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PARTICULAR SOLUTIONS OF GENERALIZED EULER-POISSON-DARBOUX EQUATION

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Abstract. In this article we consider the generalized Euler-Poisson-Darboux equation

$$u_{tt} + \frac{2\gamma}{t} u_t = u_{xx} + u_{yy} + \frac{2\alpha}{x} u_x + \frac{2\beta}{y} u_y, \quad x > 0,\ y > 0,\ t > 0.$$

We construct particular solutions in an explicit form expressed by the Lauricella hypergeometric function of three variables. Properties of each constructed solutions have been investigated in sections of surfaces of the characteristic cone. Precisely, we prove that found solutions have singularity $1/r$ at $r \to 0$, where $r^2 = (x - x_0)^2 + (y - y_0)^2 - (t - t_0)^2$.

1. Introduction

Many problems of modern mathematics and theoretical physics lead to the investigation of hypergeometric functions of many variables. In particular, problems of superstring theory [8], analytical continuations of Mellin-Barnes integrals [14] and algebraic geometry [13]. Systems of hypergeometric type differential equations have numerous applications as nontrivial model examples in realization of algorithms for symbolic calculations, which are used in modern systems of computer algebra [16]. Hypergeometric functions of many variables appear in quantum field theory as solutions of Knizhnik-Zamolodchikov equation [22]. These equations can be considered as generalized hypergeometric type equations and their solutions have integral representations, which generalize classic Euler integrals for hypergeometric functions of one variable. This approach allows us to link the special functions of hypergeometric type and challenges the theory of representations of Lie algebras and quantum groups [22].

Initially hypergeometric functions introduced by many authors with different methods, which are not related with each other. Their occurrence is determined, as a rule, by the need to solve problems, that led to a differential equation (or system of equations), insoluble in the class of elementary functions. Thus arose

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Bessel functions, Hermite functions, Gauss hypergeometric function. Hypergeometric functions occupy an important place among the special functions of mathematical physics. Many problems of gas dynamics are reduced to boundary value problems for degenerate equations of mixed type.

It is known that the degenerate equation of mixed type in the hyperbolic part of the domain is reduced to the generalized equation of Euler-Poisson-Darboux

$$u_{tt} + \frac{2\gamma}{t} u_t = u_{xx} + u_{yy} + \frac{2\alpha}{x} u_x + \frac{2\beta}{y} u_y, \quad x > 0,\ y > 0,\ t > 0, \tag{1.1}$$

where $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ are constants.

We note that Euler-Poisson-Darboux equation

$$u_{\xi\eta} - \frac{\beta}{\xi - \eta} u_\xi + \frac{\alpha}{\xi - \eta} u_\eta = 0, \quad \alpha > 0,\ \beta > 0,\ \alpha + \beta < 1, \tag{1.2}$$

was considered in [21], where the Cauchy problem for (1.2) was solved. In [17, 18, 19], non-local boundary problems for (1.2) were investigated in characteristic triangles. In [11], two confluent hypergeometric functions of three variables were introduced. Further, for introduced hypergeometric functions authors prove formulas of analytical continuation. Using the introduced confluent hypergeometric functions, they constructed the Riemann function for the generalized Euler -Poisson-Darboux equation

$$u_{\xi\eta} + [\frac{\alpha}{\eta + \xi} + \frac{\beta}{\eta - \xi}] u_\xi + [\frac{\alpha}{\eta + \xi} - \frac{\beta}{\eta - \xi}] u_\eta + \gamma u = 0. \tag{1.3}$$

Further, by the Riemann-function method the Cauchy problem for (1.3) was solved in characteristic triangle. Solution is written in an explicit form. Note that Euler -Poisson-Darboux equations (1.2) and (1.3) are written in characteristic coordinates.

In [20], the unique solvability of the Darboux problem with deviation for the Euler -Poisson-Darboux equation was proved outside of characteristic cone. Other type of the Euler -Poisson-Darboux equations were investigated in works [5, 7, 6, 23, 10, 15, 2, 3, 4].

2. Reduction of the Euler-Poisson-Darboux equation to a system of Lauricella hypergeometric functions

Solution of the Euler-Poisson-Darboux equation (1.1) is searched in the form

$$u = P\omega(\xi, \eta, \zeta), \tag{2.1}$$

where $\omega(\xi, \eta, \zeta)$ is unknown function and

$$P = (r^2)^{-\alpha-\beta-\gamma-\frac{1}{2}}, \tag{2.2}$$

$$\xi = \frac{r^2 - r_1^2}{r^2}, \quad \eta = \frac{r^2 - r_2^2}{r^2}, \quad \zeta = \frac{r^2 - r_3^2}{r^2},$$

$$r^2 = (x - x_0)^2 + (y - y_0)^2 - (t - t_0)^2, \quad r_1^2 = (x + x_0)^2 + (y - y_0)^2 - (t - t_0)^2,$$

$$r_2^2 = (x - x_0)^2 + (y + y_0)^2 - (t - t_0)^2, \quad r_3^2 = (x - x_0)^2 + (y - y_0)^2 - (t + t_0)^2.$$

Calculating necessary derivatives from (2.1) and substituting them into (1.1), we obtain

$$\begin{aligned} &A_1\omega_{\xi\xi} + A_2\omega_{\eta\eta} + A_3\omega_{\zeta\zeta} + B_1\omega_{\xi\eta} + B_2\omega_{\xi\zeta} + B_3\omega_{\eta\zeta} \\ &+ C_1\omega_\xi + C_2\omega_\eta + C_3\omega_\zeta + D\omega = 0, \end{aligned} \tag{2.3}$$

where

$$A_1 = P\xi_x^2 + P\xi_y^2 - P\xi_t^2, \quad A_2 = P\eta_x^2 + P\eta_y^2 - P\eta_t^2, \quad A_3 = P\zeta_x^2 + P\zeta_y^2 - P\zeta_t^2,$$
$$B_1 = 2P\xi_x\eta_x + 2P\xi_y\eta_y - 2P\xi_t\eta_t, \quad B_2 = 2P\xi_x\zeta_x + 2P\xi_y\zeta_y - 2P\xi_t\zeta_t,$$
$$B_3 = 2P\eta_x\zeta_x + 2P\eta_y\zeta_y - 2P\eta_t\zeta_t,$$
$$C_1 = P(\xi_{xx} + \xi_{yy} - \xi_{tt}) + 2(P_x\xi_x + P_y\xi_y - P_t\xi_t) + P\Big(\frac{2\alpha}{x}\xi_x + \frac{2\beta}{y}\xi_y - \frac{2\gamma}{t}\xi_t\Big),$$
$$C_2 = P(\eta_{xx} + \eta_{yy} - \eta_{tt}) + 2(P_x\eta_x + P_y\eta_y - P_t\eta_t) + P\Big(\frac{2\alpha}{x}\eta_x + \frac{2\beta}{y}\eta_y - \frac{2\gamma}{t}\eta_t\Big),$$
$$C_3 = P(\zeta_{xx} + \zeta_{yy} - \zeta_{tt}) + 2(P_x\zeta_x + P_y\zeta_y - P_t\zeta_t) + P\Big(\frac{2\alpha}{x}\zeta_x + \frac{2\beta}{y}\zeta_y - \frac{2\gamma}{t}\zeta_t\Big),$$
$$D = P_{xx} + P_{yy} - P_{tt} + P_x\frac{2\alpha}{x} + P_y\frac{2\beta}{y} - \frac{2\gamma}{t}P_t.$$

Now we consider A_1. Since

$$\xi_x^2 = \frac{4(x+x_0)^2(r^2)^2 - 8(x+x_0)(x-x_0)r^2r_1^2 + 4(x-x_0)^2(r_1^2)^2}{(r^2)^4},$$
$$\xi_y^2 = \frac{4(y-y_0)^2(r^2)^2 - 8(y-y_0)^2r^2r_1^2 + 4(y-y_0)^2(r_1^2)^2}{(r^2)^4},$$
$$\xi_t^2 = \frac{4(t-t_0)^2(r^2)^2 - 8(t-t_0)^2r^2r_1^2 + 4(t-t_0)^2(r_1^2)^2}{(r^2)^4},$$

we obtain

$$A_1 = 4Pr_1^2(r^2)^{-3}[(r^2) + (r_1^2) - 2(x+x_0)(x-x_0) - 2(y-y_0)^2 + 2(t-t_0)^2],$$

or $A_1 = 4Pr_1^2(r^2)^{-3}4x_0^2$. By the equality $4x_0 = -x^{-1}r^2\xi$, we obtain

$$A_1 = -4Px^{-1}x_0(r^2)^{-1}\xi(1-\xi). \tag{2.4}$$

Similarly we have

$$A_2 = -4Py^{-1}y_0(r^2)^{-1}\eta(1-\eta), \tag{2.5}$$
$$A_3 = -4Pt^{-1}t_0(r^2)^{-1}\zeta(1-\zeta). \tag{2.6}$$

Further, we calculate representation of B_1. Finding necessary derivatives from the arguments and substituting them we obtain

$$\begin{aligned}B_1 = 2P\Big\{&\frac{2(x+x_0)r^2 - 2(x-x_0)r_1^2}{(r^2)^2}\,\frac{2(x-x_0)r^2 - 2(x-x_0)r_2^2}{(r^2)^2}\\
&+ \frac{2(y-y_0)r^2 - 2(y-y_0)r_1^2}{(r^2)^2}\,\frac{2(y+y_0)r^2 - 2(y-y_0)r_2^2}{(r^2)^2}\\
&- \frac{2(t-t_0)r^2 - 2(t-t_0)r_1^2}{(r^2)^2}\,\frac{2(t-t_0)r^2 - 2(t-t_0)r_2^2}{(r^2)^2}\Big\},\end{aligned}$$

or

$$\begin{aligned}B_1 = 8P(r^2)^{-4}\Big\{&[(x+x_0)r^2 - (x-x_0)r_1^2][(x-x_0)r^2 - (x-x_0)r_2^2]\\
&+ [(y-y_0)r^2 - (y-y_0)r_1^2][(y+y_0)r^2 - (y-y_0)r_2^2]\\
&- [(t-t_0)r^2 - (t-t_0)r_1^2][(t-t_0)r^2 - (t-t_0)r_2^2]\Big\}.\end{aligned}$$

After some evaluations, we deduce

$$B_1 = 4Py^{-1}y_0(r^2)^{-1}\xi\eta + 4Px^{-1}x_0(r^2)^{-1}\xi\eta. \tag{2.7}$$

Similarly one obtains

$$B_2 = 4Pt^{-1}t_0(r^2)^{-1}\xi\zeta + 4Px^{-1}x_0(r^2)^{-1}\xi\zeta, \tag{2.8}$$

$$B_3 = 4Pt^{-1}t_0(r^2)^{-1}\eta\zeta + 4Py^{-1}y_0(r^2)^{-1}\eta\zeta. \tag{2.9}$$

Further, considering the following expressions

$$\begin{gathered}
\xi_{xx} + \xi_{yy} - \xi_{tt} = 2(r^2)^{-1}(2x^{-1}x_0\xi - \xi),\\
P_x\xi_x + P_y\xi_y - P_t\xi_t = 2P(r^2)^{-1}\big(\alpha+\beta+\gamma+\frac{1}{2}\big)(x^{-1}x_0\xi + \xi),\\
\begin{aligned}
\frac{2\alpha}{x}\xi_x + \frac{2\beta}{y}\xi_y - \frac{2\gamma}{t}\xi_t = -4(r^2)^{-1}\big[&\alpha x^{-1}x_0 + \alpha\xi + \alpha x^{-1}x_0(1-\xi) - \beta y^{-1}y_0\\
&+ \beta\xi + \beta y^{-1}y_0(1-\xi) - \gamma t^{-1}t_0 + \gamma\xi + \gamma t^{-1}t_0(1-\xi)\big],
\end{aligned}
\end{gathered}$$

we define

$$\begin{aligned}
C_1 = &-4P(r^2)^{-1}x^{-1}x_0[2\alpha - (2\alpha+\beta+\gamma+\frac{1}{2}+1)\xi]\\
&+ 4P(r^2)^{-1}y^{-1}y_0\beta\xi + 4P(r^2)^{-1}t^{-1}t_0\gamma\xi.
\end{aligned}\tag{2.10}$$

Similarly, we define

$$\begin{aligned}
C_2 = &-4P(r^2)^{-1}y^{-1}y_0[2\beta - (\alpha+2\beta+\gamma+\frac{1}{2}+1)\eta]\\
&+ 4P(r^2)^{-1}x^{-1}x_0\alpha\eta + 4P(r^2)^{-1}t^{-1}t_0\gamma\eta,
\end{aligned}\tag{2.11}$$

$$\begin{aligned}
C_3 = &-4P(r^2)^{-1}t^{-1}t_0[2\gamma - (\alpha+\beta+2\gamma+\frac{1}{2}+1)\zeta]\\
&+ 4P(r^2)^{-1}x^{-1}x_0\gamma\zeta + 4P(r^2)^{-1}y^{-1}y_0\beta\zeta.
\end{aligned}\tag{2.12}$$

After simple calculations we obtain

$$\begin{aligned}
D = &\,4(\alpha+\beta+\gamma+\frac{1}{2})P(r^2)^{-1}\alpha x^{-1}x_0 + 4(\alpha+\beta+\gamma+\frac{1}{2})P(r^2)^{-1}\beta y^{-1}y_0\\
&+ 4(\alpha+\beta+\gamma+\frac{1}{2})P(r^2)^{-1}\gamma t^{-1}t_0.
\end{aligned}\tag{2.13}$$

Substituting (2.4)-(2.13) in (2.3), we obtain

$$\begin{aligned}
&-\frac{4Px_0}{xr^2}\Big\{\xi(1-\xi)\omega_{\xi\xi} - \xi\eta\omega_{\xi\eta} - \xi\zeta\omega_{\xi\zeta} - \alpha\eta\omega_\eta - \gamma\zeta\omega_\zeta\\
&+ [2\alpha - (\alpha+\beta+\gamma+\frac{1}{2}+\alpha+1)\xi]\omega_\xi - (\alpha+\beta+\gamma+\frac{1}{2})\alpha\omega\Big\}\\
&-\frac{4Py_0}{yr^2}\Big\{\eta(1-\eta)\omega_{\eta\eta} - \xi\eta\omega_{\xi\eta} - \eta\zeta\omega_{\eta\zeta} - \beta\xi\omega_\xi - \beta\zeta\omega_\zeta\\
&\quad+ [2\beta - (\alpha+\beta+\gamma+\frac{1}{2}+\beta+1)\eta]\omega_\eta - (\alpha+\beta+\gamma+\frac{1}{2})\beta\omega\Big\}\\
&-\frac{4Pt_0}{tr^2}\Big\{\zeta(1-\zeta)\omega_{\zeta\zeta} - \xi\zeta\omega_{\xi\zeta} - \eta\zeta\omega_{\eta\zeta} - \gamma\xi\omega_\xi - \gamma\eta\omega_\eta\\
&+ [2\gamma - (\alpha+\beta+\gamma+\frac{1}{2}+\gamma+1)\zeta]\omega_\zeta - (\alpha+\beta+\gamma+\frac{1}{2})\gamma\omega\Big\} = 0.
\end{aligned}\tag{2.14}$$

Hence, the following equality is valid

$$
\begin{aligned}
&\xi(1-\xi)\omega_{\xi\xi}-\xi\eta\omega_{\xi\eta}-\xi\zeta\omega_{\xi\zeta}+[2\alpha-(\alpha+\beta+\gamma+\frac{1}{2}+\alpha+1)\xi]\omega_{\xi}\\
&-\alpha\eta\omega_{\eta}-\gamma\zeta\omega_{\zeta}-(\alpha+\beta+\gamma+\frac{1}{2})\alpha\omega=0,\\
&\eta(1-\eta)\omega_{\eta\eta}-\xi\eta\omega_{\xi\eta}-\eta\zeta\omega_{\eta\zeta}+[2\beta-(\alpha+\beta+\gamma+\frac{1}{2}+\beta+1)\eta]\omega_{\eta}\\
&-\beta\xi\omega_{\xi}-\beta\zeta\omega_{\zeta}-(\alpha+\beta+\gamma+\frac{1}{2})\beta\omega=0,\\
&\zeta(1-\zeta)\omega_{\zeta\zeta}-\xi\zeta\omega_{\xi\zeta}-\eta\zeta\omega_{\eta\zeta}+[2\gamma-(\alpha+\beta+\gamma+\frac{1}{2}+\gamma+1)\zeta]\omega_{\zeta}\\
&-\gamma\xi\omega_{\xi}-\gamma\eta\omega_{\eta}-(\alpha+\beta+\gamma+\frac{1}{2})\gamma\omega=0.
\end{aligned}
\tag{2.15}
$$

Thus, the Euler-Poisson-Darboux equation (1.1) equivalently reduced to the system (2.15).

3. Particular solutions of the Euler-Poisson-Darboux equation

In [1], system (2.15) was considered for the n-dimensional case

$$
\begin{aligned}
&x_j(1-x_j)\frac{\partial^2 F_A^{(n)}}{\partial x_j^2}-x_j\sum_{k=1,\,k\neq j}^{n}x_k\frac{\partial^2 F_A^{(n)}}{\partial x_k\partial x_j}+[c_j-(a+b_j+1)x_j]\frac{\partial F_A^{(n)}}{\partial x_j}\\
&-b_j\sum_{k=1,\,k\neq j}^{n}x_k\frac{\partial F_A^{(n)}}{\partial x_k}-ab_jF_A^{(n)}=0,\quad j=1,2,\dots,n.
\end{aligned}
$$

There were found 2^n particular solutions of this system. All of them are expressed by Lauricella hypergeometric functions $F_A^{(n)}$. In particular case, system of hypergeometric functions (2.15) has the following solutions [1]

$$\omega_1=F_A^{(3)}\Big(\alpha+\beta+\gamma+\frac{1}{2};\alpha,\beta,\gamma;2\alpha,2\beta,2\gamma;\xi,\eta,\zeta\Big),\tag{3.1}$$

$$\omega_2=\xi^{1-2\alpha}F_A^{(3)}\Big(-\alpha+\beta+\gamma+\frac{3}{2};1-\alpha,\beta,\gamma;2-2\alpha,2\beta,2\gamma;\xi,\eta,\zeta\Big),\tag{3.2}$$

$$\omega_3=\eta^{1-2\beta}F_A^{(3)}\Big(\alpha-\beta+\gamma+\frac{3}{2};\alpha,1-\beta,\gamma;2\alpha,2-2\beta,2\gamma;\xi,\eta,\zeta\Big),\tag{3.3}$$

$$\omega_4=\zeta^{1-2\gamma}F_A^{(3)}\Big(\alpha+\beta-\gamma+\frac{3}{2};\alpha,\beta,1-\gamma;2\alpha,2\beta,2-2\gamma;\xi,\eta,\zeta\Big),\tag{3.4}$$

$$
\begin{aligned}
\omega_5=\xi^{1-2\alpha}\eta^{1-2\beta}F_A^{(3)}\Big(&-\alpha-\beta+\gamma+\frac{5}{2};1-\alpha,1-\beta,\gamma;\\
&2-2\alpha,2-2\beta,2\gamma;\xi,\eta,\zeta\Big),
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
\omega_6=\xi^{1-2\alpha}\zeta^{1-2\gamma}F_A^{(3)}\Big(&-\alpha+\beta-\gamma+\frac{5}{2};1-\alpha,\beta,1-\gamma;2-2\alpha,2\beta,\\
&2-2\gamma;\xi,\eta,\zeta\Big),
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
\omega_7=\eta^{1-2\beta}\zeta^{1-2\gamma}F_A^{(3)}\Big(&\alpha-\beta-\gamma+\frac{5}{2};\alpha,1-\beta,1-\gamma;2\alpha,2-2\beta,\\
&2-2\gamma;\xi,\eta,\zeta\Big),
\end{aligned}
\tag{3.7}
$$

$$\begin{aligned}\omega_8 = \xi^{1-2\alpha}\eta^{1-2\beta}\zeta^{1-2\gamma}F_A^{(3)}\Big(&-\alpha-\beta-\gamma+\frac{7}{2};1-\alpha,1-\beta,1-\gamma;\\ &2-2\alpha,2-2\beta,2-2\gamma;\xi,\eta,\zeta\Big),\end{aligned}\tag{3.8}$$

where the Lauricella hypergeometric function is defined as

$$F_A^{(3)}(a;b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n+p}(b_1)_m(b_2)_n(b_3)_p}{(c_1)_m(c_2)_n(c_3)_p m!n!p!}x^m y^n z^p.$$

Further, substituting (3.1) - (3.8) in (2.1), we obtain

$$\begin{aligned}&q_1(x,y,t;x_0,y_0,t_0)\\ &=k_1(r^2)^{-\alpha-\beta-\gamma-\frac{1}{2}}F_A\Big(\alpha+\beta+\gamma+\frac{1}{2};\alpha,\beta,\gamma;2\alpha,2\beta,2\gamma;\xi,\eta,\zeta\Big),\end{aligned}\tag{3.9}$$

$$\begin{aligned}&q_2(x,y,t;x_0,y_0,t_0)\\ &=k_2(r^2)^{\alpha-\beta-\gamma-\frac{3}{2}}(xx_0)^{1-2\alpha}F_A^{(3)}\Big(-\alpha+\beta+\gamma+\frac{3}{2};1-\alpha,\beta,\gamma;\\ &\quad 2-2\alpha,2\beta,2\gamma;\xi,\eta,\zeta\Big),\end{aligned}\tag{3.10}$$

$$\begin{aligned}&q_3(x,y,t;x_0,y_0,t_0)\\ &=k_3(r^2)^{-\alpha+\beta-\gamma-\frac{3}{2}}(yy_0)^{1-2\beta}F_A^{(3)}\Big(\alpha-\beta+\gamma+\frac{3}{2};\alpha,1-\beta,\gamma;2\alpha,\\ &\quad 2-2\beta,2\gamma;\xi,\eta,\zeta\Big),\end{aligned}\tag{3.11}$$

$$\begin{aligned}&q_4(x,y,t;x_0,y_0,t_0)\\ &=k_4(r^2)^{-\alpha-\beta+\gamma-\frac{3}{2}}(tt_0)^{1-2\gamma}F_A^{(3)}\Big(\alpha+\beta-\gamma+\frac{3}{2};\alpha,\beta,1-\gamma;2\alpha,2\beta,\\ &\quad 2-2\gamma;\xi,\eta,\zeta\Big),\end{aligned}\tag{3.12}$$

$$\begin{aligned}&q_5(x,y,t;x_0,y_0,t_0)\\ &=k_5(r^2)^{\alpha+\beta-\gamma-\frac{5}{2}}(xx_0)^{1-2\alpha}(yy_0)^{1-2\beta}F_A^{(3)}\Big(-\alpha-\beta+\gamma+\frac{5}{2};1-\alpha,\\ &\quad 1-\beta,\gamma;2-2\alpha,2-2\beta,2\gamma;\xi,\eta,\zeta\Big),\end{aligned}\tag{3.13}$$

$$\begin{aligned}&q_6(x,y,t;x_0,y_0,t_0)\\ &=k_6(r^2)^{\alpha-\beta+\gamma-\frac{5}{2}}(xx_0)^{1-2\alpha}(tt_0)^{1-2\gamma}F_A^{(3)}\Big(-\alpha+\beta-\gamma+\frac{5}{2};1-\alpha,\beta,\\ &\quad 1-\gamma;2-2\alpha,2\beta,2-2\gamma;\xi,\eta,\zeta\Big),\end{aligned}\tag{3.14}$$

$$\begin{aligned}&q_7(x,y,t;x_0,y_0,t_0)\\ &=k_7(r^2)^{-\alpha+\beta+\gamma-\frac{5}{2}}(yy_0)^{1-2\beta}(tt_0)^{1-2\gamma}F_A^{(3)}\Big(\alpha-\beta-\gamma+\frac{5}{2};\alpha,1-\beta,\\ &\quad 1-\gamma;2\alpha,2-2\beta,2-2\gamma;\xi,\eta,\zeta\Big),\end{aligned}\tag{3.15}$$

$$\begin{aligned}&q_8(x,y,t;x_0,y_0,t_0)\\ &=k_8(xx_0)^{1-2\alpha}(yy_0)^{1-2\beta}(tt_0)^{1-2\gamma}(r^2)^{\alpha+\beta+\gamma-\frac{7}{2}}F_A^{(3)}\Big(-\alpha-\beta-\gamma+\frac{7}{2};\\ &\quad 1-\alpha,1-\beta,1-\gamma;2-2\alpha,2-2\beta,2-2\gamma;\xi,\eta,\zeta\Big)\end{aligned}\tag{3.16}$$

where k_i, $i=\overline{1,8}$ are constants.

4. Some properties of particular solutions

Theorem 4.1. *If $\alpha,\beta,\gamma>0$, then particular solutions (3.9)–(3.16) tends to infinity of the order $1/r$ at $r\to 0$.*

Proof. By the expansion for the Lauricella hypergeometric function [12]

$$\begin{aligned}
&F_A^{(3)}(a;b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)\\
&=\sum_{i,j,k=0}^{\infty}\frac{(a)_{i+j+k}(b_1)_{i+j}(b_2)_{i+k}(b_3)_{j+k}}{(c_1)_{i+j}(c_2)_{i+k}(c_3)_{j+k}i!j!k!}x^{i+j}y^{i+k}z^{j+k}\\
&\times F(a+i+j,b_1+i+j;c_1+i+j;x)F(a+i+j+k,b_2+i+k;c_2+i+k;y)\\
&\times F(a+i+j+k,b_3+j+k;c_3+j+k;z),
\end{aligned}\tag{4.1}$$

the particular solution (3.9) is rewritten as follows

$$\begin{aligned}
&q_1(x,y,t;x_0,y_0,t_0)=k_1(r^2)^{-\alpha-\beta-\gamma-\frac12}\\
&\times\sum_{i,j,k=0}^{\infty}\frac{(\alpha+\beta+\gamma+\frac12)_{i+j+k}(\alpha)_{i+j}(\beta)_{i+k}(\gamma)_{j+k}}{(2\alpha)_{i+j}(2\beta)_{i+k}(2\gamma)_{j+k}i!j!k!}\\
&\times\Big(\frac{r^2-r_1^2}{r^2}\Big)^{i+j}\Big(\frac{r^2-r_2^2}{r^2}\Big)^{i+k}\Big(\frac{r^2-r_3^2}{r^2}\Big)^{j+k}\\
&\times F\Big(\alpha+\beta+\gamma+\frac12+i+j,\alpha+i+j;2\alpha+i+j;\frac{r^2-r_1^2}{r^2}\Big)\\
&\times F\Big(\alpha+\beta+\gamma+\frac12+i+j+k,\beta+i+k;2\beta+i+k;\frac{r^2-r_2^2}{r^2}\Big)\\
&\times F\Big(\alpha+\beta+\gamma+\frac12+i+j+k,\gamma+j+k;2\gamma+j+k;\frac{r^2-r_3^2}{r^2}\Big).
\end{aligned}\tag{4.2}$$

Using the formula $F(a,b;c;x)=(1-x)^{-b}F(c-a,b;c;x/(x-1))$ [9], from (4.2) we obtain

$$q_1(x,y,t;x_0,y_0,t_0)=k_1(r^2)^{-\frac12}(r_1^2)^{-\alpha}(r_2^2)^{-\beta}(r_3^2)^{-\gamma}q_1^*(x,y,t;x_0,y_0,t_0),\tag{4.3}$$

where

$$\begin{aligned}
q_1^*(x,y,t;x_0,y_0,t_0)=&\sum_{i,j,k=0}^{\infty}\frac{(\alpha+\beta+\gamma+\frac12)_{i+j+k}(\alpha)_{i+j}(\beta)_{i+k}(\gamma)_{j+k}}{(2\alpha)_{i+j}(2\beta)_{i+k}(2\gamma)_{j+k}i!j!k!}\\
&\times\Big(\frac{r_1^2-r^2}{r_1^2}\Big)^{i+j}\Big(\frac{r_2^2-r^2}{r_2^2}\Big)^{i+k}\Big(\frac{r_3^2-r^2}{r_3^2}\Big)^{j+k}\\
&\times F\Big(\alpha-\beta-\gamma-\frac12,\alpha+i+j;2\alpha+i+j;\frac{r_1^2-r^2}{r_1^2}\Big)\\
&\times F\Big(\beta-\alpha-\gamma-\frac12-j,\beta+i+k;2\beta+i+k;\frac{r_2^2-r^2}{r_2^2}\Big)\\
&\times F\Big(\gamma-\alpha-\beta-\frac12-i,\gamma+j+k;2\gamma+j+k;\frac{r_3^2-r^2}{r_3^2}\Big).
\end{aligned}\tag{4.4}$$

Now we show that $q_1^*(x,y,t;x_0,y_0,t_0)$ is bounded aa $r\to 0$. Since

$$\lim_{r\to 0}\frac{r_1^2-r^2}{r_1^2}=\lim_{r\to 0}\frac{r_2^2-r^2}{r_2^2}=\lim_{r\to 0}\frac{r_3^2-r^2}{r_3^2}=1,$$

hence, one can find

$$\begin{aligned}&F\Big(\alpha-\beta-\gamma-\frac{1}{2},\alpha+i+j;2\alpha+i+j;1\Big)\\&=\frac{\Gamma(2\alpha)\Gamma(\beta+\gamma+\frac12)(2\alpha)_{i+j}}{\Gamma(\alpha)\Gamma(\alpha+\beta+\gamma+\frac12)(\alpha+\beta+\gamma+\frac12)_{i+j}},\end{aligned}\tag{4.5}$$

$$\begin{aligned}&F\Big(\beta-\alpha-\gamma-\frac{1}{2}-j,\beta+i+k;2\beta+i+k;1\Big)\\&=\frac{\Gamma(2\beta)\Gamma(\alpha+\gamma+\frac12)(2\beta)_{i+k}(\alpha+\gamma+\frac12)2_j}{\Gamma(\beta)\Gamma(\alpha+\beta+\gamma+\frac12)(\alpha+\beta+\gamma+\frac12)_{i+j+k}},\end{aligned}\tag{4.6}$$

$$\begin{aligned}&F\Big(\gamma-\alpha-\beta-\frac{1}{2}-i,\gamma+j+k;2\gamma+j+k;1\Big)\\&=\frac{\Gamma(2\gamma)(2\gamma)_{j+k}\Gamma(\alpha+\beta+\frac12)(\alpha+\beta+\frac12)_i}{\Gamma(\gamma)\Gamma(\alpha+\beta+\gamma+\frac12)(\alpha+\beta+\gamma+\frac12)_{i+j+k}}.\end{aligned}\tag{4.7}$$

Due to (4.5)–(4.7) at $r\to 0$ from (4.4) we obtain

$$\begin{aligned}&\lim_{r\to 0}q_1^*(x,y,t;x_0,y_0,t_0)\\&=\frac{\Gamma(2\alpha)\Gamma(2\beta)\Gamma(2\gamma)\Gamma(\alpha+\beta+\frac12)\Gamma(\alpha+\gamma+\frac12)\Gamma(\beta+\gamma+\frac12)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma^3(\alpha+\beta+\gamma+\frac12)}\\&\quad\times\sum_{i,j,k=0}^{\infty}\frac{(\alpha+\beta+\frac12)_i(\alpha+\gamma+\frac12)_j(\alpha)_{i+j}(\beta)_{i+k}(\gamma)_{j+k}}{(\alpha+\beta+\gamma+\frac12)_{i+j}(\alpha+\beta+\gamma+\frac12)_{i+j+k}i!j!k!}.\end{aligned}\tag{4.8}$$

It is easy to show that

$$\begin{aligned}&\sum_{i,j,k=0}^{\infty}\frac{(\alpha+\beta+\frac12)_i(\alpha+\gamma+\frac12)_j(\alpha)_{i+j}(\beta)_{i+k}(\gamma)_{j+k}}{(\alpha+\beta+\gamma+\frac12)_{i+j}(\alpha+\beta+\gamma+\frac12)_{i+j+k}i!j!k!}\\&=\frac{\Gamma(\frac12)\Gamma^2(\alpha+\beta+\gamma+\frac12)}{\Gamma(\alpha+\beta+\frac12)\Gamma(\alpha+\gamma+\frac12)\Gamma(\beta+\gamma+\frac12)}.\end{aligned}\tag{4.9}$$

Thus, from (4.4) we deduce

$$\lim_{r\to 0}q_1^*(x,y,t;x_0,y_0,t_0)=\frac{\sqrt{\pi}\Gamma(2\alpha)\Gamma(2\beta)\Gamma(2\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha+\beta+\gamma+\frac12)}.\tag{4.10}$$

From here considering (4.10), from (4.3) at $r\to 0$ we have the estimate

$$|q_1(x,y,t;x_0,y_0,t_0)|\le\frac{c_0}{r},\tag{4.11}$$

where

$$c_0=\frac{k_1 2^{2\alpha+2\beta+2\gamma-1}\Gamma(\alpha+\beta+\gamma)\Gamma(2\alpha)\Gamma(2\beta)\Gamma(2\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(2\alpha+2\beta+2\gamma)(r_1^2)^{\alpha}(r_2^2)^{\beta}(r_3^2)^{\gamma}}.$$

Estimate (4.11) states that function $q_1(x,y,t;x_0,y_0,t_0)$ at $r\to 0$ tends to infinity of the order $1/r$. Similarly one can prove that every function $q_i(x,y,t;x_0,y_0,t_0)$, $i=2,3,\dots,8$ has singularity $1/r$ as $r\to 0$. $\square$

Theorem 4.2. *If $\alpha, \beta, \gamma > 0$, then the found particular solutions have the following properties*

$$
\begin{gathered}
x^{2\alpha}\frac{\partial}{\partial x}q_1\Big|_{x=0}=0,\quad y^{2\beta}\frac{\partial}{\partial y}q_1\Big|_{y=0}=0,\quad t^{2\gamma}\frac{\partial}{\partial t}q_1\Big|_{t=0}=0,\\
q_2\big|_{x=0}=0,\quad y^{2\beta}\frac{\partial}{\partial y}q_2\Big|_{y=0}=0,\quad t^{2\gamma}\frac{\partial}{\partial t}q_2\Big|_{t=0}=0,\\
x^{2\alpha}\frac{\partial}{\partial x}q_3\Big|_{x=0}=0,\quad q_3\big|_{y=0}=0,\quad t^{2\gamma}\frac{\partial}{\partial t}q_3\Big|_{t=0}=0,\\
x^{2\alpha}\frac{\partial}{\partial x}q_4\Big|_{x=0}=0,\quad y^{2\beta}\frac{\partial}{\partial y}q_4\Big|_{y=0}=0,\quad q_4\big|_{t=0}=0,\\
q_5\big|_{x=0}=0,\quad q_5\big|_{y=0}=0,\quad t^{2\gamma}\frac{\partial}{\partial t}q_5\Big|_{t=0}=0,\\
q_6\big|_{x=0}=0,\quad y^{2\beta}\frac{\partial}{\partial y}q_6\Big|_{y=0}=0,\quad q_6\big|_{t=0}=0,\\
x^{2\alpha}\frac{\partial}{\partial x}q_7\Big|_{x=0}=0,\quad q_7\big|_{y=0}=0,\quad q_7\big|_{t=0}=0,\\
q_8\big|_{x=0}=0,\quad q_8\big|_{y=0}=0,\quad q_8\big|_{t=0}=0.
\end{gathered}
$$

Proofs of the above equalities are based on elementary calculations. These properties could be used in studying various boundary problems for the equation (1.1).

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