

GLOBAL STABILITY OF A DELAYED MOSQUITO-TRANSMITTED DISEASE MODEL WITH STAGE STRUCTURE

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ABSTRACT. This article presents a new eco-epidemiological deterministic delay differential equation model considering a biological controlling approach on mosquitoes, for endemic dengue disease with variable host (human) and variable vector (*Aedes aegypti*) populations, and stage structure for mosquitoes. In this model, predator-prey interaction is considered by using larvae as prey and mosquito-fish as predator. We give a complete classification of equilibria of the model, and sufficient conditions for global stability/global attractivity of some equilibria are given by constructing suitable Lyapunov functionals and using Lyapunov-LaSalle invariance principle. Also, numerical simulations are presented to show the validity of our results.

1. INTRODUCTION

Recently, many scholars have proposed and investigated various kinds of epidemic models in order to understand and describe the dynamics of infectious disease. Most of the mathematical models pertinent to epidemiology are dependents of the baseline SIR (Susceptible, Infectious and Recovered) model which was presented by Kermack and Mackendrick in 1927 based on ODE [23] with the concept of “compartment modelling”. By referring to the classical books [3, 5, 28, 32, 47], the readers can find not only the history of mathematical epidemiology but also the theories related delayed incorporated to biological systems. After Kermack and Mackendrick’s primus model numerous number of models emerged with time delay (see for example [1, 9, 10, 19, 18, 27, 31, 38, 42]), without time delay [8, 17, 46], epidemic model with stage structure [21, 45] and SVIR model with stochastic perturbation [7].

Many serious epidemics such as AIDS and dengue (here, we mentioned few of them for more details in [4, 39]) can be transmitted horizontally as well as vertically. In addition, diseases also spread due to their parental genes [4]. The disease transmission via the vectors for examples West Nile fever, malaria, dengue and Rift valley fever has been studied by many researchers [11, 14, 15, 40]. With regard to dengue, one of the most spread out flavivirus disease propagated by adult female

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Aedes aegypti mosquito species (predominantly by this type, although there are others). Prevalent throughout the year in tropical countries but transmission reaches its peak when the highest rainfall takes place. It is known that this virus spreads among host (human) after an infectious matured female *Aedes aegypti* (vector) having a blood meal from a susceptible host. On the other hand, the susceptible vector is infected after taking a blood meal from an infectious host. Although the disease spreads vertically among vectors, there is very low possibility of getting infected a new comer ([12] and references there in) which discourage us to include vertical transmission to our model. The most appropriate way to eradicate the transmission of this viral disease is to control winged stage female mosquitoes and aquatic stage of mosquitoes because in the near future, it cannot be anticipated a vaccine to prevent from dengue fever ([36] and references there in). However, information regarding the possibility of vaccine and review of the development of a vaccine is found in [30]. Contemporary, eradication and control methods of mosquitos are similar to those arranged over half a century back. In the academic article [13], author has exhibited one of the controlling strategies named Sterile Insect Technique (SIT) for the control of *Aedes aegypti* mosquitoes. Further, RIDL (Release of Insects Carrying a Dominant Lethal) based on new genetic sexing system for male mosquitoes is introduced by which allow only to born of male mosquitoes by blocking of female production of *Aedes aegypti* [20]. As a cost effective methods most countries use high toxic chemicals such as Malathion and insecticides to control mosquito population which are very dangerous for public health. Places with immovable and clear water are available; the female *Aedes aegypti* mosquitoes use those places for oviposition. By continuous awareness programs, the public can be made aware to avoid building up (source reduction) such places. Yet, it is hard to control aquatic stage of mosquitoes in the places such as lakes and ponds. As a biological control method, we can introduce mosquito-fish (predator) into water body in which immature mosquitoes (prey) are usually inhabited. In [35] the prey-dependent consumption predator prey model has been considered and some valuable results have been obtained.

Subsequently in the in 1920s, Lotka and Volterra introduced baseline model for predator-prey interaction, since then wide variety of modified and developed predator-prey models were seen in the literature. Further, in article [16], the authors have considered predator-prey model with infectious disease, models like SI, SIS and SIR with mass action incidence have been applied. In addition to that maturation time taken for the conversion process or gestation time delay for the predator or the prey in predator-prey model has been used to upgrade the model coherence with the natural world [26, 29, 37, 41, 44, 48]. It is ecologically important to look on the effects on stage-structure of immature to become mature of a certain species as all beings in the real world encounter to the stage structure. In the monograph [2], Aiello and Freedman suggested and analyzed stage structure model with constant maturation time delay for single species, for more examples one can refer [6, 24, 33, 34]. The authors in [43] considered a model with maturation delay and completely studied the stability properties and bifurcation analysis.

Motivated by the above works and as predator involvement is positively effect on controlling spread of vector transmission disease. We concern to construct a new echo-epidemic model for vector spread disease with both predator-prey interaction

and stage structure. Nonlinear functional response for predator-prey system is applied here. Biological appearance of our model is shown in Figure 1.

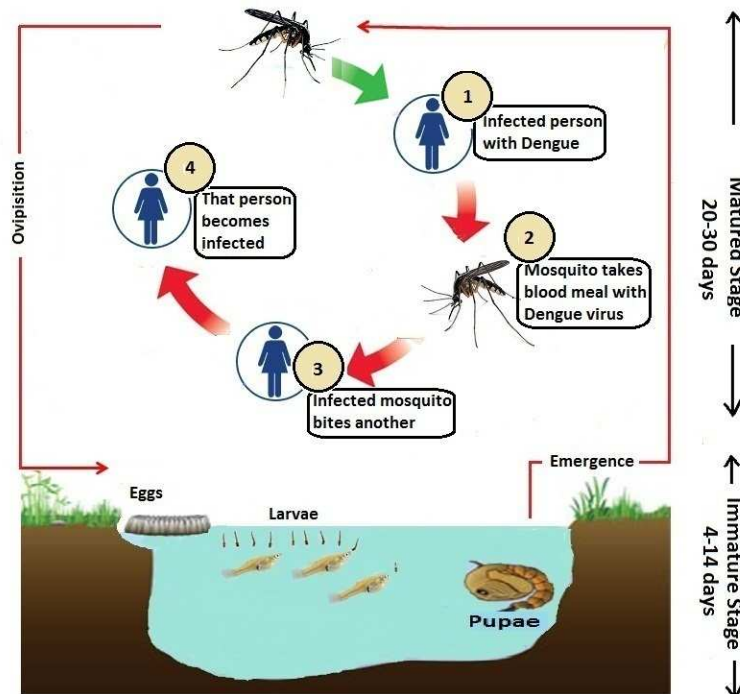


FIGURE 1. Mosquito life-cycle with predator and disease transmission among human

We form a model which is described by a system of differential equations, with the aid of schematic diagram shown in Figure 2.

Let $S_h(t)$, $I_h(t)$ and $R_h(t)$ represent the classes for the human population, indicating the number of susceptible, infective and recovered individuals at time t , respectively. Λ_h , γ_h , μ_h , q_v , τ , μ_l , μ_v , β_v , β_p , k_v , γ_v , μ_p , λ_l and λ_p are all positive constants. Here, μ_h , μ_v , μ_l , and μ_p denote the per capita mortality rate of the human, vector, larvae and predator populations respectively. Λ_h indicates the recruitment of human to susceptible class while γ_h represents the recovered rate. Further, the vector population sub-divided into two classes, namely, $S_v(t)$ is the number of susceptible and $I_v(t)$ is the number of infective. It is assumed that vectors and human populations are mixing homogeneously. It is also accepted that the infected vectors will never be recovered and they transmit the virus in their entire life-span. The birth to the immature population (larvae) is proportional to the currently available number of matured population (vector) and q_v is the conversion rate. In addition, λ_l is the rate of encounter and λ_p is the conversion efficiency. Λ_v represents the recruitment of immigrated mosquitoes to the susceptible class (see for example [22]). We assumed that larvae are the only available food for predators. The density dependent mortality rate is denoted by β_v (see, for example [2]) and the crowding rate is denoted β_p . The state variables $L_v(t)$ and $P(t)$ show up the number of larvae and number of predator at time t , respectively. The larvae who was born

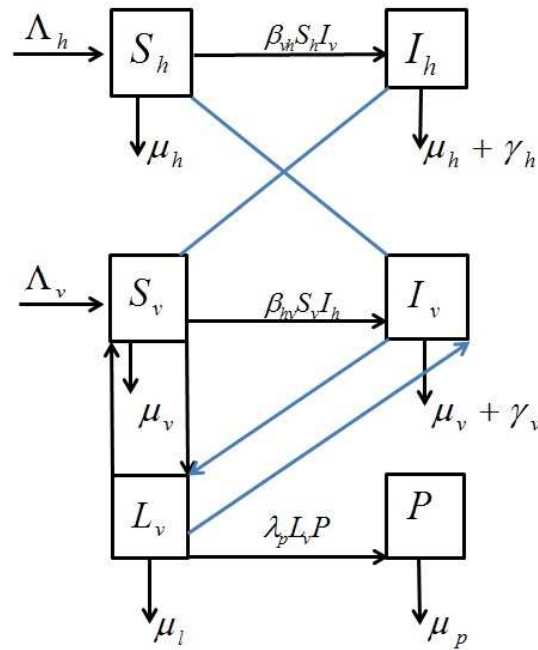


FIGURE 2. Transfer diagram of the disease in human and vectors with predator

at $t - \tau$ and still survive at time t , transforming from larvae (immature stage) to susceptible vector (matured stage) is given by the term $bq_v e^{-\tau\mu} S_v(t - \tau)$ and to infected vector (matured stage) by $aq_v e^{-\tau\mu} I_v(t - \tau)$, where a and b are positive constants such that $a + b = 1$. The number of recovered human at time t is denoted by $R_h(t)$ which has not appeared in the other equations of system (1.1). Therefore, it is excluded from further consideration. The formulated model is given below.

$$\begin{aligned}
 \dot{S}_h(t) &= \Lambda_h - \beta_{vh} I_v(t) S_h(t) - \mu_h S_h(t), \\
 \dot{I}_h(t) &= k_h \beta_{vh} I_v(t) S_h(t) - (\mu_h + \gamma_h) I_h(t), \\
 \dot{S}_v(t) &= \Lambda_v + bq_v e^{-\tau\mu} S_v(t - \tau) - \beta_{hv} I_h(t) S_v(t) - \mu_v S_v(t) - \beta_v S_v^2(t), \\
 \dot{I}_v(t) &= aq_v e^{-\tau\mu} I_v(t - \tau) + k_v \beta_{hv} I_h(t) S_v(t) - (\mu_v + \gamma_v) I_v(t), \\
 \dot{L}_v(t) &= b[S_v(t) - e^{-\tau\mu} S_v(t - \tau)] + a[I_v(t) - e^{-\tau\mu} I_v(t - \tau)] \\
 &\quad - \mu_l L_v(t) - \lambda_l f(L_v(t)) P(t), \\
 \dot{P}(t) &= P(t)(-\mu_p - \beta_p P(t)) + \lambda_p f(L_v(t)) P(t).
 \end{aligned} \tag{1.1}$$

In system (1.1) it is assumed that at time t the number of $\beta_{vh} I_v(t) S_h(t)$ is removed from susceptible human class and simultaneously the number of $k_h \beta_{vh} I_v(t) S_h(t)$ enters to infected human class. It is further assumed that at time t the number of $\beta_{hv} I_h(t) S_v(t)$ is removed from susceptible mosquito class and simultaneously the number of $k_v \beta_{hv} I_h(t) S_v(t)$ enters to infected mosquito class, where β_{vh} , β_{hv} , k_h and k_v are positive constants. $f(L_v)$ is a function such that monotonically increasing, positive and differentiable for all $L_v > 0$ and $f(0) = 0$. More general model for

system (1.1) is formulated as follows:

$$\begin{aligned}
\dot{S}_h(t) &= \Lambda_h - \beta_{vh}I_v(t)S_h(t) - \mu_h S_h(t), \\
\dot{I}_h(t) &= k_h\beta_{vh}e^{-\mu_h\sigma}I_v(t-\sigma)S_h(t-\sigma) - (\mu_h + \gamma_h)I_h(t), \\
\dot{S}_v(t) &= \Lambda_v + bq_v e^{-\tau\mu}S_v(t-\tau) - \beta_{hv}I_h(t)S_v(t) - \mu_v S_v(t) - \beta_v S_v^2(t), \\
\dot{I}_v(t) &= aq_v e^{-\tau\mu}I_v(t-\tau) + k_v\beta_{hv}e^{-\mu_v\omega}I_h(t-\omega)S_v(t-\omega) - (\mu_v + \gamma_v)I_v(t), \\
\dot{L}_v(t) &= b[S_v(t) - e^{-\tau\mu}S_v(t-\tau)] + a[I_v(t) - e^{-\tau\mu}I_v(t-\tau)] \\
&\quad - \mu_l L_v(t) - \lambda_l f(L_v(t))P(t), \\
\dot{P}(t) &= P(t)(-\mu_p - \beta_p P(t)) + \lambda_p f(L_v(t-\rho))P(t-\rho),
\end{aligned} \tag{1.2}$$

where $\sigma > 0$ is the latent delay for human, $\omega > 0$ is the latent delay for mosquito and $\rho > 0$ is the predation delay and others have the same biological meaning as in system (1.1).

This work is organized as follows. In next section, we state some lemmas that are important for our discussion and show the existence, boundedness of the solutions of system (1.1) with the initial condition (2.1). Moreover, conditions are given for existence of all kinds of equilibria of system (1.1); the reproduction number is simultaneously calculated. In Section 3, stability properties of some equilibria are established by means of Lyapunov functionals, and instability of equilibria is given by using characteristic equations. Further, the global attractiveness of other equilibria is also considered. Numerical simulations of the results are presented in Section 4. The paper ends with a brief discussion.

2. BOUNDEDNESS OF SOLUTIONS AND ANALYSIS OF EQUILIBRIA

In this section, we consider the boundedness of solutions and existence of equilibria for system (1.1). First, the initial conditions of system (1.1) are

$$\begin{aligned}
S_h(\theta) &= \varphi_1(\theta), & I_h(\theta) &= \varphi_2(\theta), & S_v(\theta) &= \varphi_3(\theta), \\
I_v(\theta) &= \varphi_4(\theta), & L_v(\theta) &= \varphi_5(\theta), & P(\theta) &= \varphi_6(\theta),
\end{aligned} \tag{2.1}$$

where $(-\tau \leq \theta \leq 0)$ and $\varphi_i(\theta)$ ($i = 1, 2, \dots, 6$) belong to Banach space $C = C([-\tau, 0], R_+)$ of continuous functions mapping from the interval $[-\tau, 0]$ into $R_+ := [0, +\infty)$, equipped with the supremum norm.

Using the basic theory of delay differential equations (see, for example, [24]), it is not difficult to show that, for the initial conditions given above, the solution $(S_h(t), I_h(t), S_v(t), I_v(t), L_v(t), P(t))$ of system (1.1) exists and unique for all time $t \geq 0$. The following lemma is used to obtain our results.

Lemma 2.1. *Consider the delay differential equation $\dot{x}(t) = \alpha + \beta x(t-\tau) - \gamma x(t) - \delta x^2(t)$ where $\alpha, \beta, \gamma > 0$, $\delta \geq 0$, $\tau \geq 0$ are constants, the initial function $\varphi \in C$ and $\varphi(\theta) > 0$.*

(i) *If $\delta > 0$ and $\gamma \geq \beta$ then unique positive equilibrium*

$$x^* = \frac{\beta - \gamma + \sqrt{(\beta - \gamma)^2 + 4\alpha\delta}}{2\delta},$$

is globally asymptotically stable.

(ii) *If $\delta = 0$ and $\gamma > \beta$, then unique positive equilibrium $x^* = \alpha/(\gamma - \beta)$ is globally asymptotically stable.*

Proof. Let us consider case (i). It is not difficult to verify that the solution $x(t)$ with any initial function φ of this equation is positive. For $t \geq 0$, let us define

$$V = x - x^* - x^* \ln \frac{x}{x^*} + \beta \int_{t-\tau}^t \left(x(\theta) - x^* - x^* \ln \frac{x(\theta)}{x^*} \right) d\theta.$$

By considering the time derivative along the solution we have that

$$\begin{aligned} \dot{V} &= (\gamma - \beta)x^* \left(2 - \frac{x}{x^*} - \frac{x^*}{x} \right) + \beta x^* \left(1 - \frac{x(t-\tau)}{x} + \ln \frac{x(t-\tau)}{x} \right) \\ &\quad - \frac{\delta}{x} (x - x^*)^2 (x + x^*). \end{aligned}$$

If $\gamma \geq \beta$, it is clear that $\dot{V} \leq 0$. Therefore, it follows easily from Corollary 5.2 of Kuang [24] that x^* is globally asymptotically stable. Proof of part (ii) can be obtained by using similar method as in part (i). \square

For biological reasons, throughout this paper we discuss the dynamical behavior of system with $\mu_v \geq \max[aq_v e^{-\tau\mu_l}, bq_v e^{-\tau\mu_l}]$.

For boundedness of the solutions to (1.1), we have the following result.

Theorem 2.2. *Every solution $(S_h(t), I_h(t), S_v(t), I_v(t), L_v(t), P(t))$ of (1.1) has the following properties:*

$$\begin{aligned} \limsup_{t \rightarrow +\infty} \left(S_h(t) + \frac{1}{k_h} I_h(t) \right) &\leq M^{S_h}, & \limsup_{t \rightarrow +\infty} I_h(t) &\leq M^{I_h}, \\ \limsup_{t \rightarrow +\infty} S_v(t) &\leq M^{S_v} & \limsup_{t \rightarrow +\infty} I_v(t) &\leq M^{I_v}, \\ \limsup_{t \rightarrow +\infty} L_v(t) &\leq M^{L_v}, & \limsup_{t \rightarrow +\infty} P(t) &\leq M^P, \end{aligned}$$

where

$$\begin{aligned} M^{S_v} &= S_v^{e_0}, & M^{I_h} &= \frac{k_h \beta_{vh} S_h^{e_0}}{\mu_h + \gamma_h} M^{I_v}, \\ M^{I_v} &= \frac{k_h k_v \beta_{hv} S_h^{e_0}}{\mu_v + \gamma_v - a q_v e^{-\tau\mu_l}} M^{S_v}, & M^{L_v} &= \frac{b M^{S_v} + a M^{I_v}}{\mu_l}, \\ M^P &= \begin{cases} \frac{\lambda_p f(M^{L_v}) - \mu_p}{\beta_p} & \lambda_p f(M^{L_v}) > \mu_p, \\ 0, & \lambda_p f(M^{L_v}) \leq \mu_p, \end{cases} \\ M^{S_h} &= S_h^{e_0} = \frac{\Lambda_h}{\mu_h}, \\ S_v^{e_0} &= \frac{1}{2\beta_v} \left(q_v b e^{-\tau\mu_l} - \mu_v + \sqrt{(q_v b e^{-\tau\mu_l} - \mu_v)^2 + 4\beta_v \Lambda_v} \right). \end{aligned}$$

Proof. From the first two equations of (1.1), for $t \geq 0$, we have

$$\left(S_h(t) + \frac{1}{k_h} I_h(t) \right)' \leq \Lambda_h - \mu_h \left(S_h(t) + \frac{1}{k_h} I_h(t) \right),$$

from which, we have

$$\limsup_{t \rightarrow +\infty} (S_h(t) + I_h(t)/k_h) \leq S_h^{e_0}, \quad \limsup_{t \rightarrow +\infty} I_h(t) \leq k_h S_h^{e_0}.$$

Hence, from (1.1) it follows that for $t \geq 0$,

$$\dot{S}_v(t) \leq \Lambda_v + b q_v e^{-\tau\mu_l} S_v(t - \tau) - \mu_v S_v(t) - \beta_v S_v^2(t).$$

The well-known comparison theorems for delay differential equations (see, for example [25]) and Lemma 2.1 imply that $\limsup_{t \rightarrow +\infty} S_v(t) \leq M^{S_v}$.

For any sufficiently small $\varepsilon > 0$, there exists some sufficiently large T , such that, for $t \geq T$, it has from system (1.1) that

$$\dot{I}_v(t) \leq a q_v e^{-\tau \mu_l} I_v(t - \tau) + k_v \beta_{hv} (k_h S_h^{e_0} + \varepsilon) (M^{S_v} + \varepsilon) - (\mu_v + \gamma_v) I_v(t).$$

Hence, from Lemma 2.1, it is easy to obtain that

$$\limsup_{t \rightarrow +\infty} I_v(t) \leq \frac{k_v \beta_{hv} (k_h S_h^{e_0} + \varepsilon) (M^{S_v} + \varepsilon)}{\mu_v + \gamma_v - a q_v e^{-\tau \mu_l}}.$$

Let $\varepsilon \rightarrow 0^+$, it has that $\limsup_{t \rightarrow +\infty} I_v(t) \leq M^{I_v}$.

With similar arguments as above, for any sufficiently small $\varepsilon > 0$, there exists some sufficiently large T_1 , such that for $t \geq T_1$, from system (1.1), we have

$$\begin{aligned} \dot{I}_h(t) &\leq k_h \beta_{vh} (M^{I_v} + \varepsilon) (S_h^{e_0} + \varepsilon) - (\mu_h + \gamma_h) I_h(t), \\ \dot{L}_v(t) &\leq b (M^{S_v} + \varepsilon) + a (M^{I_v} + \varepsilon) - \mu_l L_v(t), \end{aligned}$$

from which it is easy obtain that

$$\begin{aligned} \limsup_{t \rightarrow +\infty} I_h(t) &\leq k_h \beta_{vh} (M^{I_v} + \varepsilon) (S_h^{e_0} + \varepsilon) / (\mu_h + \gamma_h), \\ \limsup_{t \rightarrow +\infty} L_v(t) &\leq (b (M^{S_v} + \varepsilon) + a (M^{I_v} + \varepsilon)) / \mu_l. \end{aligned}$$

Further, by letting $\varepsilon \rightarrow 0^+$, one has that

$$\limsup_{t \rightarrow +\infty} I_h(t) \leq M^{I_h}, \quad \limsup_{t \rightarrow +\infty} L_v(t) \leq M^{L_v}.$$

For a sufficiently small $\varepsilon > 0$, there exists some sufficiently large T_2 , such that for $t \geq T_2$, from system (1.1), one has

$$\begin{aligned} \dot{P}(t) &\leq P(t) (-\mu_p - \beta_p P(t)) + \lambda_p f(M^{L_v} + \varepsilon) P(t) \\ &= [\lambda_p f(M^{L_v} + \varepsilon) - \mu_p - \beta_p P(t)] P(t). \end{aligned}$$

If $\lambda_p f(M^{L_v}) > \mu_p$, for sufficient small $\varepsilon > 0$, it has that $\lambda_p f(M^{L_v} + \varepsilon) - \mu_p > 0$. Hence, we can obtain that $\limsup_{t \rightarrow +\infty} P(t) \leq (\lambda_p f(M^{L_v} + \varepsilon) - \mu_p) / \beta_p$. By letting $\varepsilon \rightarrow 0^+$, one has that $\limsup_{t \rightarrow +\infty} P(t) \leq (\lambda_p f(M^{L_v}) - \mu_p) / \beta_p$.

Further, if $\lambda_p f(M^{L_v}) \leq \mu_p$, it has that for $t \geq T_2$, $\dot{P}(t) \leq (\varepsilon - \beta_p P(t)) P(t)$, which implies that $\limsup_{t \rightarrow +\infty} P(t) \leq \varepsilon / \beta_p$. By letting $\varepsilon \rightarrow 0^+$, we have that $\limsup_{t \rightarrow +\infty} P(t) = 0$. Therefore, it proves that $\limsup_{t \rightarrow +\infty} P(t) \leq M^P$. \square

Next, we study the existence of all possible nonnegative equilibria of system (1.1). We have following four cases to be considered.

(i) There exists boundary equilibrium (disease-free and predator-free) $E_0 = (S_h^{e_0}, 0, S_v^{e_0}, 0, L_v^{e_0}, 0)$, where

$$L_v^{e_0} = \frac{b}{\mu_l} (1 - e^{-\tau \mu_l}) S_v^{e_0},$$

(ii) If $f(L_v^{e_1}) > \mu_p / \lambda_p$ holds, there exists boundary equilibrium (disease-free with predator) $E_1 = (S_h^{e_1}, 0, S_v^{e_1}, 0, L_v^{e_1}, P^{e_1})$, where $S_h^{e_1} = S_h^{e_0}$, $S_v^{e_1} = S_v^{e_0}$. From last two equations of system (1.1), we have that $P^{e_1} = (\lambda_p f(L_v^{e_1}) - \mu_p) / \beta_p$ and

$$G(L_v) = b(1 - e^{-\tau \mu_l}) S_v^{e_1} - \mu_l L_v - \lambda_l f(L_v) P^{e_1} = 0,$$

where L_v is any value which satisfy the fifth equation.

It is easy to show that $G(0) = b(1 - e^{-\tau\mu_l})S_v^{e_1} > 0$ and $G(L_1) = -\lambda_l f(L_1)P^{e_1} < 0$. Therefore, there exists a unique $L_v^{e_1} \in (0, L_1)$ where $L_1 = b(1 - e^{-\tau\mu_l})S_v^{e_1}/\mu_l$.

(iii) If $S_v^{e_0} < S_v^{e_2}R_0$ holds, there exists boundary equilibrium (predator free with disease) $E_2 = (S_h^{e_2}, I_h^{e_2}, S_v^{e_2}, I_v^{e_2}, L_v^{e_2}, 0)$, where

$$S_h^{e_2} = \frac{S_v^{e_0}S_h^{e_0}}{S_v^{e_2}R_0}, \quad I_h^{e_2} = \frac{k_h\Lambda_h}{\mu_h + \gamma_h} \left(1 - \frac{S_v^{e_0}}{R_0S_v^{e_2}}\right),$$

$$I_v^{e_2} = \frac{k_v\beta_{hv}S_v^{e_2}I_h^{e_2}}{(\mu_v + \gamma_v - aq_v e^{-\tau\mu_l})}, \quad L_v^{e_2} = \frac{(bS_v^{e_2} + aI_v^{e_2})}{\mu_l} (1 - e^{-\tau\mu_l}).$$

$S_v^{e_2} = (-K + \sqrt{K^2 + 4\beta_v M})/2\beta_v$ is given by $\beta_v(S_v^{e_2})^2 + KS_v^{e_2} - M = 0$, where

$$K = \frac{\beta_{hv}k_h\Lambda_h}{\mu_h + \gamma_h} + \mu_v - bq_v e^{-\tau\mu_l}, \quad M = \Lambda_v + \frac{(\mu_v + \gamma_v - aq_v e^{-\tau\mu_l})\mu_h}{k_v\beta_{vh}},$$

and

$$R_0 = \frac{k_h\beta_{vh}}{\mu_v + \gamma_v - aq_v e^{-\tau\mu_l}} S_h^{e_0} \frac{k_v\beta_{hv}}{\mu_h + \gamma_h} S_v^{e_0}$$

is the basic reproduction number of system (1.1).

(iv) If $S_v^{e_0} < S_v^*R_0$ and $f(L_v^*) > \mu_p/\lambda_p$ hold, there exists a unique positive equilibrium (disease with predator) $E^* = (S_h^*, I_h^*, S_v^*, I_v^*, L_v^*, P^*)$, where $S_h^* = S_h^{e_2}$, $I_h^* = I_h^{e_2}$, $S_v^* = S_v^{e_2}$ and $I_v^* = I_v^{e_2}$.

From last two equations of system (1.1), we have that $P^* = (\lambda_p f(L_v^*) - \mu_p)/\beta_p$ and

$$H(L_v) = (bS_v^* + aI_v^*)(1 - e^{-\tau\mu_l}) - \mu_l L_v - \lambda_l f(L_v)P^* = 0,$$

where L_v is any value which satisfy the fifth equation. It is not difficult to show that $H(0) = (bS_v^* + aI_v^*)(1 - e^{-\tau\mu_l})S_v^* > 0$ and $H(L_2) = -\lambda_l f(L_2)P^* < 0$. Therefore, there exist a unique $L_v^* \in (0, L_2)$ where $L_2 = (bS_v^* + aI_v^*)(1 - e^{-\tau\mu_l})/\mu_l$.

3. STABILITY OF EQUILIBRIA

In this section, we analyze the stability properties of each equilibrium of system (1.1). The characteristic equation of system (1.1) at any equilibrium $E = (S_h, I_h, S_v, I_v, L_v, P)$ has the form

$$F(\lambda, \tau) \tag{3.1}$$

$$= \begin{vmatrix} \lambda + a_1 & 0 & 0 & \beta_{vh}S_h & 0 & 0 \\ -k_h\beta_{vh}I_v & \lambda + b_1 & 0 & -k_h\beta_{vh}S_h & 0 & 0 \\ 0 & \beta_{hv}S_v & \lambda + q(\lambda, \tau) & 0 & 0 & 0 \\ 0 & -k_v\beta_{hv}S_v & -k_v\beta_{hv}I_h & \lambda + c(\lambda, \tau) & 0 & 0 \\ 0 & 0 & g(\lambda, \tau) & j(\lambda, \tau) & \lambda + d & \lambda_l f(L_v) \\ 0 & 0 & 0 & 0 & -\lambda_p f'(L_v)P & \lambda + e \end{vmatrix} = 0,$$

where

$$a_1 = \beta_{vh}I_v + \mu_h, \quad b_1 = \mu_h + \gamma_h, \quad c(\lambda, \tau) = \mu_v + \gamma_v - aq_v e^{-(\lambda+\mu_l)\tau},$$

$$d = \mu_l + \lambda_l f'(L_v)P, \quad e = \mu_p + 2\beta_p P - \lambda_p f(L_v),$$

$$g(\lambda, \tau) = b(e^{-(\lambda+\mu_l)\tau} - 1), \quad q(\lambda, \tau) = h - q_v b e^{-(\lambda+\mu_l)\tau},$$

$$h = \beta_{hv}I_h + \mu_v + 2\beta_v S_v, \quad j(\lambda, \tau) = a(e^{-(\lambda+\mu_l)\tau} - 1).$$

In next theorem, we establish stability properties of the equilibrium E_0 .

Theorem 3.1. *The following conclusions hold for any time delay $\tau \geq 0$.*

(a) *If $R_0 \leq 1$ and $f(L_v^{e_0}) \leq \mu_p/\lambda_p$ then E_1, E_2 and E^* are not existent, and E_0 is globally asymptotically stable,*

(b) *Either $R_0 > 1$ or $f(L_v^{e_1}) > \mu_p/\lambda_p$ holds, then E_0 unstable.*

Proof. The characteristic equation (3.1) at E_0 is reduced to

$$(\lambda + a_1)(\lambda + q(\lambda, \tau))(\lambda + d)(\lambda + e)\Delta_1(\lambda, \tau) = 0, \tag{3.2}$$

where

$$\begin{aligned} \Delta_1(\lambda, \tau) &= (\lambda + b_1)(\lambda + c(\lambda, \tau)) - k_v\beta_{hv}S_v^{e_0}k_h\beta_{vh}S_h^{e_0} \\ &= \lambda^2 + (\mu_h + \gamma_h + \mu_v + \gamma_v - aq_ve^{-(\lambda+\mu_v)\tau})\lambda \\ &\quad + (\mu_h + \gamma_h)(\mu_v + \gamma_v - aq_ve^{-(\lambda+\mu_v)\tau})(1 - R_0). \end{aligned}$$

By following a similar procedure as the one used in [24] it easy to show for the characteristics equation (3.2) that if $f(L_v^{e_0}) < \mu_p/\lambda_p$ and $R_0 \leq 1$ hold, E_0 of system (1.1) is locally asymptotically stable.

Define a Lyapunov functional as

$$W_1 = \frac{k_1}{\beta_{vh}S_h^{e_0}}V_1 + \frac{k_1}{k_h\beta_{vh}S_h^{e_0}}I_h + \frac{1}{\beta_{hv}S_v^{e_0}}V_2 + \frac{1}{k_v\beta_{hv}S_v^{e_0}}V_3 + U,$$

where k_1 is determined later, and

$$\begin{aligned} V_1 &= S_h - S_h^{e_0} - S_h^{e_0} \ln \frac{S_h}{S_h^{e_0}}, \quad V_2 = S_v - S_v^{e_0} - S_v^{e_0} \ln \frac{S_v}{S_v^{e_0}}, \\ V_3 &= I_v + aq_ve^{-\tau\mu_i} \int_{t-\tau}^t I_v dt, \quad U = \frac{q_vbe^{-\tau\mu_i}}{\beta_{hv}S_v^{e_0}} \int_{t-\tau}^t [S_v - S_v^{e_0} - S_v^{e_0} \ln \frac{S_v}{S_v^{e_0}}] dt. \end{aligned}$$

By considering the derivative along the solution, we have

$$\begin{aligned} \dot{W}_1 &= \frac{k_1}{\beta_{vh}S_h^{e_0}}(1 - \frac{S_h^{e_0}}{S_h})(\Lambda_h - \beta_{vh}I_vS_h - \mu_hS_h) \\ &\quad + \frac{k_1}{k_h\beta_{vh}S_h^{e_0}}(k_h\beta_{vh}I_vS_h - (\mu_h + \gamma_h)I_h) \\ &\quad + \frac{1}{\beta_{hv}S_v^{e_0}}(1 - \frac{S_v^{e_0}}{S_v})(\Lambda_v + bq_ve^{-\tau\mu_i}S_v(t - \tau) - \beta_{hv}I_hS_v - \mu_vS_v - \beta_vS_v^2) \\ &\quad + \frac{1}{k_v\beta_{hv}S_v^{e_0}}(aq_ve^{-\tau\mu_i}I_v(t - \tau) + k_v\beta_{hv}I_hS_v - (\mu_v + \gamma_v)I_v) \\ &\quad + \frac{aq_ve^{-\tau\mu_i}}{k_v\beta_{hv}S_v^{e_0}}(I_v(t) - I_v(t - \tau)) \\ &\quad + \frac{q_vbe^{-\tau\mu_i}}{\beta_{hv}S_v^{e_0}}(S_v - S_v(t - \tau) + S_v^{e_0} \ln \frac{S_v(t - \tau)}{S_v}). \end{aligned}$$

By choosing $k_1 = \frac{\mu_v - aq_ve^{-\tau\mu_i} + \gamma_v}{k_v\beta_{hv}S_v^{e_0}}$ and noting that $\Lambda_v = (\mu_v - bq_ve^{-\tau\mu_i})S_v^{e_0} + \beta_v(S_v^{e_0})^2$, $\Lambda_h = \mu_hS_h^{e_0}$, we have that

$$\begin{aligned} \dot{W}_1 &= \frac{k_1\mu_h}{\beta_{vh}}(2 - \frac{S_h}{S_h^{e_0}} - \frac{S_h^{e_0}}{S_h}) + (1 - \frac{1}{R_0})I_h + \frac{(\mu_v - q_vbe^{-\tau\mu_i})}{\beta_{hv}}(2 - \frac{S_v}{S_v^{e_0}} - \frac{S_v^{e_0}}{S_v}) \\ &\quad - \frac{\beta_v}{\beta_{hv}S_vS_v^{e_0}}(S_v - S_v^{e_0})^2(S_v + S_v^{e_0}) \end{aligned}$$

$$+ \frac{q_v b e^{-\tau \mu_l}}{\beta_{hv}} \left(1 - \frac{S_v(t-\tau)}{S_v} + \ln \frac{S_v(t-\tau)}{S_v} \right).$$

It can be shown that if $R_0 \leq 1$, then $\dot{W}_1 \leq 0$. Define the subset $E = \{\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6) \mid \dot{W}_1(\varphi) = 0\}$. Further, let M be the largest invariant set in E with respect to system (1.1). Let us further show that $M = \{E_0\}$. Denote

$$\begin{aligned} S_{ht}(\theta) &= S_h(t+\theta), & S_{vt}(\theta) &= S_v(t+\theta), \\ I_{ht}(\theta) &= I_h(t+\theta), & I_{vt}(\theta) &= I_v(t+\theta), \\ L_{vt}(\theta) &= L_v(t+\theta), \\ P_t(\theta) &= P(t+\theta) \quad (-\tau \leq \theta \leq 0). \end{aligned}$$

For any solution $(S_{ht}, I_{ht}, S_{vt}, I_{vt}, L_{vt}, P_t)$ with the initial function $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6) \in M$, one has, from the invariance, that for all $t \in R$, $(S_{ht}, I_{ht}, S_{vt}, I_{vt}, L_{vt}, P_t) \in M$.

If $R_0 < 1$, then $\dot{W}_1 = 0$ if and only if $I_{ht}(0) = 0$ and

$$\frac{S_{ht}(0)}{S_h^{e_0}} = \frac{S_{vt}(0)}{S_v^{e_0}} = \frac{S_{ht}(-\tau)}{S_v} = 1.$$

Hence, from invariance of M , we can obtain that $I_{ht}(0) = I_h(t) = 0$, $S_{ht}(0) = S_h(t) = S_h^{e_0}$, $S_{vt}(0) = S_v(t) = S_v^{e_0}$. Further from first equation of (1.1), we can obtain that $I_{vt}(0) = I_v(t) = 0$.

If $R_0 = 1$, then $\dot{W}_1 = 0$ if and only if

$$\frac{S_{ht}(0)}{S_h^{e_0}} = \frac{S_{vt}(0)}{S_v^{e_0}} = \frac{S_{vt}(-\tau)}{S_v} = 1.$$

Hence, for all $t \in R$, one has that $S_{ht}(0) = S_h(t) = S_h^{e_0}$, $S_{vt}(0) = S_v(t) = S_v^{e_0}$. From first and third equations of system (1.1), we can show that $I_{ht}(0) = I_h(t) = 0$ and $I_{vt}(0) = I_v(t) = 0$ respectively, for all $t \in R$. Therefore, for all $t \in R$, in subset M last two equations of system (1.1) are reduced to

$$\begin{aligned} \dot{L}_v(t) &= b(1 - e^{-\tau \mu_l})S_v^{e_0} - \mu_l L_v(t) - \lambda_l f(L_v(t))P(t), \\ \dot{P}(t) &= P(t)(-\mu_p - \beta_p P(t)) + \lambda_p f(L_v(t))P(t). \end{aligned}$$

Define another Lyapunov functional as

$$W_2 = L_v - L_v^{e_0} - \int_{L_v^{e_0}}^{L_v} \frac{f(L_v^{e_0})}{f(\theta)} d\theta + k_2 P,$$

where k_2 due to be determined later. By taking time derivative along the solution, we have that

$$\begin{aligned} \dot{W}_2 &= \left(1 - \frac{f(L_v^{e_0})}{f(L_v)} \right) (b(1 - e^{-\tau \mu_l})S_v^{e_0} - \mu_l L_v - \lambda_l f(L_v)P) \\ &\quad + k_2 ((-\mu_p - \beta_p P)P + \lambda_p f(L_v)P). \end{aligned}$$

Letting $k_2 = \lambda_l / \lambda_p$ and noting that $b(1 - e^{-\tau \mu_l})S_v^{e_0} = \mu_l L_v^{e_0}$, we have that

$$\dot{W}_2 = \mu_l (L_v^{e_0} - L_v) \left(1 - \frac{f(L_v^{e_0})}{f(L_v)} \right) + \lambda_l P \left(f(L_v^{e_0}) - \frac{\mu_p}{\lambda_p} \right) - \frac{\lambda_l}{\lambda_p} \beta_p P^2.$$

If $f(L_v^{e_0}) \leq \mu_p / \lambda_p$ holds, and from properties of the function we have that $\dot{W}_2 \leq 0$. Further, $\dot{W}_2 = 0$ if and only if $L_v(t) = L_v^{e_0}$ and $P(t) = 0$. This proves that

$M = \{E_0\}$. By Lyapunov-LaSalle invariance principle from [24], one proves that E_0 is globally attractive. Hence, E_0 is globally asymptotically stable.

If $R_0 > 1$, we have from (3.2) that $\lim_{\lambda \rightarrow +\infty} \Delta_1(\lambda, \tau) = +\infty$ and $\Delta_1(0, \tau) < 0$. Hence, $\Delta_1(\lambda, \tau) = 0$ has at least one positive real root.

Next, we consider the factor $\lambda + e = \lambda + \mu_p - \lambda_p f(L_v^{e_0}) = 0$. Clearly, it has a positive real root when $f(L_v^{e_0}) > \mu_p/\lambda_p$. □

Remark: At equilibrium E_1 predators consume some larvae (i.e. $f(L_v^{e_1}) \leq f(L_v^{e_0})$). Hence, if $(f(L_v^{e_1}) \leq) f(L_v^{e_0}) \leq \mu_p/\lambda_p$ holds, E_0 stable. From which, it implies that E_1 is not existent. Further, it is clear from Theorem 2.2 and Theorem 3.1 that $S_v^{e_2} < S_v^{e_0}$ and if $R_0 \leq 1$ holds E_0 stable, respectively. From which, it implies that E_2 and E^* are not existent. Therefore, if E_0 stable, E_1, E_2 and E^* are not existent.

In Theorem 3.2, we establish the global stability properties of the equilibrium E_1 .

Theorem 3.2. *If $f(L_v^{e_1}) > \mu_p/\lambda_p$ (i.e. E_0 is unstable), then following conclusions hold for any time delay $\tau \geq 0$.*

(a) *If $R_0 \leq 1$ (i.e. E_2 and E^* are not existent), then E_1 is globally asymptotically stable,*

(b) *If $R_0 > 1$ holds, then E_1 is unstable.*

Proof. At equilibrium E_1 the characteristic equation (3.1) becomes

$$(\lambda + a_1)(\lambda + q(\lambda, \tau))[(\lambda + d)(\lambda + e) + \lambda_l \lambda_p P^{e_1} f'(L_v^{e_1}) f(L_v^{e_1})] \Delta_2(\lambda, \tau) = 0, \tag{3.3}$$

where

$$\begin{aligned} \Delta_2(\lambda, \tau) &= (\lambda + b_1)(\lambda + c(\lambda, \tau)) - k_v \beta_{hv} S_v^{e_1} k_h \beta_{vh} S_h^{e_1} \\ &= \lambda^2 + (\mu_h + \gamma_h + \mu_v + \gamma_v - a q v e^{-(\lambda + \mu_l)\tau}) \lambda \\ &\quad + (\mu_h + \gamma_h)(\mu_v + \gamma_v - a q v e^{-(\lambda + \mu_l)\tau})(1 - R_0). \end{aligned}$$

By following a similar procedure as in [24] it is easy to show, for the characteristics equation (3.3), that if $R_0 \leq 1$ holds, E_1 of system (1.1) is locally asymptotically stable.

We define the same Lyapunov functional (W_1) and applying the same procedure as in proof of Theorem 3.1, we can show that $S_h(t) = S_h^{e_1}, I_h(t) = 0, S_v(t) = S_v^{e_1}, I_v(t) = 0$ on M . For all $t \in R$, in subset M last two equations of system (1.1) are reduced to

$$\begin{aligned} \dot{L}_v(t) &= b(1 - e^{-\tau \mu_l}) S_v^{e_1} - \mu_l L_v(t) - \lambda_l f(L_v(t)) P(t), \\ \dot{P}(t) &= P(t)(-\mu_p - \beta_p P(t)) + \lambda_p f(L_v(t)) P(t). \end{aligned}$$

Define a Lyapunov functional as

$$W_3 = L_v - L_v^{e_1} - \int_{L_v^{e_1}}^{L_v} \frac{f(L_v^{e_1})}{f(\theta)} d\theta + \frac{\lambda_l}{\lambda_p} \left(P - P^{e_1} - P^{e_1} \ln \frac{P}{P^{e_1}} \right).$$

By taking the time derivative along the solution, we have that

$$\begin{aligned} \dot{W}_3 &= \left(1 - \frac{f(L_v^{e_1})}{f(L_v)} \right) (b(1 - e^{-\tau \mu_l}) S_v^{e_1} - \mu_l L_v - \lambda_l f(L_v) P) \\ &\quad + \frac{\lambda_l}{\lambda_p} \left(1 - \frac{P^{e_1}}{P} \right) ((-\mu_p - \beta_p P) P + \lambda_p f(L_v) P). \end{aligned}$$

Noting that $b(1 - e^{-\tau\mu_1})S_v^{e_1} = \mu_l L_v^{e_1} + \lambda_l f(L_v^{e_1})P^{e_1}$ and $\mu_p = \lambda_p f(L_v^{e_1}) - \beta_p P^{e_1}$, we have that

$$\dot{W}_3 = \mu_l(L_v^{e_1} - L_v)\left(1 - \frac{f(L_v^{e_1})}{f(L_v)}\right) + \lambda_l f(L_v^{e_1})\left(2 - \frac{f(L_v^{e_1})}{f(L_v)} - \frac{f(L_v)}{f(L_v^{e_1})}\right) - \frac{\lambda_l \beta_p}{\lambda_p}(P - P^{e_1})^2.$$

From the properties of the function f we have that $\dot{W}_3 \leq 0$. Further, $\dot{W}_3 = 0$ if and only if $L_v(t) = L_v^{e_1}$ and $P(t) = P^{e_1}$. This proves that $M = \{E_1\}$. By Lyapunov-LaSalle invariance principle from [24], E_1 is globally attractive. It proves that E_1 is globally asymptotically stable.

If $R_0 > 1$ holds, we can easily show from (3.3) that $\lim_{\lambda \rightarrow +\infty} \Delta_2(\lambda, \tau) = +\infty$ and $\Delta_2(\lambda, \tau) < 0$. Hence, $\Delta_2(\lambda, \tau) = 0$ has at least one positive real root. \square

Theorem 3.3. *If $S_v^{e_0} < S_v^{e_2} R_0$ (i.e. E_0, E_1 are unstable), then following conclusions hold for any time delay $\tau \geq 0$.*

- (a) *If $f(L_v^{e_2}) \leq \mu_p/\lambda_p$ (i.e. E^* is not existent), then E_2 is globally attractive,*
- (b) *If $f(L_v^*) > \mu_p/\lambda_p$ holds, then E_2 is unstable.*

Proof. Define a Lyapunov functional

$$W_4 = \frac{1}{\beta_{vh}I_v^{e_2}S_h^{e_2}}V_1(t) + \frac{1}{k_h\beta_{vh}I_v^{e_2}S_h^{e_2}}V_2(t) + \frac{1}{\beta_{hv}I_h^{e_2}S_v^{e_2}}(V_3(t) + U_1) + \frac{1}{k_v\beta_{hv}I_h^{e_2}S_v^{e_2}}(V_4(t) + U_2),$$

where

$$V_1(t) = S_h - S_h^{e_2} - S_h^{e_2} \ln \frac{S_h}{S_h^{e_2}}, \quad V_4(t) = I_v - I_v^{e_2} - I_v^{e_2} \ln \frac{I_v}{I_v^{e_2}},$$

$$V_2(t) = I_h - I_h^{e_2} - I_h^{e_2} \ln \frac{I_h}{I_h^{e_2}}, \quad U_1 = q_v b e^{-\tau\mu_1} \int_{t-\tau}^t [S_v - S_v^{e_2} - S_v^{e_2} \ln \frac{S_v}{S_v^{e_2}}] dt,$$

$$V_3(t) = S_v - S_v^{e_2} - S_v^{e_2} \ln \frac{S_v}{S_v^{e_2}}, \quad U_2 = a q_v e^{-\tau\mu_1} \int_{t-\tau}^t [I_v - I_v^{e_2} - I_v^{e_2} \ln \frac{I_v}{I_v^{e_2}}] dt.$$

By considering time derivative along the solution, we have

$$\begin{aligned} \dot{W}_4 &= \frac{1}{\beta_{vh}I_v^{e_2}S_h^{e_2}}\left(1 - \frac{S_h^{e_2}}{S_h}\right)(\Lambda_h - \beta_{vh}I_vS_h - \mu_hS_h) \\ &+ \frac{1}{k_h\beta_{vh}I_v^{e_2}S_h^{e_2}}\left(1 - \frac{I_h^{e_2}}{I_h}\right)(k_h\beta_{vh}I_vS_h - (\mu_h + \gamma_h)I_h) \\ &+ \frac{1}{\beta_{hv}I_h^{e_2}S_v^{e_2}}\left(1 - \frac{S_v^{e_2}}{S_v}\right)(\Lambda_v + bq_v e^{-\tau\mu_1}S_v(t - \tau) - \beta_{hv}I_hS_v - \mu_vS_v - \beta_vS_v^2) \\ &+ \frac{q_v b e^{-\tau\mu_1}}{\beta_{hv}I_h^{e_2}S_v^{e_2}}\left(S_v - S_v(t - \tau) + S_v^{e_2} \ln \frac{S_v(t - \tau)}{S_v}\right) \\ &+ \frac{1}{k_v\beta_{hv}I_h^{e_2}S_v^{e_2}}\left(1 - \frac{I_v^{e_2}}{I_v}\right)(aq_v e^{-\tau\mu_1}I_v(t - \tau) + k_v\beta_{hv}I_hS_v - (\mu_v + \gamma_v)I_v) \\ &+ \frac{aq_v e^{-\tau\mu_1}}{k_v\beta_{hv}I_h^{e_2}S_v^{e_2}}\left(I_v - I_v(t - \tau) + I_v^{e_2} \ln \frac{I_v(t - \tau)}{I_v}\right). \end{aligned}$$

Note that

$$\begin{aligned} \Lambda_h &= \beta_{vh}I_v^{e_2}S_h^{e_2} + \mu_hS_h^{e_2}, \quad \Lambda_v = (\mu_v - bq_v e^{-\tau\mu_1})S_v^{e_2} + \beta_{hv}I_h^{e_2}S_v^{e_2} + \beta_v(S_v^{e_2})^2, \\ k_h\beta_{vh}I_v^{e_2}S_h^{e_2} &= (\mu_h + \gamma_h)I_h^{e_2}, \quad k_v\beta_{hv}I_h^{e_2}S_v^{e_2} = (\mu_v - aq_v e^{-\tau\mu_1} + \gamma_v)I_v^{e_2}. \end{aligned}$$

$$\begin{aligned} \dot{W}_4 &= \frac{\mu_h}{\beta_{vh}I_v^{e_2}} \left(2 - \frac{S_h}{S_h^{e_2}} - \frac{S_h^{e_2}}{S_h} \right) + \frac{aI_v^{e_2}q_v e^{-\tau\mu_l}}{k_v\beta_{hv}I_h^{e_2}S_v^{e_2}} \left(1 - \frac{I_v(t-\tau)}{I_v} + \ln \frac{I_v(t-\tau)}{I_v} \right) \\ &+ 4 - \frac{S_h^{e_2}}{S_h} - \frac{I_v S_h}{I_v^{e_2} S_h^{e_2}} \frac{I_h^{e_2}}{I_h} - \frac{I_h S_v}{I_h^{e_2} S_v^{e_2}} \frac{I_v^{e_2}}{I_v} - \frac{S_v^{e_2}}{S_v} - \frac{\beta_v}{\beta_{hv} I_h^{e_2} S_v^{e_2} S_v} (S_v - S_v^{e_2})^2 \\ &\times (S_v + S_v^{e_2}) + \frac{q_v b e^{-\tau\mu_l}}{\beta_{hv} I_h^{e_2}} \left(1 - \frac{S_v(t-\tau)}{S_v} + \ln \frac{S_v(t-\tau)}{S_v} \right) \\ &+ \frac{(\mu_v - q_v b e^{-\tau\mu_l})}{\beta_{hv} I_h^{e_2}} \left(2 - \frac{S_v}{S_v^{e_2}} - \frac{S_v^{e_2}}{S_v} \right). \end{aligned}$$

It is easy to see that $\dot{W}_4 \leq 0$ for all $t \geq 0$. Define the subset $E = \{\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6) \mid \dot{W}_4(\varphi) = 0\}$. Let M be the largest invariant set in E with respect to (1.1). Let us further show that $M = \{E_2\}$. For any solution $(S_{ht}, I_{ht}, S_{vt}, I_{vt}, L_{vt}, P_t)$ with the initial function $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6) \in M$, it has from the invariance of M that, for all $t \in R$, $(S_{ht}, I_{ht}, S_{vt}, I_{vt}, L_{vt}, P_t) \in M$. Moreover, $\dot{W}_1 = 0$ if and only if

$$\frac{S_{ht}(0)}{S_h^{e_2}} = \frac{S_{vt}(0)}{S_v^{e_2}} = \frac{I_{ht}(0)}{I_h^{e_2}} = \frac{I_{vt}(0)}{I_v^{e_2}} = 1.$$

Hence, we can obtain $I_{ht}(0) = I_h(t) = I_h^{e_2}$, $S_{ht}(0) = S_h(t) = S_h^{e_2}$, $S_{vt}(0) = S_v(t) = S_v^{e_2}$, $I_{vt}(0) = I_v(t) = I_v^{e_2}$. Therefore, on subset M for all $t \in R$, the last two equations of (1.1) are reduced to

$$\begin{aligned} \dot{L}_v(t) &= b(1 - e^{-\tau\mu_l})S_v^{e_2} + a(1 - e^{-\tau\mu_l})I_v^{e_2} - \mu_l L_v(t) - \lambda_l f(L_v(t))P(t), \\ \dot{P}(t) &= P(t)(-\mu_p - \beta_p P(t)) + \lambda_p f(L_v(t))P(t). \end{aligned}$$

By defining similar type of Lyapunov functional (W_2) as in the proof of Theorem 3.1, one can easily show that if $f(L_v^{e_2}) \leq \mu_p/\lambda_p$, then $L_{vt}(0) = L_v(t) = L_v^{e_2}$, $P_t(0) = P(t) = 0$. This proves that $M = \{E_2\}$. Hence, by Lyapunov-LaSalle invariance principle from [24], it proves that E_2 is globally attractive.

By considering the factor $\lambda + \mu_p - \lambda_p f(L_v^{e_2})$ of the characteristic equation (3.1) at E_2 , we can show that $\lambda + \mu_p - \lambda_p f(L_v^{e_2}) = 0$ has one positive root when $f(L_v^{e_2}) > \mu_p/\lambda_p$. Hence, E_2 is unstable. \square

Theorem 3.4. *If $f(L_v^*) > \mu_p/\lambda_p$ and $S_v^{e_0} < S_v^{e_2} R_0$ (i.e. E_0, E_1, E_2 are unstable), then E^* is globally attractive for any time delay $\tau \geq 0$.*

Proof. By defining the same Lyapunov functional (W_4) that we used in proof of Theorem 3.3 and following the same procedure, we show that $S_h(t) = S_h^*$, $I_h(t) = I_h^*$, $S_v(t) = S_v^*$, $I_v(t) = I_v^*$. Then, on subset M , for any $t \in R$, the last two equations of (1.1) are reduce to

$$\begin{aligned} \dot{L}_v(t) &= b(1 - e^{-\tau\mu_l})S_v^* + a(1 - e^{-\tau\mu_l})I_v^* - \mu_l L_v(t) - \lambda_l f(L_v(t))P(t), \\ \dot{P}(t) &= P(t)(-\mu_p - \beta_p P(t)) + \lambda_p f(L_v(t))P(t). \end{aligned}$$

Again using similar type of Lyapunov functional (W_3) as in proof of Theorem 3.2, it has that if $f(L_v^{e_2})(\geq f(L_v^*)) > \mu_p/\lambda_p$, then $L_v(t) = L_v^*$, $P(t) = P^*$. This proves that $M = \{E^*\}$. Hence, E^* is globally attractive, by Lyapunov-LaSalle invariance principle from [24]. This proves that E^* is globally attractive. \square

4. NUMERICAL SIMULATIONS

In this section, we carry out some simulations of system (1.1) to illustrate the theoretical results obtained in Section 3. For convenience, we set $f(L_v) = L_v$.

We choose parameters as $\gamma_h = 0.1428$, $\mu_h = 0.000457$, $q_v = 0.9$, $\mu_l = 0.2$, $\mu_v = 0.6$, $\beta_p = 0.01$, $b_1 = 0.6$, $\alpha_H = 0.85$, $\alpha_V = 0.80$, $k_h = 1$, $\tau = 10$, $\Lambda_v = 60$, $\beta_v = 0.5$, $\gamma_v = 0$, $\lambda_l = 0.7$, $a = 0.01$, $b = 1 - a$, $\mu_p = 0.55$ and $N_H = 200$. In addition, $\beta_{vh} = b_1\alpha_H/N_H$, $\beta_{hv} = b_1\alpha_V/N_H$ and $\Lambda_h = N_H\mu_h$. Here, b_1 , denotes the biting rate of mosquitoes or the average number of bites per mosquito per day. Moreover, α_H, α_V and N_H represents transmission probability from vector to human, transmission probability from human to vector and total human population, respectively.

The trajectories of system (1.1) obtained by using Matlab software are shown in Figures 3-6. First of all, we present simulations regarding equilibrium E_0 . We can see from Figure 3 that all the trajectories converge to the equilibrium E_0 (200, 0, 10.48, 0, 44.88, 0) which implies that equilibrium E_0 is globally asymptotically stable. Further, (a) and (b) in Figure 3 depict under conditions shown in Theorem 3.1 that $R_0 = 0.7481 < 1$ and $f(L_v^{e_0}) = 44.88 < \mu_p/\lambda_p = 550$. Secondly, we give the numerical simulations of the boundary equilibrium E_1 (200, 0, 10.48, 0, 2.95, 4.06) which can be seen in Figure 3 (a) and Figure 4. In this case, $f(L_v^{e_1}) = 2.95 > \mu_p/\lambda_p = 2.75$ and $R_0 = 0.7481 < 1$ are satisfied. Hence, graphs (a) and (c) show that equilibrium E_1 is globally asymptotically stable, this confirmed Theorem 3.2. Thirdly, the equilibrium E_2 is globally attractive (see, Figure 5) in which it can be clearly seen that the trajectories approach to relevant values of the equilibrium E_2 (89.12, 0.35, 10.48, 0.28, 44.89, 0). Further, graphics (d), (e) and (f) in Figure 5 are shown under conditions noted in Theorem 3.3 that is $R_0 = 2.2443 > 1$ and $f(L_v^{e_2}) = 44.89 < \mu_p/\lambda_p = 550$. When $f(L_v^*) = 2.95 > \mu_p/\lambda_p = 2.75$ and $R_0 = 2.2443 > 1$, we depict computer simulation by (d), (e) in Figure 5 and (g) in Figure 6 for E^* , from which we can see that trajectories approached to E^* (89.12, 0.35, 10.48, 0.28, 2.95, 4.06). It means that E^* is globally attractive.

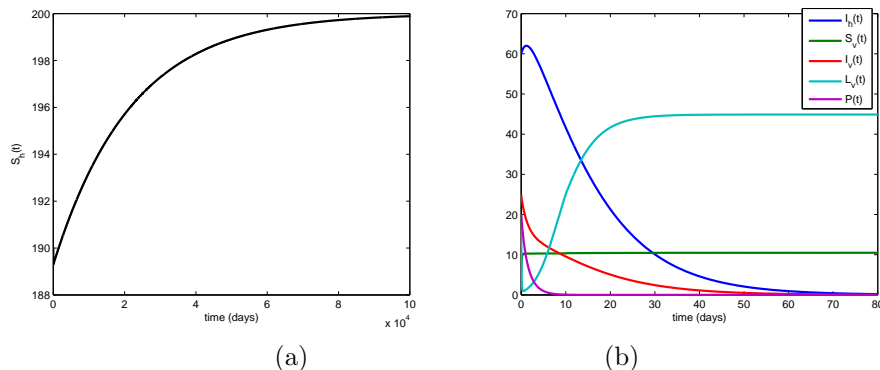


FIGURE 3. Time evolutions of system (1.1) with $k_v = 5$, $\lambda_p = 0.001$ and initial values (200; 60; 5; 25; 20; 20)

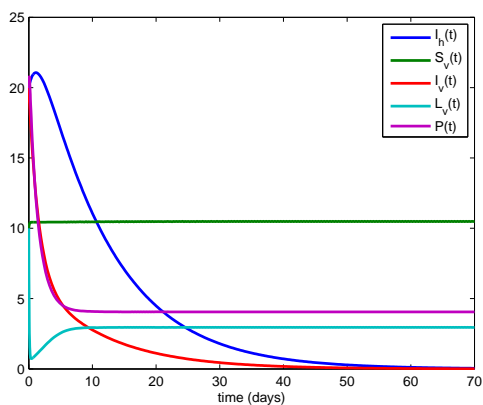


FIGURE 4. Time evolutions of system (1.1) with $k_v = 5$, $\lambda_p = 0.2$ and initial values (200; 60; 5; 25; 20; 20)

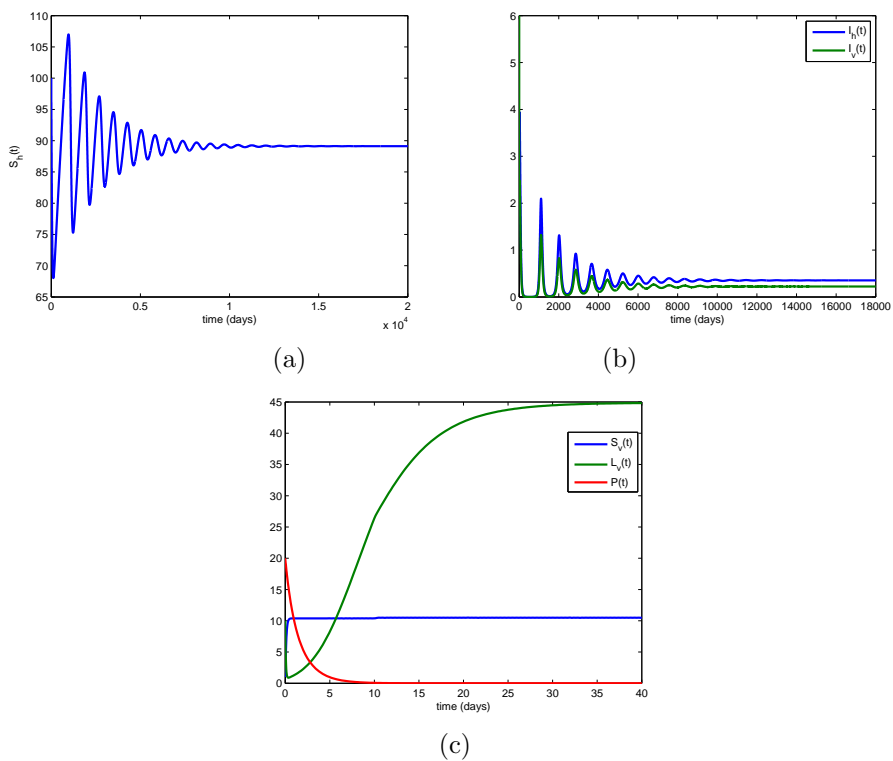


FIGURE 5. Time evolutions of system (1.1) with $k_v = 15$, $\lambda_p = 0.001$ and initial values (100; 2; 3; 6; 10; 20)

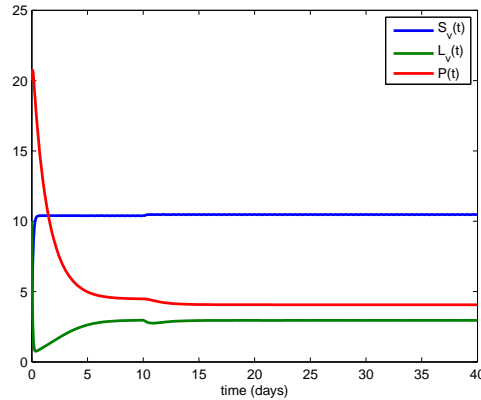


FIGURE 6. Time evolutions of system (1.1) with $k_v = 15$, $\lambda_p = 0.2$ and initial values (100; 2; 3; 6; 10; 20)

5. DISCUSSION

In this article, a vector transmission disease dynamic model is formulated with stage-structure for mosquitoes and predator prey interaction between fish and larvae in water. We investigate the stability properties of all equilibria of system (1.1). In case $R_0 < 1$, there are two distinct aspects to look into stability properties. Firstly, the expected number of secondary infection is less than one and the number of larvae is limited by some upper bound, that is $R_0 < 1$ and $f(L_v^{e_0}) < u_p/\lambda_p$, respectively. Then, equilibrium E_0 (disease-free and predator-free) is globally asymptotically stable which means that disease dies out. Further, it is shown that either $R_0 > 1$ or $f(L_v^{e_1}) > \mu_p/\lambda_p$ holds, E_0 is unstable. Biologically, $R_0 > 1$ implies the average number of secondary infections larger than unity. In fact, disease becomes endemic. If $f(L_v^{e_1}) > \mu_p/\lambda_p$ hold, there are excessive matured susceptible and infected mosquitoes emerged from larvae, then disease becomes endemic. Secondly, the expected number of secondary infection is less than one and the number of larvae is limited by some lower bound (otherwise, the predators can not be survived due to lack of enough foods), that is $R_0 < 1$ and $f(L_v^{e_1}) > u_p/\lambda_p$, respectively. Then, equilibrium E_1 (disease-free with predator) is globally asymptotically stable which means that disease dies out. In this case, although there are large number of larvae in water, they are not emerged to matured stage of mosquito as predators consume some. On the other hand, E_1 is unstable if $R_0 > 1$. In this case, disease becomes endemic as average number of secondary infections larger than unity.

It is further shown that the equilibrium E_2 (predator free with disease) is globally attractive when it exists, which means that disease becomes endemic. On the other hand, it is shown that if $f(L_v^*) > \mu_p/\lambda_p$ the equilibrium E_2 becomes unstable and E^* (disease with predator) is existent (predators have enough foods in water to survive). Further, we have shown that if the endemic equilibrium E^* exists and it is globally attractive, which concludes that disease becomes endemic. Furthermore, more general model (1.2) with several time delays is interested in considering which is left as a future work.

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