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ORTHOGONAL DECOMPOSITION AND ASYMPTOTIC BEHAVIOR FOR A LINEAR COUPLED SYSTEM OF MAXWELL AND HEAT EQUATIONS

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Abstract. We study the asymptotic behavior in time of the solutions of a coupled system of linear Maxwell equations with thermal effects. We have two basic results. First, we prove the existence of a strong solution and obtain the orthogonal decomposition of the electromagnetic field. Also, choosing a suitable multiplier, we show that the total energy of the system decays exponentially as $t \to +\infty$. The results obtained for this linear problem can serve as a first attempt to study other nonlinear problems related to this subject.

1. Introduction

It is undisputed the growing interest in understanding phenomena involving processes of reciprocal action between variations in the electromagnetic field in a region and the temperature or even other situations that are related to electromagnetic waves propagation (see [2, 4, 11, 14]).

In this work we consider a coupled system that describes interactions of the electromagnetic field with the temperature variation governed by the linear model

$$\epsilon \mathbf{E}_t - \nabla \times \mathbf{H} + \sigma(x)\mathbf{E} + \gamma \nabla\theta = 0 \quad \text{in } \Omega \times (0, +\infty), \tag{1.1}$$

$$\mu \mathbf{H}_t + \nabla \times \mathbf{E} = 0 \quad \text{in } \Omega \times (0, +\infty), \tag{1.2}$$

$$\theta_t - \operatorname{div}(\nabla\theta - \lambda \mathbf{E}) = 0 \quad \text{in } \Omega \times (0, +\infty), \tag{1.3}$$

$$\operatorname{div}(\mu \mathbf{H}) = 0 \quad \text{in } \Omega \times (0, +\infty) \tag{1.4}$$

with initial and boundary conditions

$$\mathbf{E}(x, 0) = \mathbf{E}_0(x), \quad \mathbf{H}(x, 0) = \mathbf{H}_0(x) \quad \text{and} \quad \theta(x, 0) = \theta_0(x) \quad \text{in } \Omega, \tag{1.5}$$

$$\eta \times \mathbf{E} = 0, \quad \eta \cdot \mathbf{H} = 0, \quad \theta = 0 \quad \text{on } \Gamma \times (0, +\infty). \tag{1.6}$$

Here Ω is a bounded, open, simply-connected domain of $\mathbb{R}^3$ with a regular boundary $\Gamma = \partial\Omega$. The functions $\mathbf{E} = \mathbf{E}(x,t) = (E_1(x,t), E_2(x,t), E_3(x,t))$, $\mathbf{H} = \mathbf{H}(x,t) = (H_1(x,t), H_2(x,t), H_3(x,t))$ and $\theta = \theta(x,t)$ (hereafter, a bold letter means a vector or a vector function in $\mathbb{R}^3$) represent, respectively, the electric field, the magnetic field and the difference of temperature between the actual state and a reference temperature at location $x \in \Omega$ and time t. In (1.6), η is the outward normal on

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Γ. In (1.1)-(1.2), $\nabla \times \mathbf{v}$ indicates the curl of the vectorial function $\mathbf{v}$ and ϵ and μ are positive constants characteristics of the medium considered called, respectively, the permittivity and the magnetic permeability. $\sigma = \sigma(x)$ is a real valued $L^\infty(\Omega)$-function representing the electric conductivity (see [7]), related with the Ohm's law and satisfies the hypothesis

$$\sigma_0 \le \sigma(x) \le \sigma_1, \tag{1.7}$$

where σ_0 and σ_1 are positive constants. Moreover, γ and λ are coupling constants which, for simplicity, we will assume positive.

The mathematical model (1.1)-(1.4) is motivated by considering the classical Maxwell's equations that are coupled to a heat equation, modeling an expectedly interaction of the electromagnetic field with the temperature variation in the bounded domain Ω with perfectly conducting boundary $\Gamma = \partial\Omega$. In fact, if $\mathbf{E}(x,t)$ and $\mathbf{H}(x,t)$ denote the electric and magnetic fields in Ω, respectively, and $\mathbf{D}(x,t)$ and $\mathbf{B}(x,t)$ are the electric displacement and magnetic induction in Ω, respectively, then hold (see [7]) the *Faraday's law*

$$\nabla \times \mathbf{E} = -\mathbf{B}_t, \tag{1.8}$$

the *Ampere's law*

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{D}_t, \tag{1.9}$$

where $\mathbf{J}$ represents the current density, and the *Gauss's law for magnetism*

$$\operatorname{div} \mathbf{B} = 0. \tag{1.10}$$

In our case, we assume the constitutive relations

$$D = \epsilon \mathbf{E}, \quad B = \mu \mathbf{H} \tag{1.11}$$

and *Omh's law*

$$\mathbf{J} = \sigma \mathbf{E} \tag{1.12}$$

and take, for simplicity, ϵ and μ positive constants. The boundary condition (1.6) is consistent with the fact that the boundary Γ is perfectly conducting, such that the tangential component of the electric field must vanish.

The model for the propagation of heat turns into well-known equations for the temperature θ (difference to a fixed constant reference temperature) and the heat flux vector $\mathbf{q}$,

$$\theta_t + \rho \operatorname{div} \mathbf{q} = 0 \tag{1.13}$$

and

$$\mathbf{q} + \kappa \nabla\theta = 0, \tag{1.14}$$

where ρ and κ are positive constants. Equation (1.13) represents the assumed Fourier law of heat conduction. Replacing (1.14) into (1.13) we obtain the parabolic heat equation

$$\theta_t - \rho\kappa\Delta\theta = 0. \tag{1.15}$$

The system we consider is composed by the Maxwell's equations (1.8)-(1.11) that are coupled to a heat equation (1.15) modeling an expectedly interaction effect through heat conduction. Indeed, we consider the problem (1.1)-(1.6). We point out that the results obtained for this linear problem can serve as a first attempt to study, for example, the stabilization of solutions of the nonlinear problems of inductive heating or microwave heating (see [14, 15]).

It is worth note that inductive and microwave heating processes are gaining increasing acceptance in industry (metal hardening and preheating for forging operations, for example) and in some fields of science, such as biomedical engineering (see [6, 8, 9]). Some of these processes are modeled mathematically by nonlinear systems of Maxwell's equations coupled with the heat equation (see [8, 9, 15]). Such systems have been studied by some authors not only with respect to the existence of solution, like [14, 15], but also with respect to the regularity of the solution and blow up properties. To the best of the authors knowledge, little is known about the asymptotic behavior of the energy associated with such nonlinear models. The cases analyzed, in general, are limited to those in one dimension (see [5, 10]). Hence the importance of studying the behavior of the solution of a mathematical model, even in the linear case, involving Maxwell's equations under thermal effects, which is presented in this paper.

Concerning the system (1.1)-(1.6), the *Total Energy* is given by

$$\mathcal{E}(t) = \frac{1}{2}\int_\Omega (\lambda\epsilon|\mathbf{E}|^2 + \lambda\mu|\mathbf{H}|^2 + \gamma|\theta|^2)\,dx,$$

where $|\mathbf{E}|^2 = \sum_{j=1}^3 E_j^2$ and $|\mathbf{H}|^2 = \sum_{j=1}^3 H_j^2$. Formally, an easy calculation gives us that the derivative of $\mathcal{E}(t)$ is given by

$$\frac{d\mathcal{E}(t)}{dt} = -\lambda\int_\Omega \sigma(x)|\mathbf{E}|^2\,dx - \gamma\int_\Omega |\nabla\theta|^2\,dx \le 0.$$

Therefore one may ask, "Does $\mathcal{E}(t) \to 0$ as $t \to +\infty$?", and if this is the case, "Does $\mathcal{E}(t) \to 0$ decay at a uniform rate as $t \to +\infty$?" This is not difficult to answer in the case of Maxwell's equations with the dissipation given by the conductivity σ with hypothesis (1.7). In fact, this case lead to dissipative wave equations for the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$, which have exponential decay. In our case the uniform stabilization of system (1.1)-(1.6) requires a more detailed discussion, which we present in this article.

This article is organized as follow. In section 2 we present some functional spaces and basic results. In section 3 we obtain the strong global solution of system (1.1)-(1.6). To obtain the exponential decay of the energy, in section 4 we obtain a special decomposition of the electromagnetic field in suitable Sobolev spaces. Finally, section 5 is devoted to study the exponential decay of the total energy associated to system (1.1)-(1.6).

2. Basic definitions and preliminary results

In this section we introduce some standard functional spaces as defined in [1, 2, 3]. Hereafter the bracket $(\cdot,\cdot)$ and $\|\cdot\|$ will denote, respectively, the standard inner product and norm of $L^2(\Omega)^3$ or $L^2(\Omega)$. Let

$$H(\mathrm{curl},\Omega) = \{\mathbf{v} \in L^2(\Omega)^3; \nabla\times\mathbf{v} \in L^2(\Omega)^3\},$$
$$H(\mathrm{div},\Omega) = \{\mathbf{v} \in L^2(\Omega)^3; \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\},$$

Hilbert spaces with their respective inner products

$$(\mathbf{u},\mathbf{v})_{H(\mathrm{curl},\Omega)} = (\nabla\times\mathbf{u}, \nabla\times\mathbf{v}) + (\mathbf{u},\mathbf{v}),$$
$$(\mathbf{u},\mathbf{v})_{H(\mathrm{div},\Omega)} = (\mathrm{div}\,\mathbf{u}, div\,\mathbf{v}) + (\mathbf{u},\mathbf{v}).$$

Let $H_0(\mathrm{curl}, \Omega)$ be the closure of

$$\{\mathbf{v} \in H(\mathrm{curl}, \Omega) \cap C^1(\overline{\Omega}); \mathbf{v} \times \eta = 0 \text{ on } \Gamma\}$$

in $H(\mathrm{curl}, \Omega)$ and let $H_0(\mathrm{div}, \Omega)$ be the closure of

$$\{\mathbf{v} \in H(\mathrm{div}, \Omega) \cap C^1(\overline{\Omega}); \mathbf{v} \cdot \eta = 0 \text{ on } \Gamma\}$$

in $H(\mathrm{div}, \Omega)$.

To obtain the result of existence of solution we still need to define the following spaces

$$H(\mathrm{div}\, 0, \Omega) = \{\mathbf{v} \in L^2(\Omega)^3; \mathrm{div}\, \mathbf{v} = 0\}$$

and the closed subspace of the Hilbert space $L^2(\Omega)^3$

$$H_0(\mathrm{div}\, 0, \Omega) = \{\mathbf{v} \in H(\mathrm{div}\, 0, \Omega); \mathbf{v} \cdot \eta = 0 \text{ on } \Gamma\} = H_0(\mathrm{div}, \Omega) \cap H(\mathrm{div}\, 0, \Omega).$$

Lemma 2.1. *Let $P_0 : L^2(\Omega)^3 \to H_0(\mathrm{div}\, 0, \Omega)$ be the projection operator defined by*

$$\mathbf{u} \to P_0\mathbf{u} = \mathbf{u}_1,$$

where $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, with $\mathbf{u}_1 \in H_0(\mathrm{div}\, 0, \Omega)$ and $\mathbf{u}_2 \in H_0(\mathrm{div}\, 0, \Omega)^\perp$. We have the following statements:

(i) $P_0(H(\mathrm{curl}, \Omega)) \subset H(\mathrm{curl}, \Omega) \cap H_0(\mathrm{div}\, 0, \Omega)$;
(ii) $H(\mathrm{curl}, \Omega) \cap H_0(\mathrm{div}\, 0, \Omega)$ *is dense in* $H_0(\mathrm{div}\, 0, \Omega)$.

Proof. To prove (i) it is sufficient to show that $P_0(H(\mathrm{curl}, \Omega)) \subset H(\mathrm{curl}, \Omega)$. To this we use a similar idea as in [11]. Let $\mathbf{u} \in H(\mathrm{curl}, \Omega)$. Setting $\Psi \in \mathcal{D}(\Omega)^3$, we have

$$\begin{aligned}\langle \nabla \times (P_0\mathbf{u}), \Psi\rangle &= \langle P_0\mathbf{u}, \nabla \times \Psi\rangle \\ &= \int_\Omega P_0\mathbf{u} \cdot \nabla \times \Psi \, dx = \int_\Omega \mathbf{u} \cdot \nabla \times \Psi \, dx \\ &= \langle \mathbf{u}, \nabla \times \Psi\rangle = \langle \nabla \times \mathbf{u}, \Psi\rangle,\end{aligned}$$

for all $\Psi \in \mathcal{D}(\Omega)^3$, where we have used that $\nabla \times \Psi \in H_0(\mathrm{div}\, 0, \Omega)$.

The previous identity give us $\nabla \times (P_0\mathbf{u}) = \nabla \times \mathbf{u} \in L^2(\Omega)^3$. This proves (i).

(ii) By (i) we have

$$P_0(\mathcal{D}(\Omega)^3) \subset H(\mathrm{curl}, \Omega) \cap H_0(\mathrm{div}\, 0, \Omega) \subset H_0(\mathrm{div}\, 0, \Omega),$$

so to prove (ii) it is sufficient to prove that $P_0(\mathcal{D}(\Omega)^3)$ is dense in $H_0(\mathrm{div}\, 0, \Omega)$.

Let $\mathbf{v} \in H_0(\mathrm{div}\, 0, \Omega)$. So $\mathbf{v} \in L^2(\Omega)^3$, and there exist a sequence (Ψ_n) in $\mathcal{D}(\Omega)^3$ such that

$$\Psi_n \to \mathbf{v} \quad \text{in} \quad L^2(\Omega)^3.$$

By continuity of P_0,

$$P_0(\Psi_n) \to P_0(\mathbf{v}) = \mathbf{v} \quad \text{in } H_0(\mathrm{div}\, 0, \Omega),$$

with $P_0(\Psi_n) \in P_0(\mathcal{D}(\Omega)^3)$. This concludes the proof of (ii) and Lemma 2.1. $\square$

3. Well-posedness of the problem

We rewrite system (1.1)-(1.3) in the form

$$\frac{d\Phi(t)}{dt} = A\Phi(t), \tag{3.1}$$

where $\Phi = (\mathbf{E}, \mathbf{H}, \theta)$ and A is the linear operator

$$A(\mathbf{E}, \mathbf{H}, \theta) = \left(-\sigma\epsilon^{-1}\mathbf{E} + \epsilon^{-1}\nabla \times \mathbf{H} - \gamma\epsilon^{-1}\nabla\theta, -\mu^{-1}\nabla \times \mathbf{E}, \operatorname{div}(-\lambda\mathbf{E} + \nabla\theta)\right).$$

Let us consider the Hilbert space $W = L^2(\Omega)^3 \times H_0(\operatorname{div} 0, \Omega) \times L^2(\Omega)$ with the inner product given by

$$\langle u, v\rangle_W = \epsilon\lambda(\mathbf{u}_1, \mathbf{v}_1) + \mu\lambda(\mathbf{u}_2, \mathbf{v}_2) + \gamma(u_3, v_3),$$

and induced norm

$$\|u\|_W^2 = \epsilon\lambda\|\mathbf{u}_1\|^2 + \mu\lambda\|\mathbf{u}_2\|^2 + \gamma\|u_3\|^2,$$

for any $u = (\mathbf{u}_1, \mathbf{u}_2, u_3)$ and $v = (\mathbf{v}_1, \mathbf{v}_2, v_3) \in W$.

The domain $D(A)$ of A is the set $D(A) = \{(\mathbf{E}, \mathbf{H}, \theta) \in H_0(\operatorname{curl}, \Omega) \times (H(\operatorname{curl}, \Omega) \cap H_0(\operatorname{div} 0, \Omega)) \times H_0^1(\Omega); -\lambda\mathbf{E} + \nabla\theta \in H(\operatorname{div}, \Omega)\}$, where $H_0^1(\Omega)$ denotes the usual Sobolev space.

Remark 3.1. It is easy to see that

$$\mathcal{D}(\Omega)^3 \times P_0(\mathcal{D}(\Omega)^3) \times \mathcal{D}(\Omega) \subset D(A) \subset L^2(\Omega)^3 \times H_0(\operatorname{div} 0, \Omega) \times L^2(\Omega) = W, \tag{3.2}$$

where P_0 is the orthogonal projection defined in Lemma 2.1.

Now we prove that A is the infinitesimal generator of a C_0-semigroup of contractions on W. The density of $D(A)$ in W follows by (3.2) and item (ii) of Lemma 2.1.

Lemma 3.2. *A is a dissipative operator on W.*

Proof. Let $U = (\mathbf{E}, \mathbf{H}, \theta) \in D(A)$. So by Gauss and Green's identities it follows that

$$\begin{aligned} \langle AU, U\rangle_W &= \lambda(-\sigma(x)\mathbf{E} + \nabla \times \mathbf{H} - \gamma\nabla\theta, \mathbf{E}) + \lambda(-\nabla \times \mathbf{E}, \mathbf{H}) \\ &\quad + \gamma(\operatorname{div}(\nabla\theta - \lambda\mathbf{E}), \theta) \\ &= -\lambda\int_\Omega \sigma(x)|\mathbf{E}|^2\,dx - \gamma\int_\Omega |\nabla\theta|^2\,dx \le 0. \end{aligned} \tag{3.3}$$

□

Lemma 3.3. *The range $R(I - A)$ of the operator $I - A$ is W.*

Proof. Let $w = (\mathbf{f}, \mathbf{g}, h) \in W$ and we have to prove that there exists $U = (\mathbf{E}, \mathbf{H}, \theta)$ in $D(A)$ such that $(I - A)U = w$; that is,

$$\begin{gathered} \mathbf{E} + \sigma(x)\epsilon^{-1}\mathbf{E} - \epsilon^{-1}\nabla \times \mathbf{H} + \gamma\epsilon^{-1}\nabla\theta = \mathbf{f} \\ \mathbf{H} + \mu^{-1}\nabla \times \mathbf{E} = \mathbf{g} \\ \theta - \operatorname{div}(-\lambda\mathbf{E} + \nabla\theta) = h. \end{gathered} \tag{3.4}$$

Replacing the second line in the first line of system (3.4) we obtain the equivalent system

$$\begin{gathered} (1 + \sigma(x)\epsilon^{-1})\mathbf{E} + \epsilon^{-1}\mu^{-1}\nabla \times (\nabla \times \mathbf{E}) + \gamma\epsilon^{-1}\nabla\theta = \mathbf{f} + \epsilon^{-1}\nabla \times \mathbf{g} \\ \theta - \operatorname{div}(-\lambda\mathbf{E} + \nabla\theta) = h. \end{gathered} \tag{3.5}$$

To solve (3.5) we consider the bilinear form $a : [H_0(\text{curl}, \Omega) \times H_0^1(\Omega)]^2 \to \mathbb{R}$ defined by

$$\begin{aligned} a((\mathbf{E},\theta),(\Phi,\psi)) &= \lambda(\mathbf{E},\Phi) + \lambda\epsilon^{-1}(\sigma(x)\mathbf{E},\Phi) + \lambda\epsilon^{-1}\mu^{-1}(\nabla\times\mathbf{E},\nabla\times\Phi) \\ &\quad + \lambda\gamma\epsilon^{-1}(\nabla\theta,\Phi) + \gamma\epsilon^{-1}(\theta,\psi) - \gamma\epsilon^{-1}(\lambda\mathbf{E}-\nabla\theta,\nabla\psi) \end{aligned}$$

and the linear form $F : H_0(\text{curl}, \Omega) \times H_0^1(\Omega) \to \mathbb{R}$ defined by

$$F(\Phi,\psi) = \lambda(\mathbf{f},\Phi) + \lambda\epsilon^{-1}(\mathbf{g},\nabla\times\Phi) + \gamma\epsilon^{-1}(h,\psi).$$

The bilinear form a is coercive, because

$$\begin{aligned} & a((\mathbf{E},\theta),(\mathbf{E},\theta)) \\ &= \lambda\|\mathbf{E}\|^2 + \lambda\epsilon^{-1}\|\sigma^{1/2}(x)\mathbf{E}\|^2 + \lambda\epsilon^{-1}\mu^{-1}\|\nabla\times\mathbf{E}\|^2 + \gamma\epsilon^{-1}\|\theta\|^2_{H_0^1(\Omega)} \\ &\geq C\|(\mathbf{E},\theta)\|^2_{H_0(\text{curl},\Omega)\times H_0^1(\Omega)}. \end{aligned}$$

The bilinear form a is also continuous. Indeed, Cauchy-Schwarz's inequality implies

$$\begin{aligned} & |a((\mathbf{E},\theta),(\Phi,\psi))| \\ &\leq \lambda(1+\sigma_1\epsilon^{-1})\|\mathbf{E}\|\|\Phi\| + \lambda\epsilon^{-1}\mu^{-1}\|\nabla\times\mathbf{E}\|\|\nabla\times\Phi\| + \lambda\gamma\epsilon^{-1}\|\nabla\theta\|\|\Phi\| \\ &\quad + \gamma\epsilon^{-1}\|\theta\|\|\psi\| + \gamma\lambda\epsilon^{-1}\|\mathbf{E}\|\|\nabla\psi\| + \gamma\epsilon^{-1}\|\nabla\theta\|\|\nabla\psi\| \\ &\leq \lambda(1+\sigma_1\epsilon^{-1}+\epsilon^{-1}\mu^{-1})(\|\mathbf{E}\|_{H_0(\text{curl},\Omega)}\|\Phi\|_{H_0(\text{curl},\Omega)}) \\ &\quad + \gamma\lambda\epsilon^{-1}\|\theta\|_{H_0^1(\Omega)}\|\Phi\|_{H_0(\text{curl},\Omega)} + \gamma\epsilon^{-1}\|\theta\|_{H_0^1(\Omega)}\|\psi\|_{H_0^1(\Omega)} \\ &\quad + \gamma\lambda\epsilon^{-1}\|\mathbf{E}\|_{H_0(\text{curl},\Omega)}\|\psi\|_{H_0^1(\Omega)} + \gamma\epsilon^{-1}\|\theta\|_{H_0^1(\Omega)}\|\psi\|_{H_0^1(\Omega)} \\ &\leq C\Big(\|\mathbf{E}\|^2_{H_0(\text{curl},\Omega)} + \|\theta\|^2_{H_0^1(\Omega)}\Big)^{1/2}\Big(\|\Phi\|^2_{H_0(\text{curl},\Omega)} + \|\psi\|^2_{H_0^1(\Omega)}\Big)^{1/2} \\ &= C\|(\mathbf{E},\theta)\|_{H_0(\text{curl},\Omega)\times H_0^1(\Omega)}\|(\Phi,\psi)\|_{H_0(\text{curl},\Omega)\times H_0^1(\Omega)}. \end{aligned}$$

To prove that F is continuous, we observe that

$$\begin{aligned} |F(\Phi,\psi)| &\leq \lambda\|\mathbf{f}\|\|\Phi\| + \lambda\epsilon^{-1}\|\mathbf{g}\|\|\nabla\times\Phi\| + \gamma\epsilon^{-1}\|h\|\|\psi\| \\ &\leq \lambda(1+\epsilon^{-1})(\|\mathbf{f}\|+\|\mathbf{g}\|)\|\Phi\|_{H_0(\text{curl},\Omega)} + \gamma\epsilon^{-1}\|h\|\|\psi\|_{H_0^1(\Omega)} \\ &\leq \big[\lambda(1+\epsilon^{-1})(\|\mathbf{f}\|+\|\mathbf{g}\|) + \gamma\epsilon^{-1}\|h\|\big]\big[\|\Phi\|^2_{H_0(\text{curl},\Omega)} + \|\psi\|^2_{H_0^1(\Omega)}\big]^{1/2} \\ &\leq C\|(\Phi,\psi)\|_{H_0(\text{curl},\Omega)\times H_0^1(\Omega)}. \end{aligned}$$

By Lax-Milgram's Lemma, there exists a unique $(\mathbf{E},\theta) \in H_0(\text{curl},\Omega) \times H_0^1(\Omega)$ such that

$$a((\mathbf{E},\theta),(\Phi,\psi)) = F(\Phi,\psi), \quad \forall(\Phi,\psi) \in H_0(\text{curl},\Omega) \times H_0^1(\Omega). \tag{3.6}$$

Let

$$\mathbf{H} = \mathbf{g} - \mu^{-1}\nabla\times\mathbf{E}. \tag{3.7}$$

So $\mathbf{H} \in H_0(\operatorname{div} 0,\Omega)$, because $\mathbf{g} \in H_0(\operatorname{div} 0,\Omega)$ and $\nabla\times\mathbf{E} \in H_0(\operatorname{div} 0,\Omega)$ (see [3, page 35]).

First we consider $\Phi \in \mathcal{D}(\Omega)^3$ and $\psi = 0$ in (3.6). We get

$$(1+\sigma(x)\epsilon^{-1})\mathbf{E} + \epsilon^{-1}\mu^{-1}\nabla\times(\nabla\times\mathbf{E}) + \gamma\epsilon^{-1}\nabla\theta = \mathbf{f} + \epsilon^{-1}\nabla\times\mathbf{g} \quad \text{in } \mathcal{D}'(\Omega)^3;$$

that is,

$$(1+\sigma\epsilon^{-1})\mathbf{E} - \epsilon^{-1}\nabla\times\mathbf{H} + \gamma\epsilon^{-1}\nabla\theta = \mathbf{f} \quad \text{in } \mathcal{D}'(\Omega)^3. \tag{3.8}$$

This proves $\mathbf{H} \in H(\mathrm{curl}, \Omega)$ and, hence, $\mathbf{H} \in H(\mathrm{curl}, \Omega) \cap H_0(\mathrm{div}\, 0, \Omega)$. Now, taking $\Phi = 0$ and $\psi \in \mathcal{D}(\Omega)$ in (3.6) we obtain

$$\theta + \mathrm{div}(\lambda \mathbf{E} - \nabla \theta) = h \quad \text{in } \mathcal{D}'(\Omega) \tag{3.9}$$

and this proves that $(\lambda \mathbf{E} - \nabla \theta) \in H(\mathrm{div}, \Omega)$. By (3.7)-(3.9) we have $(\mathbf{E}, \mathbf{H}, \theta) \in D(A)$ and solves (3.4). $\square$

Using the results before we have the following theorem (see [12]).

Theorem 3.4. *Let* $(\mathbf{E}_0, \mathbf{H}_0, \theta_0) \in D(A)$. *Then problem* (1.1)-(1.6) *admits a unique solution* $(\mathbf{E}, \mathbf{H}, \theta)$ *such that*

$$\begin{gathered}
\mathbf{E} \in C([0, +\infty), H_0(\mathrm{curl}, \Omega)) \cap C^1([0, +\infty), L^2(\Omega)^3), \\
\mathbf{H} \in C([0, +\infty), H(\mathrm{curl}, \Omega) \cap H_0(\mathrm{div}\, 0, \Omega)) \cap C^1([0, +\infty), H_0(\mathrm{div}\, 0, \Omega)), \\
\theta \in C([0, +\infty), H_0^1(\Omega)) \cap C^1([0, +\infty), L^2(\Omega)), \\
\lambda \mathbf{E} - \nabla \theta \in C([0, +\infty), H(\mathrm{div}, \Omega)).
\end{gathered}$$

To obtain the stability of solution of system (1.1)-(1.6) we need to a more regular solution. To this, we consider the spaces

$$\mathbb{H}_1(\Omega) = H(\mathrm{curl}\, 0, \Omega) \cap H_0(\mathrm{div}\, 0, \Omega),$$

where $H(\mathrm{curl}\, 0, \Omega) = \{\mathbf{u} \in L^2(\Omega)^3 : \nabla \times \mathbf{u} = 0\}$, and

$$\mathcal{V}_H = L^2(\Omega)^3 \times \mathbb{H}_1(\Omega)^\perp \times L^2(\Omega),$$

where $\mathbb{H}_1(\Omega)^\perp$ is the orthogonal complement of the space $\mathbb{H}_1(\Omega)$ in $L^2(\Omega)^3$.

We have the following existence result on strong solutions of system (1.1)-(1.6).

Theorem 3.5. *Let* $(\mathbf{E}_0, \mathbf{H}_0, \theta_0) \in D(A) \cap \mathcal{V}_H$. *Then the solution* $(\mathbf{E}, \mathbf{H}, \theta)$ *of* (1.1)-(1.6) *obtained in Theorem 3.4 satisfies* $(\mathbf{E}, \mathbf{H}, \theta) \in D(A) \cap \mathcal{V}_H$ *for all* $t > 0$.

Proof. It is sufficient to prove that $\mathbf{H} \in \mathbb{H}_1(\Omega)^\perp$. To this, we consider $\mathbf{h} \in \mathbb{H}_1(\Omega)$. From (1.2) we obtain

$$\int_\Omega \mu \mathbf{H}_t \cdot \mathbf{h}\, dx + \int_\Omega \nabla \times \mathbf{E} \cdot \mathbf{h}\, dx = 0. \tag{3.10}$$

Green's formula gives us

$$\int_\Omega \nabla \times \mathbf{E} \cdot \mathbf{h}\, dx = \int_\Omega \mathbf{E} \cdot (\nabla \times \mathbf{h})\, dx + \int_\Gamma (\eta \times E) \cdot \mathbf{h}\, d\Gamma = 0.$$

The above identity and (3.10) give us $(\mu \mathbf{H}, \mathbf{h}) = (\mu \mathbf{H}_0, \mathbf{h}) = 0$. So $\mathbf{H} \in \mathbb{H}_1(\Omega)^\perp$. $\square$

4. Orthogonal decomposition

Using the standard "Hodge" orthogonal decomposition of $L^2(\Omega)^3$ (see [1, 3, 13]) we can write

$$\mu \mathbf{H} = \nabla q + \mathbf{h}_1 + \nabla \times \Psi, \tag{4.1}$$

where $q \in H^1(\Omega)$, $\mathbf{h}_1 \in \mathbb{H}_1(\Omega)$, $\Psi \in H^1(\Omega)^3 \cap H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}\, 0, \Omega)$ and $\int_\Gamma \Psi \cdot \eta\, d\Gamma = 0$.

Since $\mathbf{H} \in \mathbb{H}_1(\Omega)^\perp \cap H_0(\mathrm{div}\, 0, \Omega)$, we have $\mathbf{h}_1 = 0$ and $\nabla q = 0$ (see [1]), so

$$\mu \mathbf{H} = \nabla \times \Psi. \tag{4.2}$$

Remark 4.1. It is well know (see [1, 3]) that for all $\mathbf{v} \in H(\operatorname{div} 0, \Omega) \cap H_0(\operatorname{curl}, \Omega)$ is valid the inequality

$$\|\mathbf{v}\| \leq C\|\nabla \times \mathbf{v}\|,$$

where C is a real positive constant. In our case, we obtain

$$\|\Psi\| \leq C\|\nabla \times \Psi\| = C\|\mu\mathbf{H}\|. \tag{4.3}$$

Now, we will study the $L^2(\Omega)^3$ decomposition of the electric field $\mathbf{E}$. In fact, we have (see [1])

$$\mathbf{E} = -\nabla p + \mathbf{B}, \tag{4.4}$$

where $p \in H_0^1(\Omega)$ and $\mathbf{B} \in H(\operatorname{div} 0, \Omega)$.

From equation (1.2) and decomposition (4.2) of $\mathbf{H}$ we obtain

$$0 = \mu\mathbf{H}_t + \nabla \times \mathbf{E} = \nabla \times \Psi_t + \nabla \times \mathbf{E} = \nabla \times (\Psi_t + \nabla p + \mathbf{E}). \tag{4.5}$$

Also,

$$\operatorname{div}(\Psi_t + \nabla p + \mathbf{E}) = \operatorname{div}(\Psi_t) + \operatorname{div}(\mathbf{B}) = 0, \tag{4.6}$$

because $\Psi, \mathbf{B} \in H(\operatorname{div} 0, \Omega)$.

The last two equalities give us

$$\Psi_t + \nabla p + \mathbf{E} \in H(\operatorname{curl} 0, \Omega) \cap H(\operatorname{div} 0, \Omega).$$

Now, we observe that $\Psi_t \in H_0(\operatorname{curl}, \Omega)$, $\mathbf{E} \in H_0(\operatorname{curl}, \Omega)$ and, since $p \in H_0^1(\Omega)$, $\nabla p \in H_0(\operatorname{curl} 0, \Omega) := H(\operatorname{curl} 0, \Omega) \cap H_0(\operatorname{curl}, \Omega)$ (see [3]). So

$$\Psi_t + \nabla p + \mathbf{E} \in \mathbb{H}_2(\Omega), \tag{4.7}$$

where $\mathbb{H}_2(\Omega) = H_0(\operatorname{curl} 0, \Omega) \cap H(\operatorname{div} 0, \Omega)$. From (4.7) we can write

$$\mathbf{E} = -\nabla p - \Psi_t + \mathbf{h}_2, \tag{4.8}$$

where $\mathbf{h}_2 \in \mathbb{H}_2(\Omega)$.

Finally, we can see that

$$\|\mathbf{E}\|^2 = \|\nabla p\|^2 + \|\Psi_t\|^2 + \|\mathbf{h}_2\|^2, \tag{4.9}$$

because ∇p, Ψ_t and $\mathbf{h}_2$ are two by two orthogonal vectors in $L^2(\Omega)^3$ (see [13]).

5. Exponential decay

In this section we obtain the exponential decay of the solution of system (1.1)-(1.6) obtained in section 3. To this, we use a suitable Lyapunov functional and suppose that σ satisfies hypothesis (1.7). First we present some technical lemmas and at the end of the section we prove the main result of this paper.

Lemma 5.1. *Suppose* $(\mathbf{E}_0, \mathbf{H}_0, \theta_0) \in D(A) \cap \mathcal{V}_H$ *and let* $(\mathbf{E}, \mathbf{H}, \theta)$ *solution of system* (1.1)-(1.6) *obtained in Theorem 3.5. Let*

$$\mathcal{E}(t) \equiv \frac{1}{2}\int_\Omega (\lambda\epsilon|\mathbf{E}|^2 + \lambda\mu|\mathbf{H}|^2 + \gamma|\theta|^2)\, dx.$$

Then

$$\frac{d\mathcal{E}(t)}{dt} = -\lambda\int_\Omega \sigma(x)|\mathbf{E}|^2\, dx - \gamma\int_\Omega |\nabla\theta|^2\, dx \leq 0.$$

The proof of the above lemma follows directly from the system (1.1)-(1.6) using straightforward calculation.

Lemma 5.2. *Let $G(t) = \mathcal{E}(t) - \delta F(t)$, where $\mathcal{E}(t)$ is defined in Lemma 5.1,*
$$F(t) = \epsilon \int_\Omega \mathbf{E} \cdot \Psi \, dx,$$
where $\mu \mathbf{H} = \nabla \times \Psi$, and δ is a positive parameter to be specified later. We have

(i)
$$\frac{dF(t)}{dt} = \mu \int_\Omega |\mathbf{H}|^2 \, dx - \epsilon \int_\Omega |\Psi_t|^2 \, dx - \int_\Omega \sigma(x) \mathbf{E} \cdot \Psi \, dx;$$

(ii) $\frac{1}{2}\mathcal{E}(t) \le G(t) \le 2\mathcal{E}(t)$.

Proof. To prove (i), from (1.1) and (4.8), we observe that
$$\begin{aligned}
\frac{dF(t)}{dt} &= \epsilon \int_\Omega \mathbf{E} \cdot \Psi_t \, dx + \epsilon \int_\Omega \mathbf{E}_t \cdot \Psi \, dx \\
&= \epsilon \int_\Omega (-\nabla p - \Psi_t + \mathbf{h}_2) \cdot \Psi_t \, dx + \int_\Omega (\nabla \times \mathbf{H} - \sigma(x)\mathbf{E} - \gamma \nabla \theta) \cdot \Psi \, dx \\
&= -\epsilon \int_\Omega |\Psi_t|^2 \, dx + \int_\Omega \nabla \times \mathbf{H} \cdot \Psi \, dx - \int_\Omega \sigma(x) \mathbf{E} \cdot \Psi \, dx - \gamma \int_\Omega \nabla \theta \cdot \Psi \, dx \\
&= -\epsilon \int_\Omega |\Psi_t|^2 \, dx + \int_\Omega \mathbf{H} \cdot \nabla \times \Psi \, dx - \int_\Omega \sigma(x) \mathbf{E} \cdot \Psi \, dx + \gamma \int_\Omega \theta \, div \, \Psi \, dx \\
&= -\epsilon \int_\Omega |\Psi_t|^2 \, dx + \mu \int_\Omega |\mathbf{H}|^2 \, dx - \int_\Omega \sigma(x) \mathbf{E} \cdot \Psi \, dx,
\end{aligned}$$
because $\Psi \in H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}\, 0, \Omega)$, $p \in H_0^1(\Omega)$ and $\mathbf{h}_2 \in \mathbb{H}_2(\Omega)$.

To prove (ii), we use the Cauchy-Schwarz's inequality and (4.3):
$$\begin{aligned}
|G(t) - \mathcal{E}(t)| &= \delta |F(t)| \le \delta \epsilon \|\mathbf{E}\| \|\Psi\| \\
&\le \frac{\delta \epsilon}{2} \left(\|\mathbf{E}\|^2 + \|\Psi\|^2 \right) \\
&\le \frac{\delta \epsilon}{2} \left(\|\mathbf{E}\|^2 + C^2 \|\mu \mathbf{H}\|^2 \right) \\
&= \frac{\delta}{2\lambda} \Big(\int_\Omega \lambda \epsilon |\mathbf{E}|^2 \, dx + C^2 \mu \epsilon \int_\Omega \lambda \mu |\mathbf{H}|^2 \, dx \Big) \\
&\le \delta C_1 \mathcal{E}(t),
\end{aligned}$$
where
$$C_1 = \max \Big\{ \frac{1}{\lambda}, \frac{C^2 \mu \epsilon}{\lambda} \Big\}.$$
The conclusion follows by choosing δ sufficiently small such that
$$\delta C_1 \le \frac{1}{2}. \tag{5.1}$$
□

Now, we prove the main result of this paper.

Theorem 5.3. *Suppose $(\mathbf{E}_0, \mathbf{H}_0, \theta_0) \in D(A) \cap \mathcal{V}_H$ and σ satisfies (1.7). Then the total energy $\mathcal{E}(t)$ of problem (1.1)-(1.6), defined in Lemma 5.1, satisfies*
$$\mathcal{E}(t) \le \beta \mathcal{E}(0) \exp(-\alpha t),$$
where β and α are positive constants.

Proof. From Lemmas 5.1 and 5.2 we have

$$\begin{aligned}\frac{dG(t)}{dt} &= -\lambda\int_\Omega \sigma(x)|\mathbf{E}|^2\,dx - \gamma\int_\Omega |\nabla\theta|^2\,dx \\ &\quad - \delta\int_\Omega \mu|\mathbf{H}|^2\,dx + \delta\int_\Omega \sigma(x)\mathbf{E}\cdot\Psi\,dx + \delta\epsilon\int_\Omega |\Psi_t|^2\,dx\end{aligned}$$

and from (1.7), (4.3), (4.9) and Poincaré Inequality,

$$\begin{aligned}\frac{dG(t)}{dt} &\le -\frac{\sigma_0}{\epsilon}\int_\Omega \lambda\epsilon|\mathbf{E}|^2\,dx - C_0\int_\Omega \gamma|\theta|^2\,dx - \frac{\delta}{\lambda}\int_\Omega \lambda\mu|\mathbf{H}|^2\,dx \qquad (5.2)\\ &\quad + \frac{\delta}{2}\left(\frac{\sigma_1^2}{\kappa}\|\mathbf{E}\|^2 + C^2\kappa\|\mu\mathbf{H}\|^2\right) + \delta\epsilon\int_\Omega |\mathbf{E}|^2\,dx \qquad (5.3)\\ &= -\Big[\frac{\sigma_0}{\epsilon} - \Big(\frac{\sigma_1^2}{2\kappa\epsilon\lambda} + \frac{1}{\lambda}\Big)\delta\Big]\int_\Omega \lambda\epsilon|\mathbf{E}|^2\,dx - C_0\int_\Omega \gamma|\theta|^2\,dx \qquad (5.4)\\ &\quad - \delta\Big(\frac{1}{\lambda} - \frac{1}{2\lambda}C^2\mu\kappa\Big)\int_\Omega \lambda\mu|\mathbf{H}|^2\,dx. \qquad (5.5)\end{aligned}$$

We choose $\kappa > 0$ such that

$$C_2 \equiv \frac{1}{\lambda} - \frac{1}{2\lambda}C^2\mu\kappa > 0$$

and $\delta > 0$ small satisfying (5.1) and

$$C_3 \equiv \frac{\sigma_0}{\epsilon} - \big(\frac{\sigma_1^2}{2\kappa\epsilon\lambda} + \frac{1}{\lambda}\big)\delta > 0.$$

Thus

$$\frac{dG(t)}{dt} \le -C_3\int_\Omega \lambda\epsilon|\mathbf{E}|^2\,dx - \delta C_2\int_\Omega \lambda\mu|\mathbf{H}|^2\,dx - C_0\int_\Omega \gamma|\theta|^2\,dx \le -C_4\mathcal{E}(t), \quad (5.6)$$

where $C_4 = \min\{2C_3, 2\delta C_2, 2C_0\}$. From Lemma 5.2 and the above inequality we obtain

$$\frac{dG(t)}{dt} \le -\frac{C_4}{2}G(t) \quad \text{and} \quad G(t) \le G(0)\exp(-\frac{C_4}{2}t).$$

Finally, we conclude that

$$\mathcal{E}(t) \le 2G(t) \le 4\mathcal{E}(0)\exp(-\frac{C_4}{2}t).$$

□

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