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LAPLACE TRANSFORM OF FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this article, we show that Laplace transform can be applied to fractional system. To this end, solutions of linear fractional-order equations are first derived by a direct method, without using Laplace transform. Then the solutions of fractional-order differential equations are estimated by employing Gronwall and Hölder inequalities. They are showed be to of exponential order, which are necessary to apply the Laplace transform. Based on the estimates of solutions, the fractional-order and the integer-order derivatives of solutions are all estimated to be exponential order. As a result, the Laplace transform is proved to be valid in fractional equations.

1. Introduction

Fractional calculus is generally believed to have stemmed from a question raised in the year 1695 by L'Hopital and Leibniz. It is the generalization of integer-order calculus to arbitrary order one. Frequently, it is called fractional-order calculus, including fractional-order derivatives and fractional-order integrals. Reviewing its history of three centuries, we could find that fractional calculus were mainly interesting to mathematicians for a long time, due to its lack of application background. However, in the previous decades more and more researchers have paid their attentions to fractional calculus, since they found that the fractional-order derivatives and fractional-order integrals were more suitable for the description of the phenomena in the real world, such as viscoelastic systems, dielectric polarization, electromagnetic waves, heat conduction, robotics, biological systems, finance and so on; see, for example, [1, 2, 8, 9, 10, 16, 17, 19].

Owing to great efforts of researchers, there have been rapid developments on the theory of fractional calculus and its applications, including well-posedness, stability, bifurcation and chaos in fractional differential equations and their control. Several useful tools for solving fractional-order equations have been discovered, of which Laplace transform is frequently applied. Furthermore, it is showed to be most efficient and helpful in analysis and applications of fractional-order systems, from which some results could be derived immediately. For instance, in [11, 12], the authors investigated stability of fractional-order nonlinear dynamical systems

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using Laplace transform method and Lyapunov direct method, with the introduction of Mittag-Leffler stability and generalized Mittag-Leffler stability concepts. In [5], Deng et al studied the stability of n-dimensional linear fractional differential equation with time delays by Laplace transform method. In [18], Jocelyn Sabatier et al obtained the stability conditions in the form of linear matrix inequality (LMI) for fractional-order systems by using Laplace transform. The Laplace transform was also used in [6, 7, 13, 14, 15, 21].

Although it is often used in analyzing fractional-order systems, the validity of Laplace transform to fractional systems is seldom touched upon when it is applied to fractional systems. In this paper, its validity to fractional systems will be justified. It is showed that Laplace transform could be applied to fractional systems under certain conditions. To this end, solutions of linear fractional-order equations are first derived by direct method, without using the Laplace transform. The obtained results match those obtained by the Laplace transform very well. The method provides an alternative way of solution, different from the Laplace transform. Then solutions of fractional-order differential equations are estimated. They are showed to be of exponential order, which is necessary to apply the Laplace transform. Finally, the Laplace transform is proved to be feasible in fractional equations.

The article is organized as follows. In Section 2, some preliminaries about fractional calculus are presented. In Section 3, solutions of linear fractional-order equations are expressed by the direct method, without using Laplace transform. Section 4 is devoted to the estimates of solutions of fractional-order equations. The Laplace transform is proved to be valid in fractional-order equations in Section 5. Finally, some conclusions are drawn in Section 6.

2. Preliminaries

In fractional calculus, the traditional integer-order integrals and derivatives of functions are generalized to fractional-order ones, which are commonly defined by Laplace convolution operation as follows.

Definition 2.1 ([9, p. 92])**.** Caputo fractional derivative with order α for a function $x(t)$ is defined as

$$^{C}D_{t_0}^{\alpha}x(t) = \frac{1}{\Gamma(m-\alpha)}\int_{t_0}^{t}(t-\tau)^{m-\alpha-1}x^{(m)}(\tau)d\tau,$$

where $0 \le m-1 \le \alpha < m$, $m \in Z_+$, and $t = t_0$ is the initial time and $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 ([9, p. 69])**.** Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $x : \mathbb{R}^+\mathbb{R}$ is defined as

$$I_{t_0}^{\alpha}f(t) = \frac{1}{F(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}f(\tau)d\tau,$$

where $t = t_0$ is the initial time and $\Gamma(.)$ is the Gamma function.

Definition 2.3 ([9, p. 70])**.** Riemann-Liouville fractional derivative with order α for a function $x : \mathbb{R}^+\mathbb{R}$ is defined as

$$^{RL}D_{t_0}^{\alpha}x(t) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_{t_0}^{t}(t-\tau)^{m-\alpha-1}x(\tau)d\tau,$$

where $0 \le m-1 \le \alpha < m$, $m \in Z_+$, and $t = t_0$ is the initial time and $\Gamma(.)$ is the Gamma function.

Definition 2.4 ([17, pp. 16-17])**.** The Mittag-Leffler function is defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ka+1)},$$

where $\alpha > 0$, $z \in C$. The two-parameter Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ka+\beta)},$$

where $\alpha > 0$, $\beta > 0$, $z \in C$.

There are some properties between fractional-order derivatives and fractional-order integrals, which are expressed as follows.

Lemma 2.5 ([9, pp. 75-76, 96])**.** Let $\alpha > 0$, $n = [\alpha]+1$ and $f_{n-\alpha}(t) = (I_a^{n-\alpha} f)(t)$. Then fractional integrals and fractional derivatives have the following properties. (1) If $f(t) \in L^1(a,b)$ and $f_{n-\alpha}(t) \in AC^n[a,b]$, then

$$(I_a^{\alpha\, RL} D_a^\alpha f)(t) = f(t) - \sum_{j=1}^{n} \frac{f_{n-\alpha}^{(n-j)}(a)}{\Gamma(a-j+1)} (t-a)^{\alpha-j},$$

holds almost everywhere in $[a,b]$.

(2) If $f(t) \in AC^n[a,b]$ or $f(t) \in C^n[a,b]$, then

$$(I_a^{\alpha\, C} D_a^\alpha f)(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k.$$

Note that the Laplace transform is a useful tool for analyzing and solving ordinary and partial differential equations. The definition of Laplace transform and some applications to integer-order systems are recalled from [20]. They will be useful for later analysis.

Definition 2.6 ([20, pp. 1-2])**.** The Laplace transform of f is defined as

$$F(s) = \mathcal{L}(f(t))(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{\tau\to\infty} \int_0^\tau e^{-st} f(t)dt,$$

whenever the limit exists (as a finite number).

Definition 2.7 ([20, p. 10])**.** A function f is piecewise continuous on the interval $[0,\infty)$ if (i) $\lim_{t\to 0^+} f(t) = f(0^+)$ exists and (ii) f is continuous on every finite interval $(0,b)$ except possibly at a finite number of points $\tau_1, \tau_2, \dots, \tau_n$ in $(0,b)$ at which f has a jump discontinuity.

Definition 2.8 ([20, p. 12])**.** A function f is of exponential order γ if there exist constants $M > 0$ and γ such that for some $t_0 > 0$ such that $|f(t)| \le Me^{\gamma t}$ for $t \ge t_0$.

Some existence results of Laplace transform for functions and their derivatives are listed as follows.

Theorem 2.9 ([20, p. 13])**.** *If f is piecewise continuous on $[0,\infty)$ and of exponential order γ, then the Laplace transform $\mathcal{L}(f(t))$ exists for $\mathrm{Re}(s) > \gamma$ and converges absolutely.*

Theorem 2.10 ([20, p. 56]). *If we assume that f' is continuous $[0,\infty)$ and also of exponential order, then it follows that the same is true of f.*

Theorem 2.11 ([20, p. 57]). *Suppose that $f(t), f'(t), \dots, f^{(n-1)}(t)$ are continuous on $(0,\infty)$ and of exponential order, while $f^{(n)}(t)$ is piecewise continuous on $[0,\infty)$. Then*

$$\mathcal{L}(f^{(n)}(t))(s) = s^n\mathcal{L}(f(t)) - s^{n-1}f(0+) - s^{n-2}f'(0+) - \cdots - f^{(n-1)}(0+).$$

Although the Laplace operator can be applied to many functions, there are some functions, to which it could not be applied, see for example [20, p.6]. The following inequalities will also be helpful for later analysis.

Lemma 2.12 ([21]). *Suppose $\beta > 0$, $a(t)$ is a nonnegative function locally integrable on $0 \le t < T$ (some $T \le +\infty$) and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \le t < T$, $g(t) \le M$ (constant), and suppose $u(t)$ is nonnegative and locally integrable on $0 \le t < T$ with*

$$u(t) \le a(t) + g(t)\int_0^t (t-s)^{\beta-1}u(s)\,ds$$

on this interval. Then

$$u(t) \le a(t) + \int_0^t \Big[\sum_{n=1}^{\infty}\frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}a(s)\Big]ds, 0 \le t < T.$$

Lemma 2.13 (Cauchy inequality [22]). *Let $n \in N$, and let $x_1, x_2, \dots, x_n$ be nonnegative real numbers. Then for ϑ,*

$$\Big(\sum_{i=1}^n x_i\Big)^{\vartheta} \le n^{\vartheta-1}\sum_{i=1}^n x_i^{\vartheta}.$$

Lemma 2.14 (Gronwall integral inequality [4]). *If*

$$x(t) \le h(t) + \int_{t_0}^t k(s)x(s)\,ds, \quad t \in [t_0, T),$$

where all the functions involved are continuous on $[t_0, T)$, $T \le +\infty$, and $k(s) \ge 0$, then $x(t)$ satisfies

$$x(t) \le h(t) + \int_{t_0}^t h(s)k(s)e^{\int_s^t k(u)du}\,ds, \quad t \in [t_0, T).$$

If, in addition, $h(t)$ is nondecreasing, then

$$x(t) \le h(t)e^{\int_{t_0}^t k(s)\,ds}, t \in [t_0, T).$$

3. Solutions of linear fractional-order equations by a direct method

Consider the one-dimensional linear fractional-order equation

$$D_0^{\alpha}x(t) = \lambda x(t), \tag{3.1}$$

where D denotes ${}^{RL}D$ or${}^{C}D$, $l-1 < \alpha \le l$, $l \in N$, $\lambda \in R$.

Take Laplace transform on both sides of (3.1), then the solutions of (3.1) could be figured out, see [9, pp.284, 313]. The solutions are presented as follows.

(a) When D denotes ${}^{RL}D$, the solution is represented as

$$x(t)=\sum_{j=1}^{l} d_j x_j(t), \tag{3.2}$$

where $d_j=({}^{RL}D^{\alpha-j}x)(0+)=x_{l-\alpha}^{(l-j)}(0+)$, $x_j(t)=t^{\alpha-j}E_{\alpha,\alpha+1-j}(\lambda t^{\alpha})$, and $j=1,2,\dots,l$.

(b) When D denotes ${}^{C}D$, the solution is represented as

$$x(t)=\sum_{j=0}^{l-1} b_j \widetilde{x}_j(t), \tag{3.3}$$

where $b_j=x^{(j)}(0)$, $\widetilde{x}_j(t)=t^{j}E_{\alpha,j+1}(\lambda t^{\alpha})$, and $j=1,2,\dots,l-1$.

Now we employ the direct method to derive the solutions of (3.1). The whole process will be formulated after the following theorem is introduced.

Theorem 3.1. *Suppose that $\alpha>0$, $u(t)$ and $a(t)$ are locally integrable on $0\le t<T$ (some $T\le+\infty$), and $|a(t)|\le M$ (constant). Suppose $x(t)$ is locally integrable on $0\le t<T$ with*

$$x(t)=u(t)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}a(\tau)x(\tau)d\tau$$

on this interval. Then

$$x(t)=u(t)+\int_0^t\Big[\sum_{n=1}^{\infty}\frac{1}{\Gamma(n\alpha)}(t-\tau)^{n\alpha-1}a^n(\tau)u(\tau)\Big]d\tau.$$

Proof. Let $B\phi(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}a(\tau)\phi(\tau)d\tau$, $t\ge 0$, where ϕ is the locally integrable function. Then $x(t)=u(t)+Bx(t)$ implies

$$x(t)=\sum_{k=0}^{n-1}B^k u(t)+B^n x(t).$$

Let us prove by mathematical induction that

$$B^n x(t)=\frac{1}{\Gamma(n\alpha)}\int_0^t (t-\tau)^{n\alpha-1}a^n(\tau)x(\tau)d\tau, \tag{3.4}$$

and $B^n x(t)\to 0$ as $n\to+\infty$ for each t in $0\le t<T$.

We know that the relation (3.4) is true for $n=1$. Assume that it is true for $n=k$. If $n=k+1$, then the induction hypothesis implies

$$\begin{aligned} B^{k+1}x(t)&=B(B^k x(t))\\ &=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}a(s)\Big[\frac{1}{\Gamma(k\alpha)}\int_0^s (s-\tau)^{k\alpha-1}a^k(\tau)x(\tau)d\tau\Big]ds. \end{aligned}$$

By interchanging the order of integration, we have

$$B^{k+1}x(t)=\frac{1}{\Gamma(\alpha)\Gamma(k\alpha)}\int_0^t\Big[\int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{k\alpha-1}ds\Big]a^{k+1}(\tau)x(\tau)d\tau,$$

where the integral

$$\int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{k\alpha-1}ds=(t-\tau)^{k\alpha+\alpha-1}\int_0^1 (1-z)^{\alpha-1}z^{k\alpha-1}dz$$

$$
\begin{aligned}
&= (t-\tau)^{(k+1)\alpha-1}B(k\alpha,\alpha)\\
&= \frac{\Gamma(\alpha)\Gamma(k\alpha)}{\Gamma((k+1)\alpha)}(t-\tau)^{(k+1)\alpha-1}
\end{aligned}
$$

is evaluated with the help of the substitution $s=\tau+z(t-\tau)$ and the definition of the beta function. The relation (3.4) is proved.

Since

$$
\begin{aligned}
B^n x(t) &= \frac{1}{\Gamma(n\alpha)}\int_0^t (t-\tau)^{n\alpha-1}a(\tau)x(\tau)d\tau\\
&\le \frac{1}{\Gamma(n\alpha)}\int_0^t |(t-\tau)^{n\alpha-1}a^n(\tau)x(\tau)|d\tau\\
&\le \frac{1}{\Gamma(n\alpha)}\int_0^t (t-\tau)^{n\alpha-1}|a^n(\tau)||x(\tau)|d\tau\\
&\le \frac{M^n}{\Gamma(n\alpha)}\int_0^t (t-\tau)^{n\alpha-1}|x(\tau)|d\tau\\
&\le \frac{M^nT^{n\alpha-1}}{\Gamma(n\alpha)}\int_0^t |x(\tau)|d\tau \to 0,
\end{aligned}
$$

as $n\to+\infty$ for $t\in[0,T)$. Then the proof is complete. □

The way to prove the theorem can also be found in [21] and [23]. The theorem provides a direct method to solve the linear fractional-order equation (3.1).

(A) When D denotes ${}^{RL}D$, take the operator I_0^α on both sides of (3.1), then from Lemma 2.5 we have

$$
x(t)=\sum_{j=1}^{l}\frac{x_{l-\alpha}^{(l-j)}(0+)}{\Gamma(\alpha-j+1)}t^{\alpha-j}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\lambda x(\tau)d\tau.
$$

Let

$$
\sum_{j=1}^{l}\frac{x_{l-\alpha}^{(l-j)}(0+)}{\Gamma(\alpha-j+1)}t^{\alpha-j}=\sum_{j=1}^{l}\frac{d_j}{\Gamma(\alpha-j+1)}t^{\alpha-j}=u(t),
$$

from Theorem 3.1, one obtains

$$
x(t)=u(t)+\int_0^t \Big[\sum_{n=1}^{\infty}\frac{1}{\Gamma(n\alpha)}(t-\tau)^{n\alpha-1}\lambda^n u(\tau)\Big]d\tau. \tag{3.5}
$$

Assume that

$$
\begin{aligned}
A_j &= \frac{d_j}{\Gamma(\alpha-j+1)}t^{\alpha-j}+\int_0^t \Big[\sum_{n=1}^{\infty}\frac{1}{\Gamma(n\alpha)}(t-\tau)^{n\alpha-1}\lambda^n\frac{d_j}{\Gamma(\alpha-j+1)}\tau^{\alpha-j}\Big]d\tau\\
&= \frac{d_j}{\Gamma(\alpha-j+1)}t^{\alpha-j}\\
&\quad+\sum_{n=1}^{\infty}\Big[\frac{d_j\lambda^n t^{(n+1)\alpha-j}}{\Gamma(n\alpha)\Gamma(\alpha-j+1)}\int_0^1 (1-\frac{\tau}{t})^{n\alpha-1}\big(\frac{\tau}{t}\big)^{\alpha-j}d\big(\frac{\tau}{t}\big)\Big]\\
&= \frac{d_j}{\Gamma(\alpha-j+1)}t^{\alpha-j}+\sum_{n=1}^{\infty}\Big[\frac{d_j\lambda^n t^{(n+1)\alpha-j}}{\Gamma(n\alpha)\Gamma(\alpha-j+1)}B(n\alpha,\alpha-j+1)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{d_j}{\Gamma(\alpha - j + 1)} t^{\alpha-j} + \sum_{n=1}^{\infty} \big[\frac{d_j \lambda^n t^{(n+1)\alpha - j}}{\Gamma(n\alpha + \alpha - j + 1)} \big] \\
&= d_j t^{\alpha-j} E_{\alpha,\alpha+1-j}(\lambda t^\alpha) = d_j x_j(t).
\end{aligned}
$$

Then

$$
x(t) = \sum_{j=1}^{l} A_j = \sum_{j=1}^{l} d_j x_j(t). \tag{3.6}
$$

It means that the solution of linear fractional-order differential equations with Rieman-Liouville derivative could be solved by the direct method above.

(B) When D denotes ${}^C D$, take the operator I_0^α on both sides of (3.1), then from Lemma 2.5 we have

$$
x(t) = \sum_{j=0}^{l-1} \frac{x^{(j)}(0)}{j!} t^j + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \lambda x(\tau) d\tau.
$$

Let

$$
\sum_{j=0}^{l-1} \frac{x^{(j)}(0)}{j!} t^j = u(t),
$$

from Theorem 3.1 one obtains

$$
x(t) = u(t) + \int_0^t \Big[\sum_{n=1}^{\infty} \frac{1}{\Gamma(n\alpha)} (t-\tau)^{n\alpha-1} \lambda^n u(\tau) \Big] d\tau. \tag{3.7}
$$

Assume that

$$
\begin{aligned}
B_j &= \frac{x^{(j)}(t)}{j!} t^j + \int_0^t \Big[\sum_{n=1}^{\infty} \frac{1}{\Gamma(n\alpha)} (t-\tau)^{n\alpha-1} \lambda^n \frac{x^{(j)}(0)}{j!} t^j \Big] d\tau \\
&= \frac{x^{(j)}(t)}{j!} t^j + \sum_{n=1}^{\infty} \Big[\frac{x^{(j)}(0) \lambda^n t^{n\alpha+j}}{\Gamma(n\alpha)\Gamma(j+1)} \int_0^1 (1 - \frac{\tau}{t})^{n\alpha-1} \big(\frac{\tau}{t}\big)^j d\big(\frac{\tau}{t}\big) \Big] \\
&= \frac{x^{(j)}(t)}{j!} t^j + \sum_{n=1}^{\infty} \big[\frac{x^{(j)}(0) \lambda^n t^{n\alpha+j}}{\Gamma(n\alpha)\Gamma(j+1)} B(n\alpha, j+1) \big] \\
&= \frac{x^{(j)}(0)}{j!} t^j + \sum_{n=1}^{\infty} \big[\frac{x^{(j)}(0) \lambda^n t^{n\alpha+j}}{\Gamma(n\alpha + j + 1)} \big] \\
&= x^{(j)}(0) t^j E_{\alpha,j+1}(\lambda t^\alpha) = b_j \widetilde{x}_j(t).
\end{aligned}
$$

Then

$$
x(t) = \sum_{j=0}^{l-1} B_j = \sum_{j=0}^{l-1} b_j \widetilde{x}_j(t). \tag{3.8}
$$

That is, the solution of linear fractional-order differential equations with Caputo derivative could also be solved by the direct method.

4. Estimates of solutions to fractional-order differential equations

Consider the nonlinear fractional-order differential equation

$$
D_0^\alpha x(t) = Ax(t) + f(x) + d(t), \tag{4.1}
$$

where D denotes ${}^{RL}D$ or ${}^{C}D$, $l-1<\alpha\le l$, $l\in N$, $\lambda\in R$, $x\in\mathbb{R}^n$, $f(x)$ is the nonlinear part and continuous in $x\in\mathbb{R}^n$, $f(0)=0$, $d(t)$ means the input of the equation. To obtain the main results, make the following assumptions.

- (i) $f(x)$ satisfies the Lipschitz condition, that is, there exists a constant $L>0$ such that $\|f(x)\|\le L\|x\|$;
- (ii) $d(t)$ is bounded, that is, there exists a constant $M>0$ such that $\|d(t)\|\le M$.

Then we have the following result.

Theorem 4.1. *When $t>1$, D denotes ${}^{C}D$ or ${}^{RL}D$ and* (4.1) *satisfies assumptions* (i) *and* (ii)*, then the solution of* (4.1) *satisfies*

$$\|x(t)\|\le \widetilde{M}_1 e^{p_1 t}, \tag{4.2}$$

where

$$\begin{gathered}
p_1=\frac{2^{h-1}(\|A\|+L)^h\Gamma^{\frac{h}{v}}(va-v+1)}{hv^{\alpha h-h+\frac{h}{v}}\Gamma^h(\alpha)}+\alpha+1,\\
h=1+\frac{1}{\alpha},\quad v=1+\alpha,\quad \widetilde{M}_1=\frac{1}{2}(l+1)\overline{M},\\
\overline{M}=\max\Big\{\frac{\|x_{l-1}^{(l-\alpha)}(0)\|}{\Gamma(\alpha-1+1)},\frac{\|x_{l-2}^{((l-\alpha))}(0)\|}{\Gamma(\alpha-2+1)},\dots,\frac{\|x_{0}^{(l-\alpha)}(0)\|}{\Gamma(\alpha-l+1)},\\
\frac{M}{\Gamma(\alpha+1)}\frac{\|x(0)\|}{0!},\frac{\|x'(0)\|}{1!},\dots,\frac{\|x^{(l-1)}(0)\|}{(l-1)!}\Big\}.
\end{gathered}$$

Proof. Applying the operator I_0^α on both sides of (4.1), we have

$$x(t)=u(t)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}(Ax(\tau)+f(x(\tau))+d(\tau))d\tau, \tag{4.3}$$

where

$$u(t)=\begin{cases}\sum_{j=1}^{l}\frac{x_{l-\alpha}^{(l-j)}(0)}{\Gamma(a-j+1)}t^{a-j}, & D={}^{RL}D,\\ \sum_{j=0}^{l-1}\frac{x^{(j)}(0)}{j!}t^j, & D={}^{C}D.\end{cases}$$

Taking norms of both sides of (4.3), one obtains

$$\|x(t)\|\le\|u(t)\|+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}(\|A\|\|x(\tau)\|+\|f(x(\tau))\|+\|d(\tau)\|)d\tau, \tag{4.4}$$

where

$$\|u(t)\|\le\begin{cases}\sum_{j=1}^{l}\frac{\|x_{l-\alpha}^{(l-j)}(0)\|}{\Gamma(a\alpha-j+1)}t^{\alpha-j}, & D={}^{RL}D,\\ \sum_{j=0}^{l-1}\frac{\|x^{(j)}(0)\|}{j!}t^j, & D={}^{C}D.\end{cases}$$

From assumptions (i) and (ii), one has

$$\begin{aligned}
\|x(t)\|&\le\|u(t)\|+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}[(\|A\|+L)\|x(\tau)\|+M]d\tau\\
&=\|u(t)\|+\frac{Mt^\alpha}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}(\|A\|+L)\|x(\tau)\|d\tau.
\end{aligned}\tag{4.5}$$

Let $t>1$, $\hat{M}=(l+1)\overline{M}$ and

$$\overline{M}=\max\Big\{\frac{\|x_{l-1}^{(l-\alpha)}(0)\|}{\Gamma(\alpha-1+1)},\frac{\|x_{l-2}^{((l-\alpha))}(0)\|}{\Gamma(\alpha-2+1)},\dots,\frac{\|x_{0}^{(l-\alpha)}(0)\|}{\Gamma(\alpha-l+1)},$$

$$\frac{M}{\Gamma(\alpha+1)}\frac{\|x(0)\|}{0!},\frac{\|x'(0)\|}{1!},\dots,\frac{\|x^{(l-1)}(0)\|}{(l-1)!}\Big\}.$$

Then we have

$$\begin{aligned}\|x(t)\| &\le \hat{M}t^{\alpha}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(\|A\|+L)\|x(\tau)\|d\tau\\ &= \hat{M}t^{\alpha}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{\tau-t}e^{t-\tau}(\|A\|+L)\|x(\tau)\|d\tau.\end{aligned}\tag{4.6}$$

Let $h=1+\frac{1}{\alpha}$, $v=1+\alpha$, from (4.6) and Hölder inequality, we have

$$\begin{aligned}&\|x(t)\|\\ &\le \hat{M}t^{\alpha}+\frac{\|A\|+L}{\Gamma(\alpha)}\Big[\int_0^t \big((t-\tau)^{\alpha-1}e^{\tau-t}\big)^v\,d\tau\Big]^{1/v}\Big[\int_0^t \big(e^{t-\tau}\|x(\tau)\|\big)^h\,d\tau\Big]^{1/h}\\ &= \hat{M}t^{\alpha}+\frac{\|A\|+L}{\Gamma(\alpha)}\Big[\int_0^t (t-\tau)^{v\alpha-v}e^{v\tau-vt}d\tau\Big]^{1/v}\Big[\int_0^t e^{ht-h\tau}\|x(\tau)\|^h d\tau\Big]^{1/h}.\end{aligned}\tag{4.7}$$

Note that

$$\begin{aligned}\int_0^t (t-\tau)^{v\alpha-v}e^{-(vt-v\tau)}d\tau &= \int_0^t s^{v\alpha-v}e^{-sv}ds\\ &= \frac{1}{v}\int_0^{tv} u^{v\alpha-v}e^{-u}du\\ &\le \frac{1}{v^{v\alpha-v+1}}\int_0^{+\infty} u^{v\alpha-v}e^{-u}du\\ &= \frac{1}{v^{v\alpha-v+1}}\Gamma(va-v+1),\end{aligned}\tag{4.8}$$

where $s=t-\tau$,$u=sv$. Submitting (4.8) into (4.7), one has

$$\|x(t)\|\le \hat{M}t^{\alpha}+\frac{(\|A\|+L)\Gamma^{1/v}(va-v+1)}{v^{\alpha-1+\frac{1}{v}}\Gamma(\alpha)}\Big[\int_0^t e^{ht-h\tau}\|x(\tau)\|^h d\tau\Big]^{1/h}.\tag{4.9}$$

From Lemma 2.13 and (4.9) it follows that

$$\begin{aligned}&\|x(t)\|^h\\ &\le 2^{h-1}\hat{M}^h t^{h\alpha}+2^{h-1}\frac{(\|A\|+L)^h\Gamma^{\frac{h}{v}}(va-v+1)e^{ht}}{v^{\alpha h-h+\frac{h}{v}}\Gamma^h(\alpha)}\int_0^t e^{-h\tau}\|x(\tau)\|^h d\tau,\end{aligned}\tag{4.10}$$

then we can obtain

$$\begin{aligned}&\|x(t)\|^h e^{-ht}\\ &\le 2^{h-1}\hat{M}^h t^{h\alpha}e^{-ht}+\frac{2^{h-1}(\|A\|+L)^h\Gamma^{\frac{h}{v}}(va-v+1)}{v^{\alpha h-h+\frac{h}{v}}\Gamma^h(\alpha)}\int_0^t e^{-h\tau}\|x(\tau)\|^h d\tau\\ &\le 2^{h-1}\hat{M}^h t^{h\alpha}+\frac{2^{h-1}(\|A\|+L)^h\Gamma^{\frac{h}{v}}(va-v+1)}{v^{\alpha h-h+\frac{h}{v}}\Gamma^h(\alpha)}\int_0^t e^{-h\tau}\|x(\tau)\|^h d\tau.\end{aligned}\tag{4.11}$$

From Lemma 2.14, we have

$$\|x(t)\|^h e^{-ht}\le 2^{h-1}\hat{M}^h t^{h\alpha}e^{\tilde{K}t}\le 2^{h-1}\hat{M}^h e^{(\tilde{K}+h\alpha)t},\tag{4.12}$$

where
$$\tilde{K} = \frac{2^{h-1}(\|A\| + L)^h \Gamma^{\frac{h}{v}}(va - v + 1)}{v^{\alpha h - h + \frac{h}{v}} \Gamma^h(\alpha)}.$$
Then one obtains
$$\|x(t)\| \le \frac{1}{2}\hat{M} e^{(\frac{\tilde{K}}{h} + \alpha + 1)t}. \tag{4.13}$$
Let $\widetilde{M}_1 = \frac{1}{2}\hat{M}$, $p_1 = \frac{\tilde{K}}{h} + \alpha + 1$, then $||x(t)|| \le \widetilde{M}_1 e^{p_1 t}$. The proof is complete. $\square$

Theorem 4.2. *(1) When $0 < t \le 1$, D denotes ${}^C D$ and the equation* (4.1) *satisfy assumptions* (i) *and* (ii)*, then the solution of* (4.1) *satisfies*
$$\|x(t)\| \le \widetilde{M}_2 e^{p_2 t}, \tag{4.14}$$
where $\widetilde{M}_2 = \frac{1}{2}(l+1)\overline{M}$,
$$p_2 = \frac{2^{h-1}(\|A\| + L)^h \Gamma^{\frac{h}{v}}(va - v + 1)}{h v^{\alpha h - h + \frac{h}{v}} \Gamma^h(\alpha)} + 1.$$
(2) When $0 < t \le 1$, D denotes ${}^{RL}D$ and (4.1) *satisfies assumptions* (i) *and* (ii) *for any $b > 0$, then the solution of* (4.1) *satisfies*
$$\|x(t)\| \le \widetilde{M}_3 e^{p_3 t} \quad (t > b > 0), \tag{4.15}$$
where $\widetilde{M}_3 = \frac{1}{2}(l+1)\overline{M} b^{\alpha - l}$, $p_3 = \frac{2^{h-1}(\|A\|+L)^h \Gamma^{\frac{h}{v}}(va-v+1)}{h v^{\alpha h - h + \frac{h}{v}} \Gamma^h(\alpha)} + 1$. The expressions $\overline{M}$, h and v are the same as in Theorem 4.1.

Proof. (1) When $0 < t \le 1$, D denotes ${}^C D$, we can write (4.5) as
$$\begin{aligned} \|x(t)\| &\le \hat{M} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (\|A\| + L) \|x(\tau)\| d\tau \\ &= \hat{M} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{\tau - t} e^{t - \tau} (\|A\| + L) \|x(\tau)\| d\tau. \end{aligned} \tag{4.16}$$
Following the same process as in Theorem 4.1, we have
$$\|x(t)\| \le \widetilde{M}_2 e^{p_2 t}, \tag{4.17}$$
where $\widetilde{M}_2 = \frac{1}{2}(l+1)\overline{M}$,
$$p_2 = \frac{2^{h-1}(\|A\| + L)^h \Gamma^{\frac{h}{v}}(va - v + 1)}{h v^{\alpha h - h + \frac{h}{v}} \Gamma^h(\alpha)} + 1.$$
(2) When $0 < t \le 1$, D denotes ${}^{RL}D$, and $t > b$, we can write (4.5) as
$$\begin{aligned} \|x(t)\| &\le \hat{M} b^{\alpha - l} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (\|A\| + L) \|x(\tau)\| d\tau \\ &= \hat{M} b^{\alpha - l} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{\tau - t} e^{t - \tau} (\|A\| + L) \|x(\tau)\| d\tau. \end{aligned} \tag{4.18}$$
Following the same process as in Theorem 4.1, we have
$$\|x(t)\| \le \widetilde{M}_3 e^{p_3 t}, \tag{4.19}$$
where $\widetilde{M}_3 = \frac{1}{2}(l+1)\overline{M} b^{\alpha - l}$,
$$p_3 = \frac{2^{h-1}(\|A\| + L)^h \Gamma^{\frac{h}{v}}(va - v + 1)}{h v^{\alpha h - h + \frac{h}{v}} \Gamma^h(\alpha)} + 1.$$

The proof is complete. □

The methods to prove Theorems 4.1 and 4.2 are similar to those in [24] and [3]. From Theorems 4.1 and 4.2, we can have the following theorem.

Theorem 4.3. *(1) When D denotes ${}^{C}D$, and* (4.1) *satisfies* (i) *and* (ii)*, there exist constants $\widetilde{M} > 0$ and $p > 0$ such that*

$$\|x(t)\| \le \widetilde{M} e^{pt}, \tag{4.20}$$

for all $t > 0$.

(2) When D denotes ${}^{RL}D$, and (4.1) *satisfies* (i) *and* (ii)*, there exist constants $\widetilde{M} > 0$ and $p > 0$ for any $b > 0$ such that*

$$\|x(t)\| \le \widetilde{M} e^{pt}, \tag{4.21}$$

for all $t > b > 0$.

5. Validity of Laplace transform for fractional-order equations

Consider the one-dimensional fractional-order differential equation

$$D_0^\alpha x(t) = ax(t) + f_1(x(t)) + d_1(t), \tag{5.1}$$

where D denotes ${}^{RL}D$ or ${}^{C}D$, $l-1 < \alpha \le l$, $l \in N$, $\lambda \in R$, $x \in R$, $f_1(x)$ is the nonlinear part and continuous in $x \in R$, $d_1(t)$ is the input of the equation. And f_1 and d_1 also satisfy the assumptions (i) and (ii). Before the validity of Laplace transform method is justified, some lemmas and theorems are needed.

Lemma 5.1 ([9, p. 84])**.** *Let $\mathrm{Re}(\alpha) > 0$ and $f \in L^1(0,b)$ for any $b > 0$. Also let the estimate*

$$|f(t)| \le Ae^{p_0 t} \quad (t > b > 0)$$

hold for some constants $A > 0$ and $p_0 > 0$. Then the relation $\mathcal{L}(I_0^\alpha f(t)) = s^{-\alpha}\mathcal{L}(f(t))$ is valid for $Re\{s\} > p_0$.

Theorem 5.2. *If $\alpha > 0$, $n = [\alpha]+1$, and $x(t), I_0^{n-\alpha}x(t)$, $\frac{d}{dt}I_0^{n-\alpha}x(t), \dots, \frac{d^{n-1}}{dt^{n-1}}$ $I_0^{n-\alpha}x(t)$ are continuous in $(0,\infty)$ and of exponential order, while ${}^{RL}D_0^\alpha x(t)$ is piecewise continuous on $[0,\infty)$. Then*

$$\mathcal{L}({}^{RL}D_0^\alpha x(t)) = s^\alpha \mathcal{L}(x(t)) - \sum_{k=0}^{n-1} s^{n-k-1}\frac{d^{(k-1)}}{dt^{(k-1)}} I_0^{n-\alpha}x(0+).$$

Proof. Since ${}^{RL}D_0^\alpha x(t) = \frac{d^{(n)}}{dt^{(n)}} I_0^{n-\alpha}x(t)$, let $f(t) = I_0^{n-\alpha}x(t)$, then ${}^{RL}D_0^\alpha x(t) = \frac{d^{(n)}}{dt^{(n)}} f(t)$. Under the assumptions, from theorem 2.11 we have

$$\begin{aligned}\mathcal{L}({}^{RL}D_0^\alpha x(t)) = \mathcal{L}(f^{(n)}(t)) &= s^n\mathcal{L}(f(t)) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0+)\\ &= s^\alpha \mathcal{L}(x(t)) - \sum_{k=0}^{n-1} s^{n-k-1}\frac{d^{(k-1)}}{dt^{(k-1)}} I_0^{n-\alpha}x(0+).\end{aligned}$$

This completes the proof. □

Theorem 5.3. *When D denotes ${}^{RL}D$, the Laplace transform can be taken on both sides of* (5.1)*, if assumptions* (i) *and* (ii) *are satisfied and $x(t)$ is continuous.*

Proof. From (5.1) and Theorem 4.3, there exist constants $M_1 > 0$ and $P_1 > 0$ such that

$$\begin{aligned}|{}^{RL}D_0^\alpha x(t)| &\le |a||x(t)| + |f_1(x(t))| + |d_1(t)| \\ &\le (|a| + L)x(t) + M \\ &\le M_1 e^{p_1 t}.\end{aligned}$$

This implies ${}^{RL}D_0^\alpha x(t) = \frac{d^n}{dt^n} I_0^{n-\alpha} x(t)$ begin of exponential order. Then from Theorems 2.10 and 4.3 and Lemma 5.1, we have that $x(t)$, $I_0^{n-\alpha}x(t)$, and $\frac{d}{dt}I_0^{n-\alpha}x(t)$, $\dots$, $\frac{d^{n-1}}{dt^{n-1}}I_0^{n-\alpha}x(t)$ are of exponential order. From theorem 5.2, the Laplace transform can be taken on both sides of (5.1). □

Theorem 5.4. *If $\alpha > 0$, $n = [a]+1$, and $x(t), x'(t), x''(t), x^{(n-1)}(t)$ are continuous on $[0, +\infty)$ and of exponential order, while ${}^C D_0^\alpha x(t)$ is piecewise continuous on $[0, \infty)$. Then*

$$\mathcal{L}({}^C D_0^\alpha x(t)) = s^\alpha \mathcal{L}(x(t)) - \sum_{k=0}^{n-1} s^{\alpha-k-1} x^{(k)}(0).$$

Proof. Since

$$\begin{aligned}{}^C D_0^\alpha x(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} x^{(n)}(\tau) d\tau \\ &= \frac{1}{\Gamma(n-\alpha)} t^{m-\alpha-1} * x^{(n)}(t).\end{aligned}$$

Under the assumptions in Theorem 2.11, one has

$$\begin{aligned}\mathcal{L}({}^C D_0^\alpha x(t)) &= \frac{1}{\Gamma(n-\alpha)} \mathcal{L}(t^{n-\alpha-1}) \cdot \mathcal{L}(x^{(n)}(t)) \\ &= s^\alpha \mathcal{L}(x(t)) - \sum_{k=0}^{n-1} s^{\alpha-k-1} x^{(k)}(0).\end{aligned}$$

□

Lemma 5.5. *If ${}^C D_0^\alpha x(t)$ is of exponential order and $n = [\alpha] + 1$, then $x^{(j)}(t)(j = 1, \dots, n-1)$ is also of exponential order.*

Proof. Since ${}^C D_0^\alpha x(t)$ is of exponential order, then there exist constants $M_2 > 0$ and $P_2 > 0$ such that

$$|{}^C D_0^\alpha x(t)| = |{}^C D_0^{\alpha-j} x^{(j)}(t)| \le M_2 e^{p_2 t};$$

that is,

$$-M_2 e^{p_2 t} \le {}^C D_0^{\alpha-j} x^{(j)}(t) \le M_2 e^{p_2 t}.$$

Then there exist functions $M_1(t) \le 0, M_2(t) \le 0$ such that

$$-M_2 e^{p_2 t} + M_1(t) = {}^C D_0^{\alpha-j} x^{(j)}(t) = M_2 e^{p_2 t} - M_2(t).$$

This is equivalent to

$$\sum_{i=0}^{n-i-1} \frac{x^{(i+j)}(0)}{i!} t^i - I_0^{\alpha-j}(M_2 e^{p_2 t}) + I_0^{\alpha-j} M_1(t)$$

$$= x^{(j)}(t) = \sum_{i=0}^{n-i-1} \frac{x^{(i+j)}(0)}{i!} t^i + I_0^{\alpha-j}(M_2 e^{p_2 t}) - I_0^{\alpha-j} M_2(t).$$

In view of $I_0^{\alpha-j} M_1(t) \geq 0$ and $I_0^{\alpha-j} M_2(t) \geq 0$, we have

$$\sum_{i=0}^{n-i-1} \frac{x^{(i+j)}(0)}{i!} t^i - I_0^{\alpha-j}(M_2 e^{p_2 t}) \leq x^{(j)}(t) \leq \sum_{i=0}^{n-i-1} \frac{x^{(i+j)}(0)}{i!} t^i + I_0^{\alpha-j}(M_2 e^{p_2 t}).$$

Note that

$$\begin{aligned}
I_0^{\alpha-j} e^{p_2 t} &= \frac{1}{\Gamma(\alpha-j)} \int_0^t (t-\tau)^{\alpha-j-1} e^{p_2 \tau} d\tau \\
&= \frac{1}{\Gamma(\alpha-j)} e^{p_2 t} \int_0^t (t-\tau)^{\alpha-j-1} e^{p_2(\tau-t)} d\tau \\
&= \frac{1}{\Gamma(\alpha-j)} e^{p_2 t} \int_0^t s^{\alpha-j-1} e^{-s p_2} ds \\
&= \frac{1}{\Gamma(\alpha-j)} e^{p_2 t} \frac{1}{p_2^\alpha} \int_0^{p_2 t} u^{\alpha-j-1} e^{-u} du \\
&\leq \frac{1}{\Gamma(\alpha-j)} p_2^{-\alpha+j} e^{p_2 t} \int_0^{+\infty} u^{\alpha-j-1} e^{-u} du \\
&\leq p_2^{-\alpha+j} e^{p_2 t},
\end{aligned}$$

where $s = t - \tau$, $u = p_2 s$. It is not difficult to get that there exist $M_3 > 0$ and $p_3 > 0$ such that

$$|x^{(j)}(t)| \leq M_3 e^{p_3 t},$$

where $j = 0, \dots, n-1$. □

Theorem 5.6. *When D denotes ${}^C D_0^\alpha$, the Laplace transform can be taken on both sides of* (5.1)*, if assumptions* (i) *and* (ii) *are satisfied and $x(t)$ is continuous.*

Proof. From (5.1) and Theorem 4.3, then there exist constants $M_4 > 0$ and $P_4 > 0$ such that

$$\begin{aligned}
|{}^C D_0^\alpha x(t)| &\leq |a||x(t)| + |f_1(x(t))| + |d_1(t)| \\
&\leq (|a| + L)x(t) + M \\
&\leq M_4 e^{p_4 t}.
\end{aligned}$$

It means that ${}^C D_0^\alpha x(t)$ is of exponential order. Then from Lemma 5.5 and Theorem 4.3 we have $x^{(j)}(t)(j = 0, \dots, n-1)$ is of exponential order. From Theorem 5.4, the Laplace transform can be taken on both sides of (5.1). □

Conclusions. By Gronwall and Hölder inequalities, solutions of fractional-order equations are showed to be of exponential order. Based on that, the fractional-order and integer-order derivatives are all estimated to be of exponential order. Consequently, the Laplace transform is proved to be valid for fractional-order equations under general conditions. So the validity of Laplace transform of fractional-order equations is justified.

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