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SECOND-ORDER COMPLEX LINEAR DIFFERENTIAL EQUATIONS WITH SPECIAL FUNCTIONS OR EXTREMAL FUNCTIONS AS COEFFICIENTS

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Abstract. The classical problem of finding conditions on the entire coefficients $A(z)$ and $B(z)$ guaranteeing that all nontrivial solutions of $f''+A(z)f'+B(z)f=0$ are of infinite order is discussed. Two distinct approaches are used. In the first approach the coefficient $A(z)$ itself is a solution of a differential equation $w''+P(z)w=0$, where $P(z)$ is a polynomial. This assumption yields stability on the behavior of $A(z)$ via Hille's classical method on asymptotic integration. In this case $A(z)$ is a special function of which the Airy integral is one example. The second approach involves extremal functions. It is assumed that either $A(z)$ is extremal for Yang's inequality or $B(z)$ is extremal for Denjoy's conjecture. A combination of these two approaches is also discussed.

1. Introduction and main results

It is well known that if $A(z)$ is an entire function, $B(z)\not\equiv 0$ is a transcendental entire function, and f_1, f_2 are two linearly independent solutions of the equation

$$f''+A(z)f'+B(z)f=0, \tag{1.1}$$

then at least one of f_1, f_2 must have infinite order. Hence, "most" solutions of (1.1) have infinite order. On the other hand, there are equations of the form (1.1) that possess a nontrivial solution of finite order; for example, $f(z)=e^z$ satisfies $f''+e^{-z}f'-(e^{-z}+1)f=0$. Thus a natural question is: What conditions on $A(z)$ and $B(z)$ will guarantee that every nontrivial solution of (1.1) has infinite order?

We denote the order and the lower order of an entire function f by $\rho(f)$ and $\mu(f)$, respectively. The standard notation and basic results in Nevanlinna theory of meromorphic functions can be found in [10, 14, 24].

From the work by Gundersen [8], Hellerstein, Miles and Rossi [11], and Ozawa [18], we know that if $A(z)$ and $B(z)$ are entire functions with $\rho(A)<\rho(B)$; or $A(z)$ is a polynomial and $B(z)$ is transcendental; or $\rho(B)<\rho(A)\le\frac{1}{2}$, then every nontrivial solution of (1.1) has infinite order. Therefore, the main problem left to consider is that whether every nontrivial solution of (1.1) has infinite order if

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$\rho(A) = \rho(B)$ or if $\rho(A) > \frac{1}{2}$, $\rho(B) < \rho(A)$. In general, the conclusions are false for these situations. For example, $f(z) = \exp(P(z))$ satisfies the equation

$$f'' + A(z)f' + (-P'' - (P')^2 - A(z)P')f = 0, \tag{1.2}$$

where $A(z)$ is an entire function and $P(z)$ is a nonconstant polynomial. For the case of $\rho(B) < \rho(A)$, there are also some examples [8] showing that a nontrivial solution of (1.1) has finite order.

The problem of finding conditions on $A(z)$ and $B(z)$ under which all nontrivial solutions of (1.1) are of infinite order has raised considerable interest, see, for example, [14]. Recently, this problem was studied by using a new idea that a coefficient of (1.1) is a solution of another equation.

Theorem 1.1 ([21]). *Let $A(z)$ be a nontrivial solution of $w'' + P(z)w = 0$, where $P(z) = a_n z^n + \dots + a_0, a_n \neq 0$. Let $B(z)$ be a transcendental entire function with $\rho(B) < \frac{1}{2}$. Then every nontrivial solution of* (1.1) *is of infinite order.*

From Bank and Laine's result [2, Theorem 1], we known that $\rho(A) = \frac{n+2}{2}$, and hence $\rho(A) > \rho(B)$ in Theorem 1.1. On the other hand, we known that every nontrivial solution of (1.1) is of infinite order when $\rho(A) < \rho(B)$ by Gundersen's result [8, Theorem 2]. The fact that $A(z)$ solves an equation of the form $w'' + P(z)w = 0$ makes $A(z)$ a special function. In the particular cases when $P(z) = -z$ or $P(z) = -z^n$, the solution $A(z)$ is known as the Airy integral or a generalization of the Airy integral [9]. Another special case is the Weber-Hermite function, which is a solution in the case $P(z) = \nu + \frac{1}{2} - \frac{z^2}{4}$, where ν is a constant. In the case when $P(z)$ is an arbitrary polynomial, Hille's classical method on asymptotic integration will become available. The consequences are summarized in Lemma 2.1 below. This background motivated the second and the fourth author to prove the following result.

Theorem 1.2 ([16]). *Let $A(z)$ be given as in Theorem 1.1, and let $B(z)$ be a transcendental entire function with $\mu(B) < \frac{1}{2}$ and $\rho(A) \neq \rho(B)$. Then every nontrivial solution of* (1.1) *is of infinite order.*

Theorem 1.2 is proved by using the $\cos \pi\rho$ theorem due to Barry [3], which does not work for entire functions with lower order (or order) not less than 1/2. Thus we need new ideas when the lower order (or order) of the coefficients is not less than 1/2. In the present paper, we will prove the following improvement of Theorem 1.2 by using a modification of the Phragmén-Lindelöf principle, as well as Hille's classical results on asymptotic integration.

Theorem 1.3. *Let $A(z)$ be given as in Theorem 1.1, and let $B(z)$ be a transcendental entire function with $\mu(B) < \frac{1}{2} + \frac{1}{2(n+1)}$ and $\rho(A) \neq \rho(B)$. Then every nontrivial solution of* (1.1) *is of infinite order.*

In 1988, Gundersen proved the following result.

Theorem 1.4 ([8]). *Let $\{\phi_k\}$ and $\{\theta_k\}$ be two finite collections of real numbers that satisfy $\phi_1 < \theta_1 < \phi_2 < \theta_2 < \dots < \phi_n < \theta_n < \phi_{n+1}$ where $\phi_{n+1} = \phi_1 + 2\pi$, and set $\nu = \max_{1 \le k \le n}\{\phi_{k+1} - \theta_k\}$. Suppose that $A(z)$ and $B(z)$ are entire functions such that for some constant $\alpha > 0$,*

$$|A(z)| = O(|z|^\alpha) \tag{1.3}$$

as $z \to \infty$ in $\phi_k \le \arg z \le \theta_k$ for $k = 1, 2, \dots, n$, and where $B(z)$ is transcendental with $\rho(B) < \pi/\nu$. Then every nontrivial solution of (1.1) *is of infinite order.*

The usual order $\rho(B)$ in Theorem 1.4 can be replaced with the lower order $\mu(B)(\le \rho(B))$.

Theorem 1.5. *Let $\{\phi_k\}$, $\{\theta_k\}$, ν and $A(z)$ be given as in Theorem 1.4, and let $B(z)$ be transcendental with $\mu(B) < \pi/\nu$. Then every nontrivial solution of* (1.1) *is of infinite order.*

The proof of Theorem 1.5 deviates from that of Theorem 1.4 in the sense that we require a modification of the Phragmén-Lindelöf principle, see Lemma 3.2 below. In addition, we make use of the $\cos \pi\rho$ theorem, which is not needed in proving Theorem 1.4.

We proceed to consider conditions on the coefficients $A(z)$ and $B(z)$ involving value distribution instead of just growth. We begin by recalling a conjecture due to Denjoy [4] from 1907, verified by Ahlfors [1] in 1930.

Denjoy's Conjecture. *Let f be an entire function of finite order ρ. If f has k distinct finite asymptotic values, then $k \le 2\rho$.*

An entire function f is called extremal for Denjoy's conjecture if it is of finite order ρ and has $k = 2\rho$ distinct finite asymptotic values. These functions are investigated by Ahlfors [1], Drasin [5], Kennedy [13] and Zhang [27], to mention a few. An example of a function extremal for Denjoy's conjecture is

$$f(z) = \int_0^z \frac{\sin t^q}{t^q} dt, \tag{1.4}$$

where $q > 0$ is an integer. We know that the order of f equals to q, and f has $2q$ distinct finite asymptotic values

$$a_l = e^{\frac{l\pi i}{q}} \int_0^\infty \frac{\sin r^q}{r^q} dr,$$

where $l = 1, 2, \dots, 2q$, see [28, p. 210].

Theorem 1.6. *Let $A(z)$ be given as in Theorem 1.1, and let $B(z)$ be a function extremal for Denjoy's conjecture and $\rho(A) \ne \rho(B)$. Then every nontrivial solution of* (1.1) *is of infinite order.*

We recall the definition of Borel direction as follows [25].

Definition 1.7. Let f be a meromorphic function in the finite complex plane $\mathbb{C}$ with $0 < \mu(f) < \infty$. A ray $\arg z = \theta \in [0, 2\pi)$ from the origin is called a Borel direction of order $\ge \mu(f)$ of f, if for any positive number $\varepsilon > 0$ and for any complex number $a \in \mathbb{C} \bigcup \{\infty\}$, possibly with two exceptions, the following inequality holds

$$\limsup_{r\to\infty} \frac{\log n(S(\theta - \varepsilon, \theta + \varepsilon, r), a, f)}{\log r} \ge \mu(f), \tag{1.5}$$

where $n(S(\theta - \varepsilon, \theta + \varepsilon, r), a, f)$ denotes the number of zeros, counting the multiplicities, of $f - a$ in the region $S(\theta - \varepsilon, \theta + \varepsilon, r) = \{z : \theta - \varepsilon < \arg z < \theta + \varepsilon, |z| < r\}$.

The definition of Borel direction of order $\rho(f)$ of f can be found in [28, p. 78], it is defined similarly with the only exception that "$\ge \mu(f)$" in (1.5) is to be replaced with "$= \rho(f)$".

In the sequel we will require the following result, known as Yang's inequality, on general value distribution theory.

Theorem 1.8 ([25])**.** *Suppose that f is an entire function of finite lower order $\mu > 0$. Let $q < \infty$ denote the number of Borel directions of order $\geq \mu$, and let p denote the number of finite deficient values of f. Then $p \leq q/2$.*

An entire function f is called extremal for Yang's inequality if f satisfies the assumptions of Theorem 1.8 with $p = \frac{q}{2}$. These functions were introduced in [23]. The simplest entire function extremal for Yang's inequality is e^z. A slightly more complicated example is $f(z) = \int_0^z e^{-t^n} dt$, $n \geq 2$ is teger, which has n deficient values

$$a_l = e^{i\frac{2\pi l}{n}} \int_0^\infty e^{-t^n} dt, \quad l = 1, 2, \dots, n,$$

and $q = 2n$ Borel directions $\arg z = \frac{2k-1}{2n}\pi$, $k = 1, 2, \dots, 2n$, see [24, pp. 210-211] for more details.

Theorem 1.9 ([15])**.** *Let $A(z)$ be an entire function extremal for Yang's inequality, and let $B(z)$ be a transcendental entire function such that $\rho(B) \neq \rho(A)$. Then every nontrivial solution of* (1.1) *is of infinite order.*

Also here the usual order $\rho(B)$ can be replaced with the lower order $\mu(B)$.

Theorem 1.10. *Let $A(z)$ be an entire function extremal for Yang's inequality, and let $B(z)$ be a transcendental entire function such that $\mu(B) \neq \rho(A)$. Then every nontrivial solution of* (1.1) *is of infinite order.*

Let $\lambda(A)$ be the converge exponent of the zero sequence of $A(z)$. By Lemma 2.1 below and by similar reasoning used in proving Theorems 1.9 and 1.10, we can easily obtain the following result.

Theorem 1.11. *Let $A(z)$ be given as in Theorem 1.1 with $\lambda(A) < \rho(A)$, and let $B(z)$ be a transcendental entire function satisfying one of the following conditions.*

(1) $\rho(B) \neq \rho(A)$,
(2) $\mu(B) \neq \rho(A)$.

Then every nontrivial solution of (1.1) *is of infinite order.*

2. Auxiliary results

Let $\alpha < \beta$ be such that $\beta - \alpha < 2\pi$, and let $r > 0$. Denote

$$S(\alpha, \beta) = \{z : \alpha < \arg z < \beta\},$$
$$S(\alpha, \beta, r) = \{z : \alpha < \arg z < \beta\} \cap \{z : |z| < r\}.$$

Let $\overline{F}$ denote the closure of F. Let f be an entire function of order $\rho(f) \in (0, \infty)$. For simplicity, set $\rho = \rho(f)$ and $S = S(\alpha, \beta)$. We say that f blows up exponentially in $\overline{S}$ if for any $\theta \in (\alpha, \beta)$

$$\lim_{r\to\infty} \frac{\log\log|f(re^{i\theta})|}{\log r} = \rho \tag{2.1}$$

holds. We also say that f decays to zero exponentially in $\overline{S}$ if for any $\theta \in (\alpha, \beta)$

$$\lim_{r\to\infty} \frac{\log\log|f(re^{i\theta})|^{-1}}{\log r} = \rho \tag{2.2}$$

holds.

The following lemma, originally due to Hille [12, Chapter 7.4], see also [6, 19], plays an important role in proving our results. The method used in proving the lemma is typically referred to as the method of asymptotic integration.

Lemma 2.1. *Let f be a nontrivial solution of $f'' + P(z)f = 0$, where $P(z) = a_n z^n + \dots + a_0$, $a_n \neq 0$. Set $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}$ and $S_j = S(\theta_j, \theta_{j+1})$, where $j = 0, 1, 2, \dots, n+1$ and $\theta_{n+2} = \theta_0 + 2\pi$. Then f has the following properties.*

1. *In each sector S_j, f either blows up or decays to zero exponentially.*
2. *If, for some j, f decays to zero in S_j, then it must blow up in S_{j-1} and S_{j+1}. However, it is possible for f to blow up in many adjacent sectors.*
3. *If f decays to zero in S_j, then f has at most finitely many zeros in any closed sub-sector within $S_{j-1} \cup \overline{S_j} \cup S_{j+1}$.*
4. *If f blows up in S_{j-1} and S_j, then for each $\varepsilon > 0$, f has infinitely many zeros in each sector $\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon)$, and furthermore, as $r \to \infty$,*
$$n(\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r), 0, f) = (1 + o(1)) \frac{2\sqrt{|a_n|}}{\pi(n+2)} r^{\frac{n+2}{2}},$$
where $n(\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r), 0, f)$ is the number of zeros of f in the region $\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r)$.

The Lebesgue linear measure of a set $E \subset [0, \infty)$ is $\mathrm{m}(E) = \int_E dt$, and the logarithmic measure of a set $F \subset [1, \infty)$ is $\mathrm{m_l}(F) = \int_F \frac{dt}{t}$. The upper and lower logarithmic densities of $F \subset [1, \infty)$ are given, respectively, by
$$\overline{\log \mathrm{dens}}(F) = \limsup_{r \to \infty} \frac{\mathrm{m_l}(F \cap [1, r])}{\log r},$$
$$\underline{\log \mathrm{dens}}(F) = \liminf_{r \to \infty} \frac{\mathrm{m_l}(F \cap [1, r])}{\log r}.$$

A lemma on logarithmic derivatives due to Gundersen [7] plays an important role in proving our results.

Lemma 2.2. *Let f be a transcendental meromorphic function of finite order $\rho(f)$. Let $\varepsilon > 0$ be a given real constant, and let k and j be integers such that $k > j \geq 0$. Then there exists a set $E \subset [0, 2\pi)$ of linear measure zero, such that if $\psi_0 \in [0, 2\pi) - E$, then there is a constant $R_0 = R_0(\psi_0) > 0$ such that for all z satisfying $\arg z = \psi_0$ and $|z| \geq R_0$, we have*
$$\Big| \frac{f^{(k)}(z)}{f^{(j)}(z)} \Big| \leq |z|^{(k-j)(\rho(f) - 1 + \varepsilon)}. \tag{2.3}$$

The following result is due to Barry [3].

Lemma 2.3. *Let f be an entire function with $0 \leq \mu(f) < 1$. Then, for every $\alpha \in (\mu(f), 1)$, $\overline{\mathrm{logdens}}(\{r \in [1, \infty) : m(r) > M(r) \cos \pi\alpha\}) \geq 1 - \frac{\mu(f)}{\alpha}$, where $m(r) = \inf_{|z|=r} \log |f(z)|$, and $M(r) = \sup_{|z|=r} \log |f(z)|$.*

The following result was proved in [8].

Lemma 2.4. *Let $A(z)$ and $B(z) \not\equiv 0$ be two entire functions such that for real constants $\alpha, \beta, \theta_1, \theta_2$, where $\alpha > 0$, $\beta > 0$ and $\theta_1 < \theta_2$, we have*
$$|A(z)| \geq \exp\{(1 + o(1)) \alpha |z|^{\beta}\}, \tag{2.4}$$

$$|B(z)| \leq \exp\{o(1)|z|^{\beta}\} \tag{2.5}$$

as $z \to \infty$ *in* $\overline{S}(\theta_1, \theta_2) = \{z : \theta_1 \leq \arg z \leq \theta_2\}$. *Let* $\varepsilon > 0$ *be a given small constant, and let* $\overline{S}(\theta_1 + \varepsilon, \theta_2 - \varepsilon) = \{z : \theta_1 + \varepsilon \leq \arg z \leq \theta_2 - \varepsilon\}$.

If f *is a nontrivial solution of* (1.1) *with* $\rho(f) < \infty$, *then the following conclusions hold:*

(1) *There exists a constant* $b(\neq 0)$ *such that* $f(z) \to b$ *as* $z \to \infty$ *in* $\overline{S}(\theta_1 + \varepsilon, \theta_2 - \varepsilon)$. *Furthermore,*
$$|f(z) - b| \leq \exp\{-(1 + o(1))\alpha|z|^{\beta}\} \tag{2.6}$$
as $z \to \infty$ *in* $\overline{S}(\theta_1 + \varepsilon, \theta_2 - \varepsilon)$.

(2) *For each integer* $k > 1$,
$$|f^{(k)}(z)| \leq \exp\{-(1 + o(1))\alpha|z|^{\beta}\}$$
as $z \to \infty$ *in* $\overline{S}(\theta_1 + \varepsilon, \theta_2 - \varepsilon)$.

3. Modified Phragmén-Lindelöf principle

We recall a result due to Phragmén and Lindelöf [17, Theorem 7.5].

Lemma 3.1. *Let* f *be an analytic function in* D *and continuous in* $\overline{D}$, *where* $D = S(\alpha, \beta) \cap \{z : |z| > r_0\}$, *and* α, β, r_0 *are constants such that* $0 < \beta - \alpha \leq 2\pi$ *and* $r_0 > 0$. *Suppose that there exists a constant* $M > 0$ *such that* $|f(z)| \leq M$ *for* $z \in \partial D$. *If*
$$\liminf_{r \to \infty} \frac{\log \log M(r, D, f)}{\log r} < \frac{\pi}{\beta - \alpha},$$
where $M(r, D, f) = \max_{\substack{|z|=r \\ z \in D}} |f(z)|$, *then* $|f(z)| \leq M$ *for all* $z \in D$.

Next we introduce a key lemma in which the Phragmén-Lindelöf principle is tailored to suit for our purposes.

Lemma 3.2. *Let* f *be an entire function of lower order* $\mu(f) \in [\frac{1}{2}, \infty)$. *Then there exists a sector* $S(\alpha, \beta) = \{z : \alpha < \arg z < \beta\}$ *with* $\beta - \alpha \geq \frac{\pi}{\mu(f)}$, *such that*
$$\limsup_{r \to \infty} \frac{\log \log |f(re^{i\theta})|}{\log r} \geq \mu(f)$$
holds for all the rays $\arg z = \theta \in (\alpha, \beta)$, *where* $0 \leq \alpha < \beta \leq 2\pi$.

Proof. Suppose on the contrary to the assertion that any sector $S(\alpha, \beta)$ with $\beta - \alpha \geq \frac{\pi}{\mu(f)}$ there exists at least one ray $\arg z = \psi_1 \in (\alpha, \beta)$ such that
$$\limsup_{r \to \infty} \frac{\log \log |f(re^{i\psi_1})|}{\log r} = \mu_1 < \mu(f),$$
where μ_1 is a constant.

Let $\psi_1' = \psi_1 + \frac{\pi}{\mu(f)}$. From our assumption, there exists at least one ray $\arg z = \psi_2 \in (\psi_1, \psi_1')$, such that
$$\limsup_{r \to \infty} \frac{\log \log |f(re^{i\psi_2})|}{\log r} = \mu_2 < \mu(f), \tag{3.1}$$
where μ_2 is a constant. Let $\eta_1 = \max\{\mu_1, \mu_2\}$, and let $\lambda_1 \in (\eta_1, \mu(f)) \cap \mathbb{Q}$ be a constant, where $\mathbb{Q}$ denotes the set of rational numbers. Suppose that $\omega_0 = z^{\lambda_1}$ is the principal branch of $\omega = z^{\lambda_1}$. Then $S(\psi_1, \psi_2)$ is mapped onto a subsector of

the right half-plane by transformation $\zeta = e^{i\theta_1} z^{\lambda_1}$, where $\theta_1 \in (0, 2\pi)$ is a constant depending on λ_1, ψ_1 and ψ_2.

Let $H(z) = \frac{f(z)}{\exp(e^{i\theta_1} z^{\lambda_1})}$. Then we obtain

$$\lim_{r\to\infty} |H(re^{i\psi_1})| = \lim_{r\to\infty} |H(re^{i\psi_2})| = 0 \tag{3.2}$$

and

$$\begin{aligned} \liminf_{r\to\infty} \frac{\log\log M(r, S(\psi_1,\psi_2), H)}{\log r} &\le \liminf_{r\to\infty} \frac{\log\log M(r, S(\psi_1,\psi_2), f)}{\log r} \\ &\le \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r} \\ &= \mu(f) < \frac{\pi}{\psi_2 - \psi_1}. \end{aligned} \tag{3.3}$$

By Lemma 3.1, there exists a constant $M_1 > 0$ such that

$$|H(z)| \le M_1$$

holds for all $z \in S(\psi_1, \psi_2)$; that is,

$$|f(z)| \le M_1 \exp(|z|^{\lambda_1}) \tag{3.4}$$

holds for all $z \in S(\psi_1, \psi_2)$.

Let $\psi_2' = \psi_2 + \frac{\pi}{\mu(f)}$. From our assumption, there exists at least one ray $\arg z = \psi_3 \in (\psi_2, \psi_2')$ such that

$$\limsup_{r\to\infty} \frac{\log\log |f(re^{i\psi_3})|}{\log r} = \mu_3 < \mu(f),$$

where μ_3 is a constant. Similarly as above, there exist constants $M_2 > 0, \lambda_2 < \mu(f)$ and $\theta_2 \in (0, 2\pi)$ such that

$$|f(z)| \le M_2 \exp(|z|^{\lambda_2}) \tag{3.5}$$

holds for all $z \in S(\psi_2, \psi_3)$. We proceed in this way until there exists a ray $\arg z = \psi_m$ such that

$$\limsup_{r\to\infty} \frac{\log\log |f(re^{i\psi_m})|}{\log r} = \mu_m < \mu(f) \tag{3.6}$$

and $\psi_1 + 2\pi - \psi_m < \frac{\pi}{\mu(f)}$, where μ_m is a constant. By the discussion above, there exist constants $M_{m-1} > 0, \lambda_{m-1} < \mu(f)$ and $\theta_{m-1} \in (0, 2\pi)$ such that

$$|f(z)| \le M_{m-1} \exp(|z|^{\lambda_{m-1}}) \tag{3.7}$$

holds for all $z \in S(\psi_{m-1}, \psi_m)$. By (3.4), (3.5), (3.7) and Lemma 3.1, we have

$$|f(z)| \le M \exp(|z|^{\lambda}) \quad \text{for all } z \in \mathbb{C}, \tag{3.8}$$

where $M = \max\{M_1, M_2, \dots, M_{m-1}\}$ and $\lambda = \max\{\lambda_1, \lambda_2, \dots, \lambda_{m-1}\} < \mu(f)$. By (3.8), we obtain

$$\mu(f) = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r} \le \lim_{r\to\infty} \frac{\log\log M + \lambda \log r}{\log r} = \lambda,$$

which is a contradiction with the fact $\lambda < \mu(f)$. This completes the proof. $\square$

4. Proofs of Theorems 1.3 and 1.5

We rely heavily on Phragmén-Lindelöf principle and modified Phragmén-Lindelöf principle.

Proof of Theorem 1.3. Since the case $\rho(A) < \rho(B)$ is proved in [8], we may assume $\rho(A) > \rho(B)$. Suppose on the contrary to the assertion that there exists a nontrivial solution f of (1.1) with $\rho(f) < \infty$. We aim for a contradiction. Set $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}$ and $S_j = \{z : \theta_j < \arg z < \theta_{j+1}\}$, where $j = 0, 1, 2, \dots, n+1$ and $\theta_{n+2} = \theta_0 + 2\pi$. We consider two cases appearing in Lemma 2.1.

Case 1: Suppose that $A(z)$ blows up exponentially in each sector S_j, where $j = 0, 1, \dots, n+1$; that is, for any $\theta \in (\theta_j, \theta_{j+1})$, we have

$$\lim_{r\to\infty} \frac{\log\log|A(re^{i\theta})|}{\log r} = \rho(A) = \frac{n+2}{2}. \tag{4.1}$$

Then for any given constant $\varepsilon \in (0, \frac{\pi}{4\rho(A)})$ and $\eta \in (0, \frac{\rho(A)-\rho(B)}{4})$, we have

$$|A(z)| \geq \exp\{(1+o(1))\alpha|z|^{\frac{n+2}{2}-\eta}\}, \tag{4.2}$$

$$|B(z)| \leq \exp(|z|^{\rho(B)+\eta}) \leq \exp(|z|^{\rho(A)-2\eta}) \leq \exp\{o(1)|z|^{\frac{n+2}{2}-\eta}\} \tag{4.3}$$

as $z \to \infty$ in $S_j(\varepsilon) = \{z : \theta_j + \varepsilon < \arg z < \theta_{j+1} - \varepsilon\}$, $j = 0, 1, \dots, n+1$, where α is a positive constant depending on ε. Combining (4.2), (4.3), and Lemma 2.4, there exist corresponding constants $b_j \neq 0$ such that

$$|f(z) - b_j| \leq \exp\{-(1+o(1))\alpha|z|^{\frac{n+2}{2}-\eta}\} \tag{4.4}$$

as $z \to \infty$ in $S_j(2\varepsilon)$, $j = 0, 1, \dots, n+1$. Therefore, f is bounded in the whole complex plane by the Phragmén-Lindelöf principle. So f is a nonzero constant in the whole complex plane by Liouville's theorem. This contradicts with the fact that equation (1.1) doesn't have nonzero constant solutions.

Case 2: There exists at least one sector of the $n+2$ sectors, such that $A(z)$ decays to zero exponentially, say $S_{j_0} = \{z : \theta_{j_0} < \arg z < \theta_{j_0+1}\}$, $0 \leq j_0 \leq n+1$. That is, for any $\theta \in (\theta_{j_0}, \theta_{j_0+1})$, we have

$$\lim_{r\to\infty} \frac{\log\log\frac{1}{|A(re^{i\theta})|}}{\log r} = \frac{n+2}{2}. \tag{4.5}$$

If $\mu(B) < \frac{1}{2}$, the assertion follows by Theorem 1.2.

If $\frac{1}{2} \leq \mu(B) < \frac{1}{2} + \frac{1}{2(n+1)}$, then by Lemma 3.2, there exists a sector $S(\alpha,\beta) = \{z : \alpha < \arg z < \beta\}$ with $\beta - \alpha \geq \frac{\pi}{\mu(B)} > \frac{\pi}{\frac{1}{2}+\frac{1}{2(n+1)}} = 2\pi - \frac{2\pi}{n+2}$, such that

$$\limsup_{r\to\infty} \frac{\log\log|B(re^{i\theta})|}{\log r} \geq \mu(B) \tag{4.6}$$

holds for any $\theta \in (\alpha, \beta)$. Thus, there exists a subsector $S(\alpha', \beta')$, such that for any $\theta \in (\alpha', \beta')$ we have (4.5) and (4.6).

By Lemma 2.2, there exists a set $E_1 \subset [0, 2\pi)$ of linear measure zero, such that if $\psi_0 \in [0, 2\pi) - E_1$, then there is a constant $R_0 = R_0(\psi_0) > 1$ such that for all z satisfying $\arg z = \psi_0$ and $|z| \geq R_0$, we have

$$\Big|\frac{f^{(k)}(z)}{f(z)}\Big| \leq |z|^{2\rho(f)}, \quad k = 1, 2. \tag{4.7}$$

Thus, there exists a sequence of points $z_n = r_n e^{i\theta}$ with $r_n \to \infty$ as $n \to \infty$ and $\theta \in (\alpha', \beta') - E_1$, such that (4.5), (4.6) and (4.7) hold.

Combining (4.5), (4.6), (4.7) and (1.1), for every $n > n_0$, we have

$$\begin{aligned} \exp(r_n^{\mu(B)-\varepsilon}) &\le |B(r_n e^{i\theta})| \\ &\le \Big|\frac{f''(r_n e^{i\theta})}{f(r_n e^{i\theta})}\Big| + |A(r_n e^{i\theta})|\Big|\frac{f'(r_n e^{i\theta})}{f(r_n e^{i\theta})}\Big| \\ &\le r_n^{2\rho(f)}(1+o(1)). \end{aligned} \tag{4.8}$$

Obviously, this is a contradiction for sufficiently large n and arbitrary small ε. Therefore we have $\rho(f) = \infty$ for every nontrivial solution f of (1.1). This completes the proof. □

Proof of Theorem 1.5. Assume on the contrary to the assertion that there is a nontrivial solution f of (1.1) with $\rho(f) = \rho < \infty$.

Case 1: Suppose first that $\mu(B) > 0$. By Lemma 2.2, there exists a set $E_2 \subset [0, 2\pi)$ of linear measure zero, such that if $\varphi \in [\phi_k, \theta_k] - E_2$ for some k, $1 \le k \le n$, we have

$$\Big|\frac{f'(re^{i\varphi})}{f(re^{i\varphi})}\Big| = O(r^{\rho}), \quad \Big|\frac{f''(re^{i\varphi})}{f(re^{i\varphi})}\Big| = O(r^{2\rho}), \tag{4.9}$$

as $r \to \infty$ along $\arg z = \varphi$. Combining (4.9), (1.1), and our assumption, we have

$$|B(re^{i\varphi})| \le \Big|\frac{f''(re^{i\varphi})}{f(re^{i\varphi})}\Big| + |A(re^{i\varphi})|\Big|\frac{f'(re^{i\varphi})}{f(re^{i\varphi})}\Big| = O(r^{\sigma}) \tag{4.10}$$

in each $[\phi_k, \theta_k] - E_2$, $1 \le k \le n$, as $r \to \infty$, where $\sigma = \alpha + 2\rho$.

For any given $\varepsilon \in (0, \min\{\frac{\pi}{2\mu(B)} - \frac{\nu}{2}, \frac{\mu(B)}{4}\})$, for any $\varphi' \in (\theta_k - \varepsilon, \theta_k) - E_2$, $\varphi'' \in (\phi_{k+1}, \phi_{k+1} + \varepsilon) - E_2$, $k = 1, 2, \dots, n$, we obtain

$$|B(re^{i\varphi'})| = O(r^{\sigma}), \ |B(re^{i\varphi''})| = O(r^{\sigma}),$$

as $r \to \infty$.

Similarly as in the proof of Lemma 3.2, let $H(z) = \frac{B(z)}{\exp(az^{\lambda})}$, where $\lambda \in (0, \mu(B) - 4\varepsilon) \cap \mathbb{Q}$, $\mathbb{Q}$ denotes the set of rational numbers, $a = e^{i\tau}$, $\tau \in (0, 2\pi)$ is a constant depending on λ, φ' and φ'', and z^{λ} denotes the principal branch. Note that

$$\varphi'' - \varphi' < \phi_{k+1} - \theta_k + 2\varepsilon < \nu + 2\varepsilon < \frac{\pi}{\mu(B)} < \frac{\pi}{\lambda}.$$

So, $S(\varphi', \varphi'')$ is mapped onto a subsector of the right half-plane by $\zeta = az^{\lambda}$. Thus, we obtain

$$\lim_{r\to\infty} |H(re^{i\varphi'})| = \lim_{r\to\infty} |H(re^{i\varphi''})| = 0$$

and

$$\begin{aligned} \liminf_{r\to\infty} \frac{\log\log M(r, S(\varphi', \varphi''), H)}{\log r} &\le \liminf_{r\to\infty} \frac{\log\log M(r, S(\varphi', \varphi''), B)}{\log r} \\ &\le \liminf_{r\to\infty} \frac{\log\log M(r, B)}{\log r} \\ &= \mu(B) < \frac{\pi}{\varphi'' - \varphi'}. \end{aligned}$$

By Lemma 3.1, there exists a constant $M > 0$ such that

$$|H(z)| \le M$$

for all $z \in S(\varphi', \varphi'')$; that is,

$$|B(z)| \leq M \exp(|z|^\lambda) \tag{4.11}$$

for all $z \in S(\varphi', \varphi'')$. So we have (4.11) in each $(\theta_k - \varepsilon, \phi_{k+1} + \varepsilon) - E_2$, $1 \leq k \leq n$. By (4.10), (4.11) and Phragmén-Lindelöf principle, we obtain

$$|B(z)| \leq M \exp(|z|^\lambda)$$

for all $z \in \mathbb{C}$. Thus, we have

$$\mu(B) = \liminf_{r\to\infty} \frac{\log\log M(r,B)}{\log r} \leq \lim_{r\to\infty} \frac{\log\log M + \lambda \log r}{\log r} = \lambda,$$

which contradicts with the fact that $\lambda < \mu(B)$.

Case 2: Suppose that $\mu(B) = 0$. By using Lemma 2.3, there exists a set $E_3 \subset [1,\infty)$ with $\overline{\log \operatorname{dens}}(E_3) = 1$ such that for all z satisfying $|z| = r \in E_3$, we have

$$\log|B(z)| > \frac{\sqrt{2}}{2} \log M(r,B), \tag{4.12}$$

where $M(r,B) = \max_{|z|=r} |B(z)|$.

It follows from (1.1), (1.3), (4.9) and (4.12) that there exists a sequence (R_n) with $R_n \to \infty$ as $n \to \infty$, such that

$$M(R_n, B)^{\sqrt{2}/2} < |B(R_n e^{i\varphi})| \leq R_n^{2\rho(f)}(1 + R_n^\alpha), \tag{4.13}$$

as $n \to \infty$, $\varphi \in \cup_{k=1}^n [\phi_k, \theta_k] - E_2$. However, $B(z)$ is a transcendental entire function, so that

$$\liminf_{r\to\infty} \frac{\log M(r,B)}{\log r} = \infty,$$

which contradicts with (4.13). This completes the proof. □

5. Proof of Theorem 1.6

We begin by recalling some properties satisfied by entire functions that are extremal for Denjoy's conjecture.

Lemma 5.1 ([28, Theorem 4.11])**.** *Let f be an entire function extremal for Denjoy's conjecture. Then, for any $\theta \in (0, 2\pi)$, either $\Delta(\theta)$ is a Borel direction of order $\rho(f)$ of f or there exists a constant $\sigma (0 < \sigma < \frac{\pi}{4})$, such that*

$$\lim_{\substack{|z|\to\infty \\ z \in (S(\theta-\sigma, \theta+\sigma) - E)}} \frac{\log\log|f(z)|}{\log r} = \rho(f),$$

where $\Delta(\theta)$ is a half-line from the origin, E denotes a subset of $S(\theta - \sigma, \theta + \sigma)$, and satisfies

$$\lim_{r\to\infty} \mathrm{m}(S(\theta-\sigma, \theta+\sigma; r, \infty) \cap E) = 0,$$

where $S(\theta - \sigma, \theta + \sigma; r, \infty) = \{z : \theta - \sigma < \arg z < \theta + \sigma, r < |z| < \infty\}$.

Lemma 5.2. *Let f be an entire function of order $\rho \in (0, \infty)$, and let $S(\phi_1, \phi_2) = \{z : \phi_1 < \arg z < \phi_2\}$ be a sector with $\phi_2 - \phi_1 < \frac{\pi}{\rho}$. If there exists a Borel direction of order ρ of f in $S(\phi_1, \phi_2)$, then for at least one of the two rays $L_j : \arg z = \phi_j (j = 1, 2)$, say L_2, we have*

$$\limsup_{r\to\infty} \frac{\log\log|f(re^{i\phi_2})|}{\log r} = \rho.$$

Lemma 5.2 is [26, Lemma 1], which can be proved by using a result in [20, pp. 119-120].

We may assume $\rho(A) > \rho(B)$ due to Gundersen's result [8, Theorem 2]. If $A(z)$ blows up exponentially in each sector S_j, $j = 0, 1, \dots, n+1$, then the assertion follows by the proof of Theorem 1.3. Suppose there exists at least one sector of the $n+2$ sectors, such that $A(z)$ decays to zero exponentially, say $S_{j_0} = \{z : \theta_{j_0} < \arg z < \theta_{j_0+1}\}$, $0 \le j_0 \le n+1$. That is, for any $\theta \in (\theta_{j_0}, \theta_{j_0+1})$, we have

$$\lim_{r\to\infty} \frac{\log\log \frac{1}{|A(re^{i\theta})|}}{\log r} = \frac{n+2}{2}.$$

Suppose on the contrary to the assertion that there is a nontrivial solution f of (1.1) with $\rho(f) < \infty$. By Lemma 2.2, there exists a set $E_1 \subset [0, 2\pi)$ of linear measure zero, such that if $\psi_0 \in [0, 2\pi) - E_1$, then there is a constant $R_0 = R_0(\psi_0) > 1$ such that for all z satisfying $\arg z = \psi_0$ and $|z| \ge R_0$, we have (4.7). Next we consider the two cases appearing in Lemma 5.1.

Case 1: Suppose that the ray $\arg z = \theta$ is a Borel direction of order $\rho(B)$ of $B(z)$, where $\theta_{j_0} < \theta < \theta_{j_0+1}$. Choose $\phi_1 \in (\theta_{j_0}, \theta) - E_1$ and $\phi_2 \in (\theta, \theta_{j_0+1}) - E_1$. Then $\phi_2 - \phi_1 < \frac{\pi}{\rho(A)} < \frac{\pi}{\rho(B)}$. By Lemma 5.2, at least one of two rays $L_1 : \arg z = \phi_1$ and $L_2 : \arg z = \phi_2$, say L_1, satisfies

$$\limsup_{r\to\infty} \frac{\log\log |B(re^{i\phi_1})|}{\log r} = \rho(B).$$

Thus, there exists a sequence of points $z_n = r_n e^{i\phi_1}$ with $r_n \to \infty$ as $n \to \infty$, such that

$$\lim_{n\to\infty} \frac{\log\log |B(r_n e^{i\phi_1})|}{\log r_n} = \rho(B), \tag{5.1}$$

$$\lim_{n\to\infty} \frac{\log\log \frac{1}{|A(r_n e^{i\phi_1})|}}{\log r_n} = \frac{n+2}{2}, \tag{5.2}$$

$$\Big| \frac{f^{(k)}(r_n e^{i\phi_1})}{f(r_n e^{i\phi_1})} \Big| \le r_n^{2\rho(f)}, \quad k = 1, 2. \tag{5.3}$$

Combining (5.1)-(5.3) and (1.1), we arrive at a contradiction as in the proof of Theorem 1.3. Thus, we have that every nontrivial solution f of (1.1) satisfies $\rho(f) = \infty$.

Case 2: Suppose that the ray $\arg z = \theta$ is not a Borel direction of order $\rho(B)$ of $B(z)$, where $\theta_{j_0} < \theta < \theta_{j_0+1}$. By Lemma 5.1, there exists a constant $\sigma \in \big(0, \min\{\frac{\theta - \theta_{j_0}}{2}, \frac{\theta_{j_0+1} - \theta}{2}, \frac{\pi}{4}\}\big)$, such that

$$\lim_{\substack{|z|\to\infty \\ z\in(S(\theta-\sigma,\theta+\sigma)-E_2)}} \frac{\log\log |B(z)|}{\log r} = \rho(B), \tag{5.4}$$

where E_2 denotes a subset of $S(\theta - \sigma, \theta + \sigma)$, and satisfies

$$\lim_{r\to\infty} \mathrm{m}(S(\theta - \sigma, \theta + \sigma; r, \infty) \cap E_2) = 0.$$

Let $\Delta = \{z : \arg z = \psi, \psi \in E_1\}$. We can easily see that there exists a sequence of points z_n with $z_n \to \infty$ as $n \to \infty$, $\{z_n\} \subset (S(\theta - \sigma, \theta + \sigma) - E_2) \cap (S_{j_0} - \Delta)$, such that

$$\lim_{n\to\infty}\frac{\log\log|B(z_n)|}{\log|z_n|}=\rho(B), \tag{5.5}$$

$$\lim_{n\to\infty}\frac{\log\log\frac{1}{|A(z_n)|}}{\log|z_n|}=\frac{n+2}{2}, \tag{5.6}$$

$$\Big|\frac{f^{(k)}(z_n)}{f(z_n)}\Big|\le|z_n|^{2\rho(f)},\quad k=1,2. \tag{5.7}$$

Combining (5.5)-(5.7) and (1.1), we arrive at a contradiction as in the proof of Theorem 1.3. Thus, we have $\rho(f)=\infty$ for every nontrivial solution f of (1.1). This completes the proof.

6. Proof of Theorem 1.10

We begin by recalling some basic properties satisfied by entire functions that are extremal for Yang's inequality. To this end, if A is a function extremal for Yang's inequality, then the rays $\arg z=\theta_k$, denote the q distinct Borel directions of order $\ge\mu(A)$ of A, where $k=1,2,\dots,q$ and $0\le\theta_1<\theta_2<\dots<\theta_q<\theta_{q+1}=\theta_1+2\pi$.

Lemma 6.1 ([23]). *Suppose that A is a function extremal for Yang's inequality. Then $\mu(A)=\rho(A)$. Moreover, for every deficient value a_i, $i=1,2,\dots,p$, there exists a corresponding sector domain $S(\theta_{k_i},\theta_{k_i+1})=\{z:\theta_{k_i}<\arg z<\theta_{k_i+1}\}$ such that for every $\varepsilon>0$ the inequality*

$$\log\frac{1}{|A(z)-a_i|}>C(\theta_{k_i},\theta_{k_i+1},\varepsilon,\delta(a_i,A))T(|z|,A) \tag{6.1}$$

holds for $z\in S(\theta_{k_i}+\varepsilon,\theta_{k_i+1}-\varepsilon;r,+\infty)=\{z:\theta_{k_i}+\varepsilon<\arg z<\theta_{k_i+1}-\varepsilon,r<|z|<\infty\}$, where $C(\theta_{k_i},\theta_{k_i+1},\varepsilon,\delta(a_i,A))$ is a positive constant depending only on $\theta_{k_i},\theta_{k_i+1},\varepsilon$ and $\delta(a_i,A)$.

In the sequel, we shall say that A decays to a_i exponentially in $S(\theta_{k_i},\theta_{k_i+1})$, if (6.1) holds in $S(\theta_{k_i},\theta_{k_i+1})$. Note that if A is a function extremal for Yang's inequality, then $\mu(A)=\rho(A)$. Thus, for these functions, we need only to consider the Borel directions of order $\rho(A)$.

Lemma 6.2 ([15]). *Let A be an entire function extremal for Yang's inequality. Suppose that there exists $\arg z=\theta$ with $\theta_j<\theta<\theta_{j+1}$, $1\le j\le q$, such that*

$$\limsup_{r\to\infty}\frac{\log\log|A(re^{i\theta})|}{\log r}=\rho(A). \tag{6.2}$$

Then $\theta_{j+1}-\theta_j=\frac{\pi}{\rho(A)}$.

We state one more auxiliary result that covers one particular case of the proof of Theorem 1.10.

Lemma 6.3 ([22]). *Let $A(z)$ be a finite order entire function having a finite deficient value, and let $B(z)$ be a transcendental entire function with $\mu(B)<\frac{1}{2}$. Then every nontrivial solution of (1.1) is of infinite order.*

By [8, Theorem 2] and Lemma 6.3, we just need prove the case $1/2\le\mu(B)<\rho(A)$. Suppose on the contrary to the assertion that there is a nontrivial solution f of (1.1) with $\rho(f)=\rho<\infty$. We aim for a contradiction.

Suppose that a_i, $i = 1, 2, \dots, p$, are all the finite deficient values of $A(z)$. Thus we have $2p$ sectors $S_j = \{z | \theta_j < \arg z < \theta_{j+1}\}$, $j = 1, 2, \dots, 2p$, such that $A(z)$ has the following properties. In each sector S_j, either there exists some a_i such that

$$\log \frac{1}{|A(z) - a_i|} > C(\theta_j, \theta_{j+1}, \varepsilon, \delta(a_i, A)) T(|z|, A) \tag{6.3}$$

holds for $z \in S(\theta_j + \varepsilon, \theta_{j+1} - \varepsilon; r, +\infty)$, where $C(\theta_j, \theta_{j+1}, \varepsilon, \delta(a_i, A))$ is a positive constant depending only on $\theta_j, \theta_{j+1}, \varepsilon$ and $\delta(a_i, A)$, or there exists $\arg z = \theta \in (\theta_j, \theta_{j+1})$ such that

$$\limsup_{r \to \infty} \frac{\log \log |A(re^{i\theta})|}{\log r} = \rho(A)\,. \tag{6.4}$$

For the sake of simplicity, in the sequel we use C to represent $C(\theta_j, \theta_{j+1}, \varepsilon, \delta(a_i, A))$. Note that if there exists some a_i such that (6.3) holds in S_j, then there exists $\arg z = \theta$ such that (6.4) holds in S_{j-1} and S_{j+1}. If there exists $\theta \in (\theta_j, \theta_{j+1})$ such that (6.4) holds, then there are a_i ($a_{i'}$) such that (6.3) holds in S_{j-1} and S_{j+1}, respectively.

Without loss of generality, we assume that there is a ray $\arg z = \theta$ in S_1 such that (6.4) holds. Therefore, there exists a ray in each sector $S_3, S_5, \dots, S_{2p-1}$, such that (6.4) holds. By using Lemma 6.2, we know that all the sectors have the same magnitude $\frac{\pi}{\rho(A)}$.

Note that $B(z)$ is an entire function of lower order $1/2 \le \mu(B) < \rho(A)$. By Lemma 3.2 we see that there exists a sector $S(\alpha, \beta)$ with $\beta - \alpha \ge \frac{\pi}{\mu(B)}$, $0 \le \alpha < \beta \le 2\pi$, such that for all the rays $\arg z = \theta \in (\alpha, \beta)$ we have

$$\limsup_{r \to \infty} \frac{\log \log |B(re^{i\theta})|}{\log r} \ge \mu(B). \tag{6.5}$$

Note that $\rho(A) > \mu(B)$. It is not hard to see that there exists a sector $S(\alpha', \beta')$, where $\alpha < \alpha' < \beta' < \beta$, such that there is an a_{j_0} such that

$$\log \frac{1}{|A(re^{i\theta}) - a_{j_0}|} > CT(r, A) \tag{6.6}$$

holds for all $\theta \in [\alpha', \beta']$. By using Lemma 2.2, there exists a $\theta_0 \in [\alpha', \beta']$ and $R > 1$ such that for $k = 1, 2$,

$$\Big| \frac{f^{(k)}(re^{i\theta_0})}{f(re^{i\theta_0})} \Big| \le r^{2\rho(f)} \tag{6.7}$$

holds for all $r > R$. Note that (6.5) holds for $\theta = \theta_0$. Thus there is a sequence (r_n) with $r_n \to \infty$ as $n \to \infty$, such that

$$|B(r_n e^{i\theta_0})| \ge \exp(r_n^{\mu(B)-\varepsilon}) \tag{6.8}$$

holds for every $\varepsilon \in (0, \mu(B))$. Therefore, we deduce from (6.6)-(6.8) that

$$\begin{aligned}
\exp(r_n^{\mu(B)-\varepsilon}) &\le |B(r_n e^{i\theta_0})| \\
&\le \Big| \frac{f''(r_n e^{i\theta_0})}{f(r_n e^{i\theta_0})} \Big| + \Big| \frac{f'(r_n e^{i\theta_0})}{f(r_n e^{i\theta_0})} \Big| (|A(r_n e^{i\theta_0}) - a_{j_0}| + |a_{j_0}|) \\
&\le r_n^{2\rho(f)} (1 + |a_{j_0}| + \exp(-CT(r_n, A)))
\end{aligned}$$

holds for all sufficiently large n. Obviously, this is a contradiction. Hence Theorem 1.10 holds.

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References

[1] L. Ahlfors; Untersuchungen zur Theorie der konformen abbildung und der Theorie der ganzen Funktionen, *Acta Soc. Sci. Fenn.* 1 (1930), 1-40.
[2] S. Bank, I. Laine; On the oscillation theory of $f'' + Af = 0$ where A is entire, *Trans. Amer. Math. Soc.* 273 (1982), 351-363.
[3] P. D. Barry; Some theorems related to the $\cos \pi\rho$ theorem, *Proc. London. Math. Soc.* 21 (1970), no. 3, 334-360.
[4] A. Denjoy; Sur les fonctions entieres de genre fini, *C. R. Acad. Sci. Paris* 45 (1907), 106-109.
[5] D. Drasin; On asymptotic curves of functions extremal for Denjoy's conjecture, *Proc. London Math. Soc.* 26 (1973), 142-166.
[6] G. G. Gundersen; On the real zeros of solutions of $f'' + A(z)f = 0$ where $A(z)$ is entire, *Ann. Acad. Sci. Fenn. Math.* 11 (1986), 275-294.
[7] G. G. Gundersen; Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates, *J. London Math. Soc.* 37 (1988), 88-104.
[8] G. G. Gundersen; Finite order solutions of second order linear differential equations, *Trans. Amer. Math. Soc.* 305 (1988), 415-429.
[9] G. G. Gundersen, M. Steinbart; A generalization of the Airy integral for $f'' - z^n f = 0$, *Trans. Amer. Math. Soc.* 337 (1993), no. 2, 737-755.
[10] W. Hayman; *Meromorphic Functions*, Clarendon Press, Oxford, 1964.
[11] S. Hellerstein, J. Miles, J. Rossi; On the growth of solutions of $f'' + gf' + hf = 0$, *Trans. Amer. Math. Soc.* 324 (1991), 693-705.
[12] E. Hille; *Lectures On Ordinary Differential Equations*, Addison-Wesley Publiching Company, Reading, Massachusetts-Menlo Park, California-London-Don Mills, Ontario, 1969.
[13] P. B. Kennedy; A class of integral functions bounded on certain curves, *Proc. London Math. Soc.* 6 (1956), 518-547.
[14] I. Laine; *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin, New York, 1993.
[15] J. R. Long, P. C. Wu, Z. Zhang; On the growth of solutions of second order linear differential equations with extremal coefficients, *Acta Math. Sinica, English Series* 29 (2013), no. 2, 365-372.
[16] J. R. Long, K. E. Qiu; On the growth of solutions of a class of second order complex linear differential equations, *J. Math. Pract. Theory*, 45 (2015), no. 2, 243-247. (In Chinese)
[17] A. I. Markushevich; *Theory of Functions of a Complex Variable*, Vol. Π, Revised English Edition Translated and Edited by Richard A. Silverman. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1965.
[18] M. Ozawa; On a solution of $w'' + e^{-z}w' + (az + b)w = 0$, *Kodai Math. J.* 3 (1980), 295-309.
[19] K. C. Shin; New polynomials P for which $f'' + P(z)f = 0$ has a solution with almost all real zeros, *Ann. Acad. Sci. Fenn. Math.* 27 (2002), 491-498.
[20] M. Tsuji; *Potential Theory in Modern Function Theory*, Tokyo, Maruzen, 1959.
[21] X. B. Wu, P. C. Wu; On the growth of solutions of $f'' + Af' + Bf = 0$, where A is a solution of a second order linear differential equation, *Acta Math. Scientia* 33 (2013), no. 3, 46-52. (In Chinese)
[22] P. C. Wu, J. Zhu; On the growth of solutions of the complex differential equation $f'' + Af' + Bf = 0$, *Sci. China Ser. A* 54 (2011), no. 5, 939-947.
[23] S. J. Wu; Some results on entire functions of finite lower order, *Acta Math. Sinica* 10 (1994), no. 2, 168-178.
[24] L. Yang; *Value Distribution Theory*, Springer-Verlag, Berlin, 1993.
[25] L. Yang; Deficient values and angular distibution of entire functions, *Trans. Amer. Math. Soc.* 308 (1988), no. 2, 583-601.

[26] L. Yang, G. H. Zhang; The distribution of Borel directions of entire functions, *Acta Math. Sinica* 19 (1976), no. 3, 157-168. (In Chinese)
[27] G. H. Zhang; On entire function extremal for Denjoy's confecture, *Sci. Sinica (I)* 24 (1981), 885-898.
[28] G. H. Zhang; *Theory of Entire and Meromorphic Functions-Deficient Values*, Asymptotic values and Singular Directions, Springer-Verlag, Berlin, 1993.

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