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POSITIVE GROUND STATE SOLUTIONS TO SCHRÖDINGER-POISSON SYSTEMS WITH A NEGATIVE NON-LOCAL TERM

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Abstract. In this article, we study the Schrödinger-Poisson system

$$-\Delta u + u - \lambda K(x)\phi(x)u = a(x)|u|^{p-1}u, \quad x \in \mathbb{R}^3,$$
$$-\Delta\phi = K(x)u^2, \quad x \in \mathbb{R}^3,$$

with $p \in (1,5)$. Assume that $a : \mathbb{R}^3 \to \mathbb{R}^+$ and $K : \mathbb{R}^3 \to \mathbb{R}^+$ are nonnegative functions and satisfy suitable assumptions, but not requiring any symmetry property on them, we prove the existence of a positive ground state solution resolved by the variational methods.

1. Introduction and main results

In this article we study the Schrödinger-Poisson system

$$\begin{gathered} -\Delta u + V(x)u + \lambda K(x)\phi(x)u = f(x,u), \quad x \in \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2, \quad x \in \mathbb{R}^3, \end{gathered} \tag{1.1}$$

where $V(x) = 1$, $\lambda < 0$, $f(x,s) = a(x)s^p$ and $a(x), K(x)$ satisfying some suitable assumptions, we will prove problem (1.1) exists a positive ground state solution.

Similar problems have been widely investigated and it is well known they have a strong physical meaning because they appear in quantum mechanics models (see e.g. [9]) and in semiconductor theory [7, 8, 14, 15]. Variational methods and critical point theory are always powerful tools in studying nonlinear differential equations. In recent years, system (1.1) has been studied widely via modern variational methods under the various hypotheses, see [2, 4, 14, 19, 16] and the references therein. Many researches have been devoted to the study of problem (1.1), but they mainly concern either the autonomous case or, in the non-autonomous case, the search of the so-called semi-classical states. We refer the reader interested in a detailed bibliography to the survey paper [2]. All these works deal with systems like (1.1) with $\lambda > 0$ and the nonlinearity $f(x,s) = s^p$ with p subcritical.

To the best of our knowledge, there are only a few article on the existence of ground state solutions to (1.1) with the negative coefficient of the non-local term. Recently, in [17], the author obtained a ground state solution. In [18], the

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author considered the nonlinearity $f(x,s) = a(x)s^2$ and obtained a ground state solution. In this article, we consider the nonlinearity $f(x,s) = a(x)s^p$ for following Schrödinger-Poisson system

$$\begin{gathered} -\Delta u + u - \lambda K(x)\phi(x)u = a(x)|u|^{p-1}u, \quad x \in \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2, \quad x \in \mathbb{R}^3. \end{gathered} \tag{1.2}$$

It is worth noticing that there are few works concerning on this case up to now.

As we shall see in Section 2, problem (1.2) can be easily transformed in a non-linear Schrödinger equation with a non-local term. Briefly, the Poisson equation is solved by using the Lax-Milgram theorem, then, for all $u \in H^1(R^3)$, a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ is obtained, such that $-\Delta\phi = K(x)u^2$ and that, inserted into the first equation, gives

$$-\Delta u + u - \lambda K(x)\phi_u(x)u^2 = a(x)|u|^{p-1}u, \quad x \in \mathbb{R}^3. \tag{1.3}$$

This problem is variational and its solutions are the critical points of the functional defined in $H^1(\mathbb{R}^3)$ by

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2dx - \frac{1}{p+1}\int_{\mathbb{R}^3} a(x)|u|^{p+1}dx. \tag{1.4}$$

In our research, we deal with the case in which $p \in (1,5)$, moreover we always assume that $a(x)$ and $K(x)$ satisfy:

- (A1) There exists a constant $c > 0$, such that $a(x) > c$ for all $x \in \mathbb{R}^3$ and $a(x) - c \in L^{\frac{6}{5-p}}(\mathbb{R}^3)$;
- (K1) $K \in L^2(\mathbb{R}^3)$.

Our main result reads as follows.

Theorem 1.1. *Suppose $a, K : \mathbb{R}^3 \to \mathbb{R}^+$, $\lambda > 0$ and $p \in (1,5)$. Let* (A1), (K1) *hold. Then problem* (1.2) *has a positive ground state solution.*

Remark 1.2. To the best of our knowledge, there are only two articles [17, 18] on the existence of ground state solutions to (1.1) with the negative coefficient of the non-local. In [17], the author discusses the negative coefficient of the non-local term under symmetry assumption, but we get the positive ground state solution without any symmetry assumption. Compared with the [18], we do not need conditions

$$\lim_{|x|\to+\infty} a(x) = a_\infty \quad \text{and} \quad \lim_{|x|\to+\infty} K(x) = K_\infty.$$

The remainder of this paper is organized as follows. In Section 2, notation and preliminaries are presented. In Section 3, we give the proof of Theorem 1.1.

2. Notation and preliminaries

Hereafter we use the following notation:

$H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the standard scalar product and norm

$$(u,v) = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv)dx; \quad \|u\|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)dx.$$

$D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{D^{1,2}} = \Big(\int_{\mathbb{R}^3} |\nabla u|^2dx\Big)^{1/2}.$$

$L^p(\Omega)$, $1 \le p \le +\infty$, $\Omega \subseteq \mathbb{R}^3$, denotes a Lebesgue space, the norm in $L^p(\Omega)$ is denoted by $\|u\|_{L^p(\Omega)} = |u|_{p,\Omega}$ when Ω is a proper subset of $\mathbb{R}^3$, by $\|u\|_{L^p(\Omega)} = |\cdot|_p$ when $\Omega = \mathbb{R}^3$.

$L^\infty(\Omega)$ is the space of measurable functions in Ω; that is,

$$\operatorname{ess\,sup}_{x\in\Omega} |u(x)| = \inf\{C > 0 : |u(x)| \le C \text{ a. e. in } \Omega\} < +\infty.$$

For any $\rho > 0$ and for any $z \in \mathbb{R}^3$, $B_\rho(z)$ denotes the ball of radius ρ centered at z, and $|B_\rho(z)|$ denotes its Lebesgue measure. C, C_0, C_1, C_2 are various positive constants which can change from line to line.

From the embeddings, $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ and $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, we obtain the inequalities

$$|u|_6 \le C_1\|u\| \quad \forall u \in H^1(\mathbb{R}^3)\backslash\{0\},$$
$$|u|_6 \le C_2\|u\| \quad \forall u \in D^{1,2}(\mathbb{R}^3)\backslash\{0\}.$$

It is well known and easy to show that problem (1.2) can be reduced to a single equation with a non-local term. Actually, considering for all $u \in H^1(\mathbb{R}^3)$, the linear functional L_u defined in $D^{1,2}(\mathbb{R}^3)$ by

$$L_u(v) = \int_{\mathbb{R}^3} K(x)u^2 v\,dx,$$

the Hölder and Sobolev inequalities imply

$$L_u(v) \le |K|_2|u^2|_3|v|_6 = |K|_2|u|_6^2|v|_6 \le C_2|K|_2 \cdot |u|_6^2\|v\|_{D^{1,2}}. \tag{2.1}$$

Hence, from the Lax-Milgram theorem, for every $u \in H^1(\mathbb{R}^3)$, the Poisson equation $-\Delta\phi = K(x)u^2$ exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} K(x)u^2 v\,dx = \int_{\mathbb{R}^3} \nabla\phi_u \cdot \nabla v\,dx, \tag{2.2}$$

for any $v \in D^{1,2}(\mathbb{R}^3)$. Using integration by parts, we get

$$\int_{\mathbb{R}^3} \nabla\phi_u \cdot \nabla v\,dx = -\int_{\mathbb{R}^3} v\Delta\phi_u dx,$$

therefore,

$$-\Delta\phi_u = K(x)u^2,$$

in a weak sense and the representation formula

$$\phi_u = \int_{\mathbb{R}^3} \frac{K(y)}{|x-y|}u^2(y)dy = \frac{1}{|x|} * Ku^2 \tag{2.3}$$

holds. Moreover, $\phi_u > 0$ when $u \ne 0$, because K does, and by (2.1), (2.2) and the Sobolev inequality, the relations

$$\|\phi_u\|_{D^{1,2}} \le C_2C_1^2 \cdot |K|_2\|u\|^2, \quad |\phi_u|_6 \le C_2\|\phi_u\|_{D^{1,2}}, \tag{2.4}$$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(x)K(y)}{|x-y|}u^2(x)u^2(y)dxdy = \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx \le C_2^2C_1^4 \cdot |K|_2^2\|u\|^4 \tag{2.5}$$

hold. Substituting ϕ_u in problem (1.2), we are led to (1.3), whose solutions can be obtained by looking for critical points of the functional $I : H^1(\mathbb{R}^3) \to \mathbb{R}$ where I is defined in (1.4). Indeed, (2.4) and (2.5) imply that I is a well-defined C^2 functional, and that

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^3} \Big(\nabla u \cdot \nabla v + uv - \lambda K(x)\phi_u uv - a(x)|u|^{p-1}uv\Big)dx. \tag{2.6}$$

Hence, if $u \in H^1(\mathbb{R}^3)$ is a critical point of I, then the pair (u, ϕ_u), with ϕ_u as in (2.3), is a solution of (1.2).

Let us define the operator $\Phi: H^1(\mathbb{R}^3) \to D^{1,2}(\mathbb{R}^3)$ as

$$\Phi[u] = \phi_u.$$

In the next lemma we summarize some properties of Φ, useful for the study our problem.

Lemma 2.1 ([11]). *(1) Φ is continuous;*
(2) Φ maps bounded sets into bounded sets;
(3) if $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ then $\Phi[u_n] \rightharpoonup \Phi[u]$ in $D^{1,2}(\mathbb{R}^3)$;
(4) $\Phi[tu] = t^2\Phi[u]$ for all $t \in \mathbb{R}$.

Lemma 2.2 ([13]). *Suppose $r > 0$, $2 < q < 2^*(= 6)$. If $\{u_n\} \subset H^1(\mathbb{R}^3)$ is bounded and*

$$\sup_{y\in\mathbb{R}^3} \int_{B(y,r)} |u_n|^q dx \to 0, \quad \text{as } n \to +\infty,$$

then $u_n \to 0$ in $L^q(\mathbb{R}^3)$ for $2 < q < 2^*$.

3. Proof of main results

First wee give some properties of the nonlinear Schrödinger equation

$$-\Delta u + u = c|u|^{p-1}u, \tag{3.1}$$

that has been broadly studied in [13, 12]. We set

$$\mathcal{N}_\infty := \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \|u\|^2 = c|u|_{p+1}^{p+1}\}. \tag{3.2}$$

Then for any $u \in \mathcal{N}_\infty$, we have

$$I_\infty(u) = \frac{1}{2}\|u\|^2 - \frac{c}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}dx = \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|u\|^2, \tag{3.3}$$

and $m_\infty := \inf\{I_\infty(u) : u \in \mathcal{N}_\infty\}$.

It is well known that (3.1) has at least a ground state solution which we denote w_∞. By using (3.2) and (3.3), we know that

$$m_\infty = I_\infty(w_\infty) = \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|w_\infty\|^2$$

and

$$\|w_\infty\|^2 = c\int_{\mathbb{R}^3} |w_\infty|^{p+1}dx. \tag{3.4}$$

For (1.2), it is not difficult to verify that the functional I is bounded either from below or from above. So it is convenient to consider I restricted to a natural constraint, the Nehari manifold, that contains all the nonzero critical points of I and on which I turns out to be bounded from below. We set

$$\mathcal{N} := \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : G(u) = 0\}$$

where

$$G(u) = \langle I'(u), u\rangle = \|u\|^2 - \lambda\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \int_{\mathbb{R}^3} a(x)|u|^{p+1}dx. \tag{3.5}$$

The following lemma states the main properties of $\mathcal{N}$.

Lemma 3.1. *I is bounded from below on $\mathcal{N}$ by a positive constant.*

Proof. Let $u \in \mathcal{N}$, from (A1) and Hölder's inequality, we have

$$\begin{aligned} 0 &= \|u\|^2 - \lambda \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \int_{\mathbb{R}^3} a(x)|u|^{p+1} dx \\ &\geq \|u\|^2 - C\|u\|^4 - C_0\|u\|^{p+1} \end{aligned} \tag{3.6}$$

from which we have

$$\|u\| \geq C_1 > 0, \quad \forall u \in \mathcal{N} \tag{3.7}$$

Using this inequality, $\lambda > 0$, $K > 0$, $a > 0$, when $1 < p < 3$, we obtain

$$\begin{aligned} I(u) &= \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|u\|^2 + \Big(\frac{1}{p+1} - \frac{1}{4}\Big)\lambda \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx \\ &\geq \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|u\|^2 \\ &\geq \Big(\frac{1}{2} - \frac{1}{p+1}\Big)C_1^2 > 0, \end{aligned} \tag{3.8}$$

when $3 \leq p < 5$,

$$\begin{aligned} I(u) &= \frac{1}{4}\|u\|^2 + \Big(\frac{1}{4} - \frac{1}{p+1}\Big) \int_{\mathbb{R}^3} a(x)|u|^{p+1} dx \\ &\geq \frac{1}{4}\|u\|^2 \\ &\geq \frac{1}{4}C_1^2 > 0. \end{aligned} \tag{3.9}$$

Setting $m := \inf\{I(u) : u \in \mathcal{N}\}$, as a consequence of Lemma 3.1, m turns out to be a positive number. Then we obtain a sequence $\{u_n\} \subset \mathcal{N}$, such that

$$\lim_{n\to\infty} I(u_n) = m. \tag{3.10}$$

□

Now we give the proof of our main result.

Proof of Theorem 1.1. First, we prove that

$$m < m_\infty. \tag{3.11}$$

We know that $w_\infty \in \mathcal{N}_\infty$ and $I_\infty(w_\infty) = m_\infty$. We claim that there exists $t_0 > 0$ such that $t_0 w_\infty \in \mathcal{N}$. Indeed, from (3.5), for all $t \geq 0$ one has

$$G(tw_\infty) = t^2\|w_\infty\|^2 - \lambda t^4 \int_{\mathbb{R}^3} K(x)\phi_{w_\infty} w_\infty^2 dx - t^{p+1} \int_{\mathbb{R}^3} a(x)|w_\infty|^{p+1} dx,$$

then $G(0) = 0$ and $G(tw_\infty) \to -\infty$ as $t \to +\infty$. Moreover,

$$G_t'(tw_\infty) = t\Big(2\|w_\infty\|^2 - 4\lambda t^2 \int_{\mathbb{R}^3} K(x)\phi_{w_\infty} w_\infty^2 dx - (p+1)t^{p-1} \int_{\mathbb{R}^3} a(x)|w_\infty|^{p+1} dx\Big),$$

then there exists $t_{\max} > 0$ such that $G_t'(tw_\infty) > 0$ for all $0 < t < t_{\max}$ and $G_t'(tw_\infty) < 0$ for all $t > t_{\max}$. Then $G(tw_\infty)$ is increasing for all $0 < t < t_{\max}$

and $G(tw_\infty)$ decreasing for all $t > t_{\max}$. Thus there exists $t_0 > 0$ such that $G(t_0w_\infty) = 0$. That is, $t_0w_\infty \in \mathcal{N}$. Our claim is true. It follows that

$$
\begin{aligned}
m &\le I(t_0w_\infty) \\
&= \frac{t_0^2}{2}\|w_\infty\|^2 - \frac{t_0^4}{4}\lambda\int_{\mathbb{R}^3} K(x)\phi_{w_\infty}(x)w_\infty^2 dx - \frac{t_0^{p+1}}{p+1}\int_{\mathbb{R}^3} a(x)|w_\infty|^{p+1}dx \\
&< \frac{t_0^2}{2}\|w_\infty\|^2 - \frac{t_0^{p+1}}{p+1}\int_{\mathbb{R}^3} c|w_\infty|^{p+1}dx \\
&\le \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|w_\infty\|^2 \\
&= I_\infty(w_\infty) = m_\infty.
\end{aligned} \tag{3.12}
$$

We assume that $\{u_n\}$ is what obtained in (3.10). From (2.3), we can get $\{|u_n|\}$ is also a minimize sequence. Setting $u_n(x) \ge 0$ in $\mathbb{R}^3$ a.e. by (3.8) and (3.9), we have if $p \in (1, 3)$, then

$$
I(u_n) \ge \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|u_n\|^2,
$$

and if $p \in [3, 5)$, then

$$
I(u_n) \ge \frac{1}{4}\|u_n\|^2.
$$

In both cases, being $I(u_n)$ is bounded, $\{u_n\}$ is also bounded.

On the other hand, since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, there exists $\overline{u} \in H^1(\mathbb{R}^3)$ such that, up to a subsequence,

$$
u_n \rightharpoonup \overline{u}, \quad \text{in } H^1(\mathbb{R}^3); \tag{3.13}
$$

$$
u_n \to \overline{u}, \quad \text{in } L^{p+1}_{\rm loc}(\mathbb{R}^3); \tag{3.14}
$$

$$
u_n(x) \to \overline{u}(x), \quad \text{a.e. in } \mathbb{R}^3. \tag{3.15}
$$

Setting

$$
z_n^1(x) = u_n(x) - \overline{u}(x).
$$

Obviously, $z_n^1 \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$, but not strongly. A direct computation gives

$$
\|u_n\|^2 = \|z_n^1 + \overline{u}\|^2 = \|z_n^1\|^2 + \|\overline{u}\|^2 + o(1). \tag{3.16}
$$

According to the Brezis-Lieb Lemma [10], we deduce

$$
|u_n|_{p+1}^{p+1} = |\overline{u}|_{p+1}^{p+1} + |z_n^1|_{p+1}^{p+1} + o(1). \tag{3.17}
$$

Then, we claim that, for any $h \in H^1(\mathbb{R}^3)$, we have

$$
\int_{\mathbb{R}^3} |u_n|^{p-1}u_nh\,dx \to \int_{\mathbb{R}^3} |\overline{u}|^{p-1}\overline{u}h\,dx. \tag{3.18}
$$

For every $h \in C_0^\infty(\mathbb{R}^3)$, there exists a bounded open subset $\Omega \subset \mathbb{R}^3$, such that $\operatorname{supp} h \subset \Omega$, where $\operatorname{supp} h = \overline{\{x \in \mathbb{R}^3 : h(x) \ne 0\}}$. From (3.14), we have

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^3} |u_n|^{p-1}u_nh\,dx - |\overline{u}|^{p-1}\overline{u}h\,dx\Big| \\
&< \int_{\mathbb{R}^3} \big||u_n|^{p-1}u_nh - |\overline{u}|^{p-1}\overline{u}h\big|dx \\
&\le \int_{\mathbb{R}^3} p(|u_n|^{p-1} + |\overline{u}|^{p-1})|u_n - \overline{u}||h|dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} p|u_n|^{p-1}|u_n-\overline{u}||h|dx + \int_{\mathbb{R}^3} p|\overline{u}|^{p-1}|u_n-\overline{u}||h|dx \\
&< p|u_n|_{p+1}|u_n-\overline{u}|_{p+1,\Omega}|h|_{p+1} + p|\overline{u}|_{p+1}|u_n-\overline{u}|_{p+1,\Omega}|h|_{p+1} < \varepsilon
\end{aligned}
$$

which proves (3.18). Let us show that

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2dx = \int_{\mathbb{R}^3} K(x)\phi_{\overline{u}}\overline{u}^2dx + o(1), \tag{3.19}$$

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_nh\,dx = \int_{\mathbb{R}^3} K(x)\phi_{\overline{u}}\overline{u}h\,dx + o(1). \tag{3.20}$$

First let us observe that, in view of the Sobolev embedding theorem, (3.13) and (3) of Lemma 2.1, $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$ implies

$$u_n \rightharpoonup \overline{u}, \quad \text{in } L^6(\mathbb{R}^3); \tag{3.21}$$

$$u_n^2 \to \overline{u}^2, \quad \text{in } L^3_{\rm loc}(\mathbb{R}^3); \tag{3.22}$$

$$\phi_{u_n} \rightharpoonup \phi_{\overline{u}}, \quad \text{in } D^{1,2}(\mathbb{R}^3); \tag{3.23}$$

$$\phi_{u_n} \to \phi_{\overline{u}}, \quad \text{in } L^6_{\rm loc}(\mathbb{R}^3). \tag{3.24}$$

Furthermore, considering (3.22) and (3.24) respectively, we can assert that for any choice of $\varepsilon > 0$ and $\rho > 0$, the relations

$$|u_n^2 - \overline{u}^2|_{3,B_\rho(0)} < \varepsilon, \tag{3.25}$$

$$|\phi_{u_n} - \phi_{\overline{u}}|_{6,B_\rho(0)} < \varepsilon \tag{3.26}$$

hold for large n.

On the other hand, u_n being bounded in $H^1(\mathbb{R}^3)$, ϕ_{u_n} is bounded in $D^{1,2}(\mathbb{R}^3)$ and in $L^6(\mathbb{R}^3)$, because of (2) of Lemma 2.1 and the continuity of the Sobolev embedding of $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$. Moreover $K \in L^2(\mathbb{R}^3)$, for any $\varepsilon > 0$, there exists $\overline{\rho} = \overline{\rho}(\varepsilon)$ such that

$$|K|_{2,\mathbb{R}^3\setminus B_\rho(0)} < \varepsilon, \quad \forall \rho \geq \overline{\rho}. \tag{3.27}$$

Hence, by (3.25) and (3.27), for large n, we deduce that

$$
\begin{aligned}
&\Big| \int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2dx - \int_{\mathbb{R}^3} K(x)\phi_{\overline{u}}\overline{u}^2dx \Big| \\
&\leq \Big| \int_{\mathbb{R}^3} K(x)\phi_{u_n}(u_n^2-\overline{u}^2)dx + \int_{\mathbb{R}^3} K(x)(\phi_{u_n}-\phi_{\overline{u}})\overline{u}^2dx \Big| \\
&\leq \Big| \int_{\mathbb{R}^3} K(x)\phi_{u_n}(u_n^2-\overline{u}^2)dx \Big| + \Big| \int_{\mathbb{R}^3} K(x)(\phi_{u_n}-\phi_{\overline{u}})\overline{u}^2dx \Big| \\
&\leq \Big| \int_{\mathbb{R}^3\setminus B_\rho(0)} K(x)\phi_{u_n}(u_n^2-\overline{u}^2)dx \Big| + \Big| \int_{B_\rho(0)} K(x)\phi_{u_n}(u_n^2-\overline{u}^2)dx \Big| \\
&\quad + \Big| \int_{\mathbb{R}^3\setminus B_\rho(0)} K(x)(\phi_{u_n}-\phi_{\overline{u}})\overline{u}^2dx \Big| + \Big| \int_{B_\rho(0)} K(x)(\phi_{u_n}-\phi_{\overline{u}})\overline{u}^2dx \Big| \\
&\leq |K|_{2,\mathbb{R}^3\setminus B_\rho(0)} \Big(|\phi_{u_n}|_{6,\mathbb{R}^3\setminus B_\rho(0)} |u_n^2-\overline{u}^2|_{3,\mathbb{R}^3\setminus B_\rho(0)} \\
&\quad + |\phi_{u_n}-\phi_{\overline{u}}|_{6,\mathbb{R}^3\setminus B_\rho(0)} |\overline{u}^2|_{3,\mathbb{R}^3\setminus B_\rho(0)} \Big) + |K|_{2,B_\rho(0)} |\phi_{u_n}|_{6,B_\rho(0)} |u_n^2-\overline{u}^2|_{3,B_\rho(0)} \\
&\quad + |K|_{2,B_\rho(0)} |\phi_{u_n}-\phi_{\overline{u}}|_{6,B_\rho(0)} |\overline{u}^2|_{3,B_\rho(0)} \\
&\leq C\varepsilon
\end{aligned}
$$

which proves (3.19).

Analogously, by (3.26) and (3.27), for large n, we infer that

$$\Big|\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n h\,dx - \int_{\mathbb{R}^3} K(x)\phi_{\overline{u}}\overline{u}h\,dx\Big| \le \varepsilon$$

which proves (3.20). Therefore, by (3.16), (3.17) and (3.19) respectively, we obtain

$$\begin{aligned} I(u_n) &= \frac{1}{2}\|u_n\|^2 - \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2dx - \frac{1}{p+1}\int_{\mathbb{R}^3} a(x)|u_n|^{p+1}dx \\ &= \frac{1}{2}\|z_n^1\|^2 + \frac{1}{2}\|\overline{u}\|^2 - \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{\overline{u}}\overline{u}^2dx - \frac{1}{p+1}\int_{\mathbb{R}^3} a(x)|\overline{u}|^{p+1}dx \\ &\quad - \frac{c}{p+1}\int_{\mathbb{R}^3}|z_n^1|^{p+1}dx + o(1) \\ &= I(\overline{u}) + I_\infty(z_n^1) + o(1). \end{aligned} \tag{3.28}$$

By (3.18) and (3.20) for any $h \in C_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned} \langle I'(u_n), h\rangle &= \int_{\mathbb{R}^3}(\nabla u_n\cdot\nabla h + u_n h - \lambda K(x)\phi_{u_n}u_n h - a(x)|u_n|^{p-1}u_n h)dx \\ &= \int_{\mathbb{R}^3}(\nabla \overline{u}\cdot\nabla h + \overline{u}h - \lambda K(x)\phi_{\overline{u}}\overline{u}h - a(x)|\overline{u}|^{p-1}\overline{u}h)dx + o(1) \\ &= \langle I'(\overline{u}), h\rangle + o(1). \end{aligned} \tag{3.29}$$

We now claim that

$$\nabla I(u_n) \to 0, \quad \text{in } H^1(\mathbb{R}^3). \tag{3.30}$$

By Lagrange's multiplier theorem, we know that there exists $\lambda_n \in \mathbb{R}$ such that

$$o(1) = \nabla I|_{\mathcal{N}}(u_n) = \nabla I(u_n) - \lambda_n \nabla G(u_n). \tag{3.31}$$

So, taking the scalar product with u_n, we obtain

$$o(1) = (\nabla I(u_n), u_n) - \lambda_n(\nabla G(u_n), u_n).$$

G turns out to be a C^1 functional. Using (3.6) and $\lambda > 0, K > 0, a > 0$, when $1 < p \le 3$, we deduce

$$\begin{aligned} \langle G'(u), u\rangle &= 2\|u\|^2 - 4\lambda\int_{\mathbb{R}^3} K(x)\phi_u u^2dx - (p+1)\int_{\mathbb{R}^3} a(x)|u|^{p+1}dx \\ &= (1-p)\|u\|^2 + \lambda(p-3)\int_{\mathbb{R}^3} k(x)\phi_u u^2dx \\ &\le (1-p)\|u\|^2 \\ &\le -(p-1)C_1 < 0, \end{aligned} \tag{3.32}$$

when $3 < p < 5$,

$$\begin{aligned} \langle G'(u), u\rangle &= 2\|u\|^2 - 4\lambda\int_{\mathbb{R}^3} K(x)\phi_u u^2dx - (p+1)\int_{\mathbb{R}^3} a(x)|u|^{p+1}dx \\ &= -2\|u\|^2 + (3-p)\int_{\mathbb{R}^3} a(x)|u|^{p+1}dx \\ &\le -2\|u\|^2 \\ &\le -2C_2 < 0. \end{aligned} \tag{3.33}$$

Since $u_n \in \mathcal{N}$, we have $(\nabla I(u_n), u_n) = 0$; by inequalities (3.32) and (3.33), we have $(\nabla G(u_n), u_n) < C < 0$. Thus $\lambda_n \to 0$ for $n \to +\infty$. Moreover, by the boundedness of $\{u_n\}$, $\nabla G(u_n)$ is bounded and this implies $\lambda_n \nabla G(u_n) \to 0$, so (3.31) follows from (3.30). By (3.29) and (3.30), we have $\langle I'(\overline{u}), h\rangle = 0$, so $\overline{u}$ is a solution of problem (1.2). By (3.19), we have

$$\begin{aligned}\langle I'(u_n), u_n\rangle &= \|u_n\|^2 - \lambda \int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} a(x)|u_n|^{p+1}) dx \\ &= \|\overline{u}\|^2 + \|z_n^1\|^2 - \lambda \int_{\mathbb{R}^3} K(x)\phi_{\overline{u}}\overline{u}^2 dx - \int_{\mathbb{R}^3} a(x)|\overline{u}|^{p+1} dx \\ &\quad - c\int_{\mathbb{R}^3} |z_n^1|^{p+1} dx + o(1) \\ &= \langle I'(\overline{u}), \overline{u}\rangle + \langle I'_\infty(z_n^1), z_n^1\rangle + o(1),\end{aligned}$$

which implies that

$$o(1) = \langle I'_\infty(z_n^1), z_n^1\rangle = \|z_n^1\|^2 - c|z_n^1|_{p+1}^{p+1}. \tag{3.34}$$

Setting

$$\delta := \limsup_{n\to+\infty} \Big(\sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |z_n^1|^{p+1} dx \Big).$$

We claim $\delta = 0$. By [19, Lemma 1.21], one has

$$z_n^1 \to 0, \quad \text{in } L^{p+1}(\mathbb{R}^3). \tag{3.35}$$

From (3.34) and (3.35), we obtain

$$\begin{aligned}o(1) &= \langle I'_\infty(z_n^1), z_n^1\rangle \\ &= \|z_n^1\|^2 - c|z_n^1|_{p+1}^{p+1} \\ &= \|z_n^1\|^2 + o(1) \\ &= \|u_n - \overline{u}\|^2 + o(1),\end{aligned}$$

so $u_n \to \overline{u}$ in $H^1(\mathbb{R}^3)$. Let $u = \overline{u}$, so $I(u) = m$, $I'(u) = 0$ and $u(x) > 0$ a.e. in $\mathbb{R}^3$.

Let us prove $\delta = 0$. Actually, if $\delta > 0$, there exists sequence $\{y_n^1\} \subset \mathbb{R}^3$, such that

$$\int_{B_1(y_n^1)} |z_n^1|^{p+1} dx > \frac{\delta}{2}.$$

Let us now consider $z_n^1(\cdot + y_n^1)$. We assume that $z_n^1(\cdot + y_n^1) \rightharpoonup u^1$ in $H^1(\mathbb{R}^3)$ and, then, $z_n^1(x + y_n^1) \to u^1(x)$ a.e. on $\mathbb{R}^3$. Since

$$\int_{B_1(0)} |z_n^1(x + y_n^1)|^{p+1} dx > \frac{\delta}{2},$$

from the Rellich theorem it follows that

$$\int_{B_1(0)} |u^1(x)|^{p+1} dx \ge \frac{\delta}{2},$$

and thus $u^1 \neq 0$. Finally, let us set

$$z_n^2(x) = z_n^1(x + y_n^1) - u^1(x).$$

Then, using (3.16), (3.17) and the Brezis-Lieb Lemma, we have

$$\|z_n^2\|^2 = \|z_n^1\|^2 - \|u^1\|^2 + o(1), \tag{3.36}$$

$$|z_n^2|_{p+1}^{p+1} = |u_n|_{p+1}^{p+1} - |\overline{u}|_{p+1}^{p+1} - |u^1|_{p+1}^{p+1} + o(1). \tag{3.37}$$

These equalities imply

$$I_\infty(z_n^2) = I_\infty(z_n^1) - I_\infty(u^1) + o(1),$$

hence, by using (3.28), we obtain

$$\begin{aligned} I(u_n) &= I(\overline{u}) + I_\infty(z_n^1) + o(1) \\ &= I(\overline{u}) + I_\infty(u^1) + I_\infty(z_n^2) + o(1). \end{aligned} \tag{3.38}$$

Using (3.34), (3.36) and (3.37), we obtain

$$\begin{aligned} \langle I'_\infty(z_n^1), z_n^1\rangle &= \|z_n^1\|^2 - c|z_n^1|_{p+1}^{p+1} \\ &= \|u^1\|^2 - c|u^1|_{p+1}^{p+1} + \|z_n^2\|^2 - c|z_n^2|_{p+1}^{p+1} + o(1) \\ &= \langle I'_\infty(u^1), u^1\rangle + \langle I'_\infty(z_n^2), z_n^2\rangle + o(1), \end{aligned}$$

which implies

$$o(1) = \langle I'_\infty(z_n^2), z_n^2\rangle = \|z_n^2\|^2 - c|z_n^2|_{p+1}^{p+1}.$$

Moreover, we obtain

$$I_\infty(z_n^2) = \frac{1}{2}\|z_n^2\|^2 - \frac{c}{p+1}|z_n^2|_{p+1}^{p+1} = \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|z_n^2\|^2 + o(1). \tag{3.39}$$

Since $z_n^1 \rightharpoonup u^1$ in $H^1(\mathbb{R}^3)$ and $u^1 \neq 0$, according to (3.34), one has $u^1 \in \mathcal{N}_\infty$. Because of $\overline{u} \in \mathcal{N}$, from Lemma 3.1, we obtain $I(\overline{u}) > 0$. Thus, using (3.38) and (3.39), we obtain

$$\begin{aligned} m &= \liminf_{n\to\infty} I(u_n) \\ &\geq I(\overline{u}) + I_\infty(u^1) + \liminf_{n\to\infty} I_\infty(z_n^2) \\ &\geq I_\infty(u^1) \geq m_\infty \end{aligned}$$

which contradicts with (3.11). □

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