation

$$\phi_1''(t) + b(t)\,\phi_1'(t) + a(t)\,\phi_1(t) = \lambda_1 c(t)\,\phi_1(t), \quad t \in \mathbb{R}. \tag{2.11}$$

Thus, λ_1 is the minimum positive real eigenvalue of the linear equation (2.1) under the ω-periodic condition.

Let $f : \mathbb{R} \times C_B \to [0,\infty)$ satisfy Assumption (H3). For every $u \in K$, set

$$F(u)(t) := f(t, u_t), \quad t \in \mathbb{R}. \tag{2.12}$$

Since $j : t \mapsto u_t$ maps $\mathbb{R}$ into C_B and it is continuous, by Assumption (H3), $F(u) \in C^+_\omega(\mathbb{R})$ and $F : K \to C^+_\omega(\mathbb{R})$ is continuous and maps every bounded set of K into a bounded set of $C^+_\omega(\mathbb{R})$. Hence, by Lemma 2.2 the composite mapping

$$A = T \circ F. \tag{2.13}$$

maps K into K and $A : K \to K$ is completely continuous. Thus we have

Lemma 2.5. *Assume that* (H1)-(H3) *hold. Then $A = T \circ F : K \to K$ is completely continuous.*

By the definition of operator T, the positive ω-periodic solution of (1.1) is equivalent to the nontrivial fixed point of A. We will find the nonzero fixed point of A by using the fixed point index in cones.

We recall some concepts and conclusions on the fixed point index in [3, 5]. For the details, see [3, Chapter 6] or [5, Chapter 3]. Let X be a Banach space and $K \subset X$ be a closed convex cone in X. Assume Ω is a bounded open subset of X with boundary $\partial\Omega$, and $K \cap \Omega \neq \emptyset$. Let $A : K \cap \overline{\Omega} \to K$ be a completely continuous mapping. If $Au \neq u$ for every $u \in K \cap \partial\Omega$, then the fixed point index $i(A, K \cap \Omega, K)$ is well defined. One important fact is that if $i(A, K \cap \Omega, K) \neq 0$, then A has a fixed point in $K \cap \Omega$. The following two lemmas are needed in our argument. The proofs of these lemmas can be found in [3, 5].

Lemma 2.6. *Let X be a Banach space, $K \subset X$ be a closed convex cone, $\Omega \subset X$ be a bounded open subset, and $A : K \cap \overline{\Omega} \to K$ be a completely continuous mapping. Then the following conclusions hold:*

- (i) *If there exists $e \in K \setminus \{\theta\}$ such that $u - Au \neq \mu e$ for every $u \in K \cap \partial\Omega$ and $\mu \geq 0$, then $i(A, K \cap \Omega, K) = 0$.*
- (ii) *If $\theta \in \Omega$ and $Au \neq \mu u$ for every $u \in K \cap \partial\Omega$ and $\mu \geq 1$, then $i(A, K \cap \Omega, K) = 1$.*

Lemma 2.7. *Let X be a Banach space, $K \subset X$ be a closed convex cone, $\Omega \subset X$ be a bounded open subset with $\theta \in \Omega$, $A : K \cap \overline{\Omega} \to K$ be a completely continuous mapping and it satisfies that $Au \neq u$ for every $u \in K \cap \partial\Omega$. Then the following conclusions hold:*

- (i) *If $\|Au\| \geq \|u\|$ for $u \in K \cap \partial\Omega$, then $i(A, K \cap \Omega, K) = 0$.*
- (ii) *If $\|Au\| \leq \|u\|$ for $u \in K \cap \partial\Omega$, then $i(A, K \cap \Omega, K) = 1$.*

3. Main results

Suppose that $f : \mathbb{R} \times C_B \to [0,\infty)$ satisfies Assumption (H3). We consider the existence and multiplicity of positive ω-periodic solutions of Equation (1.1). Define a closed convex cone $\mathbb{K}$ in C_B by

$$\mathbb{K} = \{\phi \in C_B : \phi(s) \geq \sigma\|\phi\|_B,\ s \in (-\infty, 0]\}. \tag{3.1}$$

Let K be the cone in $C_\omega(\mathbb{R})$ defined by (2.6). We easily see that for every $u \in K$ and $t \in \mathbb{R}$, $u_t \in \mathbb{K}$ and $\|u_t\|_B = \|u\|_C$. For $r > 0$, set

$$\mathbb{K}_r = \{\phi \in \mathbb{K} : \|\phi\|_B < r\}, \quad \partial\mathbb{K}_r = \{\phi \in \mathbb{K} : \|\phi\|_B = r\}, \tag{3.2}$$

$$K_r = \{u \in K : \|u\|_C < r\}, \quad \partial K_r = \{u \in K : \|u\|_C = r\}. \tag{3.3}$$

For convenience, we introduce the following symbols:

$$\begin{aligned} f_0 &= \liminf_{\phi\in\mathbb{K}\|\phi\|_B\to 0^+} \min_{t\in[0,\omega]} \frac{f(t,\phi)}{\|\phi\|_B}, \\ f^0 &= \limsup_{\phi\in\mathbb{K}\|\phi\|_B\to 0^+} \max_{t\in[0,\omega]} \frac{f(t,\phi)}{\|\phi\|_B}, \\ f_\infty &= \liminf_{\phi\in\mathbb{K}\|\phi\|_B\to \infty} \min_{t\in[0,\omega]} \frac{f(t,\phi)}{\|\phi\|_B}, \\ f^\infty &= \limsup_{\phi\in\mathbb{K}\|\phi\|_B\to \infty} \max_{t\in[0,\omega]} \frac{f(t,\phi)}{\|\phi\|_B}. \end{aligned}$$

Our main results are as follows.

Theorem 3.1. *Suppose that* (H1)–(H3) *hold. If f satisfies the condition*

(F1) $f^0 < \sigma\lambda_1$, $f_\infty > \lambda_1$,

then (1.1) *has at least one positive ω-periodic solution.*

Theorem 3.2. *Suppose that* (H1)–(H3) *hold. If f satisfies the condition*

(F2) $f_0 > \lambda_1$, $f^\infty < \sigma\lambda_1$,

then (1.1) *has at least one positive ω-periodic solution.*

Theorem 3.3. *Suppose that* (H1)–(H3) *hold. If f satisfies the following conditions*

(F3) $f^0 < \sigma\lambda_1$, $f^\infty < \sigma\lambda_1$;

(F4) *there exists $\alpha > 0$ such that*

$$f(t,\phi) > \frac{\alpha}{\underline{G}\int_0^\omega c(s)ds}, \quad \text{for } \phi \in \partial\mathbb{K}_\alpha,\ t \in [0,\omega],$$

then (1.1) *has at least two positive ω-periodic solutions.*

Theorem 3.4. *Suppose that* (H1)–(H3) *hold. If f satisfies the following conditions*

(F5) $f_0 > \lambda_1$, $f_\infty > \lambda_1$;

(F6) *there exists $\beta > 0$ such that*

$$f(t,\phi) < \frac{\beta}{\overline{G}\int_0^\omega c(s)ds}, \quad \text{for} \quad \phi \in \partial\mathbb{K}_\beta,\ t \in [0,\omega],$$

then (1.1) *has at least two positive periodic solutions.*

Proof of Theorem 3.1. Choose the working space $X = C_\omega(\mathbb{R})$. Let K be the closed convex cone in $C_\omega(\mathbb{R})$ defined by (2.6) and $A : K \to K$ be the operator defined by (2.13). Then the positive ω-periodic solution of Equation (1.1) is equivalent to the nontrivial fixed point of A. Let $0 < r < R < +\infty$ and set

$$\Omega_r = \{u \in C_\omega(\mathbb{R}) : \|u\|_C < r\}, \quad \Omega_R = \{u \in C_\omega(\mathbb{R}) : \|u\|_C < R\}. \tag{3.4}$$

We show that the operator A has a fixed-point in $K \cap (\Omega_R \setminus \overline{\Omega}_r)$ when r is small enough and R large enough.

Since $f^0 < \sigma\lambda_1$, by the definition of f^0, there exist $\eta \in (0, \sigma\lambda_1)$ and $\delta > 0$ such that

$$f(t, \phi) \le \eta \|\phi\|_B, \quad t \in [0, \omega],\ \phi \in \mathbb{K}_\delta. \tag{3.5}$$

Choosing $r \in (0, \delta)$, we prove that A satisfies the condition of Lemma 2.6 (ii) in $K \cap \partial\Omega_r$, namely $Au \neq \mu u$ for every $u \in K \cap \partial\Omega_r$ and $\mu \ge 1$. In fact, if it is not true, there exist $u_0 \in K \cap \partial\Omega_r = \partial K_r$ and $\mu_0 \ge 1$ such that $Au_0 = \mu_0 u_0$. From the definitions of ∂K_r and $\partial\mathbb{K}_r$, we easily see that $u_{0t} \in \partial\mathbb{K}_r \subset \mathbb{K}_\delta$ and $\|u_{0t}\|_B = \|u_0\|_C$ for every $t \in \mathbb{R}$. From this and (3.4) it follows that

$$f(t, u_{0t}) \le \eta \|u_{0t}\|_B = \eta \|u_0\|_C \le \frac{\eta}{\sigma}\, u_0(t), \quad t \in [0, \omega]. \tag{3.6}$$

By this and the definition of A and (2.8), we have

$$\begin{aligned} u_0(t) &= \frac{1}{\mu_0}\, Au_0(t) \le Au_0(t) \\ &= \int_0^\omega G(t,s)c(s)f(s, u_{0s})ds \\ &\le \frac{\eta}{\sigma} \int_0^\omega G(t,s)c(s)u_0(s)ds \\ &= \frac{\eta}{\sigma}\, Tu_0(t), \quad t \in [0, \omega]. \end{aligned}$$

Hence, we have

$$\theta \le u_0 \le \frac{\eta}{\sigma}\, Tu_0.$$

By the positivity of T, inductively, we obtain that

$$u_0 \le \big(\frac{\eta}{\sigma}\big)^k T^k u_0, \quad k = 1, 2, 3, \dots. \tag{3.7}$$

So we have

$$\|u_0\|_C \le \big(\frac{\eta}{\sigma}\big)^k \|T^k u_0\|_C \le \big(\frac{\eta}{\sigma}\big)^k \|T^k\| \cdot \|u_0\|_C. \quad k = 1, 2, 3, \dots.$$

From this it follows that

$$\|T^k\| \ge \big(\frac{\sigma}{\eta}\big)^k, \quad k = 1, 2, 3, \dots.$$

By this and the Gelfand's formula of spectral radius, we have

$$r(T) = \lim_{k \to \infty} \|T^k\|^{1/k} \ge \frac{\sigma}{\eta} > \frac{1}{\lambda_1} = r(T),$$

which is a contradiction. Hence A satisfies the condition of Lemma 2.6 (ii) in $K \cap \partial\Omega_r$. By Lemma 2.6 (ii), we have

$$i(A, K \cap \Omega_r, K) = 1. \tag{3.8}$$

On the other hand, since $f_\infty > \lambda_1$, by the definition of f_∞, there exist $\eta_1 > \lambda_1$ and $H > 0$ such that

$$f(t, \phi) \ge \eta_1 \|\phi\|_B, \quad t \in [0, \omega],\ \phi \in \mathbb{K},\ \|\phi\|_B > H. \tag{3.9}$$

Choose $R > \max\{H/\sigma,\ \delta\}$ and $e(t) \equiv 1$. Clearly, $e \in K \setminus \{\theta\}$. We show that A satisfies the condition of Lemma 2.6 (i) in $K \cap \partial\Omega_R$, namely $u - Au \neq \mu e$ for every $u \in K \cap \partial\Omega_R$ and $\mu \ge 0$. In fact if it is not true, there exist $u_1 \in K \cap \partial\Omega_R = \partial K_R$

and $\mu_1 \geq 0$ such that $u_1 - Au_1 = \mu_1 e$. For every $t \in \mathbb{R}$, since $u_{1t} \in \mathbb{K}$, from the definition of $\mathbb{K}$ it follows that

$$u_{1t}(s) \geq \sigma\|u_{1t}\|_B = \sigma\|u_1\|_C = \sigma R > H, \quad s \in (-\infty, 0],$$

and hence $\|u_{1t}\|_B > H$. By (3.9), we have

$$f(t, u_{1t}) \geq \eta_1 |u_{1t}\|_B = \eta_1 |u_1\|_C \geq \eta_1 u_1(t), \quad t \in [0, \omega]. \tag{3.10}$$

By this and the definition of A and (2.8), we have

$$\begin{aligned} u_1(t) &= Au_1(t) + \mu_1 e(t) \geq Au_1(t) \\ &= \int_0^\omega G(t,s)c(s)f(s, u_{1s})ds \\ &\geq \eta_1 \int_0^\omega G(t,s)c(s)u_1(s)ds \\ &= \eta_1 Tu_1(t), \quad t \in [0, \omega]. \end{aligned}$$

This implies $u_1 \geq \eta_1 Tu_1$. By the positivity of T, inductively, we obtain that

$$u_1 \geq {\eta_1}^k T^k u_1, \quad k = 1, 2, 3, \dots. \tag{3.11}$$

Since $u_1(t) \geq \sigma\|u_1\|_C$ for $t \in \mathbb{R}$, by the positivity of T^k, we have

$$T^k u_1 \geq T^k(\sigma\|u_1\|_C) = \sigma\|u_1\|_C T^k(1), \quad k = 1, 2, 3, \dots.$$

From this and (3.11) it follows that

$$\|u_1\|_C \geq {\eta_1}^k \,\sigma\|u_1\|_C \,\|T^k(1)\|_C \quad k = 1, 2, 3, \dots.$$

Thus, we have

$$\|T^k(1)\|_C \leq \frac{1}{\sigma {\eta_1}^k}, \quad k = 1, 2, 3, \dots. \tag{3.12}$$

Next we show that

$$\|T^k\| \leq \|T^k(1)\|_C, \quad k = 1, 2, 3, \dots. \tag{3.13}$$

Given $k \in \mathbb{N}$, for every $h \in C_\omega(\mathbb{R})$, since $-\|h\|_C \leq h(t) \leq \|h\|_C$ for every $t \in \mathbb{R}$, by the positivity of T^k we have

$$-\|h\|_C T^k(1)(t) \leq T^k h(t) \leq \|h\|_C T^k(1)(t), \quad t \in \mathbb{R},$$

and hence,

$$\|T^k h\|_C \leq \|T^k(1)\|_C |h\|_c.$$

This means that (3.13) holds.

Now from (3.12) and (3.13) it follows that

$$\|T^k\| \leq \frac{1}{\sigma {\eta_1}^k}, \quad k = 1, 2, 3, \dots. \tag{3.14}$$

By this and the formula of spectral radius we have

$$r(T) = \lim_{k\to\infty} \|T^k\|^{1/k} \leq \frac{1}{\eta_1} < \frac{1}{\lambda_1} = r(T),$$

which is a contradiction. Hence A satisfies the condition of Lemma 2.6 (i) in $K \cap \partial\Omega_R$. By Lemma 2.6 (i), we have

$$i(A, K \cap \Omega_R, K) = 0. \tag{3.15}$$

Now, by the additivity of fixed point index, (3.8) and (3.15), we have

$$i(A, K \cap (\Omega_R \setminus \overline{\Omega}_r), K) = i(A, K \cap \Omega_R, K) - i(A, K \cap \Omega_r, K) = -1.$$

Hence A has a fixed point in $K \cap (\Omega_R \setminus \overline{\Omega}_r)$, which is a positive ω-periodic solution of Equation (1.1). □

Proof of Theorem 3.2. Let $\Omega_r, \Omega_R \subset C_\omega(\mathbb{R})$ be defined by (3.3). We prove that the operator A defined by (2.13) has a fixed point in $K \cap \Omega_R \setminus \overline{\Omega_r}$ when r is small enough and R large enough.

Since $f_0 > \lambda_1$, by the definition of f_0, there exist $\eta_1 > \lambda_1$ and $\delta > 0$ such that

$$f(t,\phi) \geq \eta_1 \|\phi\|_B, \quad t \in [0,\omega],\ \phi \in \mathbb{K}_\delta. \tag{3.16}$$

Choose $r \in (0,\delta)$ and $e(t) \equiv 1$. Clearly, $e \in K \setminus \{\theta\}$. We show that A satisfies the condition of Lemma 2.6 (i) in $K \cap \partial\Omega_r$, namely $u - Au \neq \mu e$ for every $u \in K \cap \partial\Omega_r$ and $\mu \geq 0$. In fact if it's not true, there exist $u_0 \in K \cap \partial\Omega_r = \partial K_r$ and $\mu_0 \geq 0$ such that $u_0 - Au_0 = \mu_0\, e$. Since $u_{0t} \in \partial\mathbb{K}_r \subset \mathbb{K}_\delta$ and $\|u_{0t}\|_B = \|u_0\|_C$ for every $t \in \mathbb{R}$, from (3.16) it follows that

$$f(t,u_{0t}) \geq \eta_1 |u_{0t}\|_B = \eta_1 |u_0\|_C \geq \eta_1 u_0(t), \quad t \in [0,\omega]. \tag{3.17}$$

By this and the definition of A and (2.8), we have

$$\begin{aligned} u_0(t) &= Au_1(t) + \mu_0\, e(t) \geq Au_0(t) \\ &= \int_0^\omega G(t,s)c(s)f(s,u_{0s})ds \\ &\geq \eta_1 \int_0^\omega G(t,s)c(s)u_0(s)ds \\ &= \eta_1 Tu_0(t), \quad t \in [0,\omega]. \end{aligned}$$

This implies $u_0 \geq \eta_1 Tu_0$. By the positivity of T, inductively, we obtain that

$$u_0 \geq {\eta_1}^k T^k u_0, \quad k = 1,2,3,\dots. \tag{3.18}$$

Using this and a demonstration similar to (3.15), we obtain that

$$i(A, K \cap \Omega_r, K) = 0. \tag{3.19}$$

Since $f^\infty < \sigma\lambda_1$, by the definition of f^∞, there exist $\eta \in (0,\lambda_1)$ and $H > 0$ such that

$$f(t,\phi) \leq \eta |\phi\|_B, \quad t \in [0,\omega],\ \phi \in \mathbb{K},\ \|\phi\|_B > H. \tag{3.20}$$

Choosing $R > \max\{\frac{H}{\sigma}, \delta\}$, we prove that A satisfies the condition of Lemma 2.6 (ii) in $K \cap \partial\Omega_R$, namely $Au \neq \mu u$ for every $u \in K \cap \partial\Omega_R$ and $\mu \geq 1$. In fact, if it's not true, there exist $u_1 \in K \cap \partial\Omega_R = \partial K_R$ and $\mu_1 \geq 1$ such that $Au_1 = \mu_1 u_1$. For every $t \in \mathbb{R}$, since $u_{1t} \in \mathbb{K}$, from the definition of $\mathbb{K}$ it follows that

$$u_{1t}(s) \geq \sigma \|u_{1t}\|_B = \sigma \|u_1\|_C = \sigma R > H, \quad s \in (-\infty, 0],$$

and hence $\|u_{1t}\|_B > H$. By (3.20), we have

$$f(t,u_{1t}) \leq \eta |u_{1t}\|_B = \eta |u_1\|_C \leq \frac{\eta}{\sigma}\, u_1(t), \quad t \in [0,\omega]. \tag{3.21}$$

By this and the definition of A and (2.8), we have

$$\begin{aligned} u_1(t) = \frac{1}{\mu_1} Au_1(t) \leq Au_1(t) &= \int_0^\omega G(t,s)c(s)f(s,u_{1s})ds \\ \leq \frac{\eta}{\sigma} \int_0^\omega G(t,s)c(s)u_1(s)ds \\ = \frac{\eta}{\sigma} Tu_1(t), \quad t \in [0,\omega]. \end{aligned}$$

This implies

$$\theta \le u_1 \le \frac{\eta}{\sigma} T u_1.$$

By the positivity of T, inductively, we obtain that

$$u_1 \le \big(\frac{\eta}{\sigma}\big)^k T^k u_1, \quad k = 1, 2, 3, \dots. \tag{3.22}$$

Using this and a demonstration similar to (3.8), we can obtain that

$$i(A, K \cap \Omega_R, K) = 1. \tag{3.23}$$

Now, from (3.19) and (3.23) it follows that

$$i(A, K \cap (\Omega_R \setminus \overline{\Omega}_r), K) = i(A, K \cap \Omega_R, K) - i(A, K \cap \Omega_r, K) = 1.$$

Hence A has a fixed point in $K \cap (\Omega_R \setminus \overline{\Omega}_r)$, which is a positive ω-periodic solution of (1.1). □

Proof of Theorem 3.3. Set $\Omega_\alpha = \{u \in C_\omega(\mathbb{R}) : \|u\|_C < \alpha\}$, we show that

$$\|Au\|_C > \|u\|_C, \quad u \in K \cap \partial\Omega_\alpha. \tag{3.24}$$

Let $u \in K \cap \partial\Omega_\alpha = \partial K_\alpha$. Since $\|u_{0t}\|_B = \|u_0\|_C = \alpha$ and $u_t \in \partial\mathbb{K}_\alpha$ for every $t \in \mathbb{R}$, by the assumption (F5), we have

$$f(t, u_t) > \frac{\alpha}{\underline{G} \int_0^\omega c(s) ds}, \quad t \in [0, \omega].$$

By the definition of A and (2.8), we have

$$Au(t) = \int_0^\omega G(t,s)c(s)f(s,u_s)ds > \frac{\alpha}{\underline{G} \int_0^\omega c(s) ds} \int_0^\omega G(t,s)c(s)\, ds \ge \alpha,$$

from which it follows that $\|Au\|_C > \alpha = \|u\|_C$. Hence (3.24) holds. By Lemma 2.7 (i), we have

$$i(A, K \cap \Omega_\alpha, K) = 0. \tag{3.25}$$

Since $f^0 < \sigma\lambda_1$, by the proof of Theorem 3.1, there exists $r < \alpha$ such that (3.8) holds, and since $f^\infty < \sigma\lambda_1$, by the proof of Theorem 3.2, there exists $R > \alpha$ such that (3.23) holds. Using the additivity of fixed point index, by (3.8), (3.23) and (3.25) we have

$$i(A, K \cap (\Omega_\alpha \setminus \overline{\Omega}_r), K) = i(A, K \cap \Omega_\alpha, K) - i(A, K \cap \Omega_r, K) = -1,$$
$$i(A, K \cap (\Omega_R \setminus \overline{\Omega}_\alpha), K) = i(A, K \cap \Omega_R, K) - i(A, K \cap \Omega_\alpha, K) = 1.$$

Hence A has two fixed points $u_1 \in K \cap (\Omega_\alpha \setminus \overline{\Omega}_r)$ and $u_2 \in K \cap (\Omega_R \setminus \overline{\Omega}_\alpha)$, and u_1 and u_2 are two positive ω-periodic solutions of (1.1). □

Proof of Theorem 3.4. Set $\Omega_\beta = \{u \in C_\omega(\mathbb{R}) : \|u\|_C < \beta\}$, we show that

$$\|Au\|_C < \|u\|_C, \quad u \in K \cap \partial\Omega_\beta. \tag{3.26}$$

Let $u \in K \cap \partial\Omega_\beta = \partial K_\beta$. Since $\|u_{0t}\|_B = \|u_0\|_C = \beta$ and $u_t \in \partial\mathbb{K}_\beta$ for every $t \in \mathbb{R}$, by the assumption (F6), we have

$$f(t, u_t) < \frac{\beta}{\overline{G} \int_0^\omega c(s) ds}, \quad t \in [0, \omega].$$

By the definition of A and (2.8), we have

$$Au(t) = \int_0^\omega G(t,s)c(s)f(s,u_s)ds < \frac{\beta}{\overline{G} \int_0^\omega c(s) ds} \int_0^\omega G(t,s)c(s)\, ds \le \beta,$$

from which it follows that $\|Au\|_C < \beta = \|u\|_C$. Hence (3.24) holds. By Lemma 2.7 (ii), we have

$$i(A, K \cap \Omega_\beta, K) = 1. \tag{3.27}$$

Since $f_0 > \lambda_1$, by the proof of Theorems 3.2, there exists $r < \beta$ such that (3.19) holds, and since $f_\infty > \lambda_1$, by the proof of Theorems 3.1, there exists $R > \beta$ such that (3.15) holds. Using the additivity of fixed point index, by (3.19), (3.27) and (3.15) we have

$$i(A, K \cap (\Omega_\beta \setminus \overline{\Omega}_r), K) = i(A, K \cap \Omega_\beta, K) - i(A, K \cap \Omega_r, K) = 1,$$
$$i(A, K \cap (\Omega_R \setminus \overline{\Omega}_\beta), K) = i(A, K \cap \Omega_R, K) - i(A, K \cap \Omega_\beta, K) = -1.$$

Hence A has two fixed points $u_1 \in K \cap (\Omega_\beta \setminus \overline{\Omega}_r)$ and $u_2 \in K \cap (\Omega_R \setminus \overline{\Omega}_\beta)$, and u_1 and u_2 are two positive ω-periodic solutions of Equation (1.1). □

We also have the following multiplicity result.

Theorem 3.5. *Suppose that* (H1)–(H3) *hold. If f satisfies one of the following conditions*

- (i) (F1) *holds, and there exist positive constants α, β satisfying $\alpha < \beta$, such that* (F5) *and* (F6) *hold;*
- (ii) (F2) *holds, and there exist positive constants β, α satisfying $\beta < \alpha$, such that* (F6) *and* (F5) *hold,*

then (1.1) *has at least three positive ω-periodic solutions.*

Proof. We prove only the case of that the condition (i) holds. The case of that the condition (ii) holds can be proved by the same method.

Since $f^0 < \sigma\lambda_1$ and $f_\infty > \lambda_1$, by the proof of Theorem 3.1, there exist $r < \alpha$ and $R > \beta$ such that (3.8) and (3.15) hold. By the proofs of Theorems 3.3–3.4, (3.25) and (3.27) hold. Hence by the additivity of fixed point index, we have

$$i(A, K \cap (\Omega_\alpha \setminus \overline{\Omega}_r), K) = i(A, K \cap \Omega_\alpha, K) - i(A, K \cap \Omega_r, K) = -1,$$
$$i(A, K \cap (\Omega_\beta \setminus \overline{\Omega}_\alpha), K) = i(A, K \cap \Omega_\beta, K) - i(A, K \cap \Omega_\alpha, K) = 1,$$
$$i(A, K \cap (\Omega_R \setminus \overline{\Omega}_\beta), K) = i(A, K \cap \Omega_R, K) - i(A, K \cap \Omega_\beta, K) = -1.$$

From these we conclude that A has three fixed point $u_1 \in K \cap (\Omega_\alpha \setminus \overline{\Omega}_r)$, $u_2 \in K \cap (\Omega_\beta \setminus \overline{\Omega}_\alpha)$ and $u_3 \in K \cap (\Omega_R \setminus \overline{\Omega}_\beta)$. Hence u_1, u_2 and u_3 satisfy

$$r < \|u_1\|_C < \alpha < \|u_1\|_C < \beta < \|u_1\|_C < R, \tag{3.28}$$

and are three positive ω-periodic solutions of (1.1). □

Example 3.6. Consider the second-order differential equation with infinite delay

$$u''(t) + b(t)u'(t) + a(t)u(t) = c(t) \int_{-\infty}^{t} e^{\alpha(s-t)} u^2(s) ds, \quad t \in \mathbb{R}, \tag{3.29}$$

where $a(t), b(t), c(t)$ are continuous ω-periodic functions on $\mathbb{R}$ and they satisfy assumptions (H1) and (H2), $\alpha > 0$ is a constant. We show that (3.29) has at least one positive ω-periodic solution.

For $u \in C_\omega(\mathbb{R})$, since

$$\int_{-\infty}^{t} e^{\alpha(s-t)} u^2(s) ds = \int_{-\infty}^{0} e^{\alpha s} u^2(t+s)\, ds = \int_{-\infty}^{0} e^{\alpha s} {u_t}^2(s)\, ds,$$

we define the mapping $f : \mathbb{R} \times C_B \to [0, \infty)$ by

$$f(t, \phi) = \int_{-\infty}^{0} e^{\alpha s} \phi^2(s)\, ds, \quad t \in \mathbb{R},\ \phi \in C_B, \tag{3.30}$$

then (3.29) is rewritten to the form of Equation (1.1). By the definition (3.30), $f : \mathbb{R} \times C_B \to [0, \infty)$ is continuous and it satisfies the assumption (H3). We show f satisfies the condition (F1) of Theorem 3.1.

For every $\phi \in \mathbb{K}$, since $\sigma \|\phi\|_B \le \phi(s) \le \|\phi\|_B$ for $s \in (\infty, 0]$, we have

$$f(t, \phi) = \int_{-\infty}^{0} e^{\alpha s} \phi^2(s) ds \le \frac{1}{\alpha} \|\phi\|_B{}^2, \tag{3.31}$$

$$f(t, \phi) = \int_{-\infty}^{0} e^{\alpha s} \phi^2(s) ds \ge \frac{\sigma^2}{\alpha} \|\phi\|_B{}^2. \tag{3.32}$$

From (3.31) and (3.32) it follows that

$$f^0 = \limsup_{\phi \in \mathbb{K} \|\phi\|_B \to 0^+} \max_{t \in [0, \omega]} \frac{f(t, \phi)}{\|\phi\|_B} = 0,$$

$$f_\infty = \liminf_{\phi \in \mathbb{K} \|\phi\|_B \to \infty} \min_{t \in [0, \omega]} \frac{f(t, \phi)}{\|\phi\|_B} = +\infty.$$

Hence, f satisfies the condition (F1). By Theorem 3.1, equation (3.29) has at least one positive ω-periodic solution.

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