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EXISTENCE AND NON-EXISTENCE OF SOLUTIONS FOR A $p(x)$-BIHARMONIC PROBLEM

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Abstract. In this article, we study the following problem with Navier boundary conditions

$$\Delta(|\Delta u|^{p(x)-2}\Delta u) + |u|^{p(x)-2}u = \lambda|u|^{q(x)-2}u + \mu|u|^{\gamma(x)-2}u \quad \text{in } \Omega,$$
$$u = \Delta u = 0 \quad \text{on } \partial\Omega.$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$. $p(x), q(x)$ and $\gamma(x)$ are continuous functions on $\overline{\Omega}$, λ and μ are parameters. Using variational methods, we establish some existence and non-existence results of solutions for this problem.

1. Introduction

In recent years, the study of differential equations and variational problems with $p(x)$-growth conditions was an interesting topic, which arises from nonlinear electrorheological fluids and elastic mechanics. In that context we refer the reader to Ruzicka [11], Zhikov [16] and the reference therein; see also [2, 4, 5, 7].

Fourth-order equations appears in many context. Some of theses problems come from different areas of applied mathematics and physics such as Micro Electro-Mechanical systems, surface diffusion on solids, flow in Hele-Shaw cells (see [8]). In addition, this type of equations can describe the static from change of beam or the sport of rigid body. El Amrouss et al [1] studied a class of $p(x)$-biharmonic of the form

$$\Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda|u|^{p(x)-2}u + f(x,u) \quad \text{in } \Omega,$$
$$u = \Delta u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$, with smooth boundary $\partial\Omega$, $N \geq 1$, $\lambda \leq 0$ and some assumptions on the Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$. They obtained the existence and multiplicity of solutions.

In a recent article, Lin Li et al [9] considered the above problem and using variational methods, by the assumptions on the Carathéodory function f, they establish the existence of at least one solution and infinitely many solutions of the problem.

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Inspired by the above references and the work of Jinghua Yao [13], the aim of this article is to study the existence and multiplicity of weak solutions of the following fourth-order elliptic equation with Navier boundary conditions

$$\begin{gathered} \Delta(|\Delta u|^{p(x)-2}\Delta u)+|u|^{p(x)-2}u=\lambda|u|^{q(x)-2}u+\mu|u|^{\gamma(x)-2}u \quad \text{in } \Omega, \\ u=\Delta u=0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N\geq 1$, $p(x),q(x)$ and $\gamma(x)$ are continuous functions on $\overline{\Omega}$ with $\inf_{x\in\overline{\Omega}}p(x)>1, \inf_{x\in\overline{\Omega}}q(x)>1$, $\inf_{x\in\overline{\Omega}}\gamma(x)>1$ and λ and μ are parameters. Throughout the paper, we assume that $\lambda^2+\mu^2\neq 0$.

2. Preliminaries

To study $p(x)$-Laplacian problems, we need some results on the spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, and properties of $p(x)$-Laplacian, which we use later. Let Ω be a bounded domain of $\mathbb{R}^N$, denote

$$C_+(\overline{\Omega})=\{h(x);h(x)\in C(\overline{\Omega}),h(x)>1,\forall x\in\overline{\Omega}\}.$$

For any $h\in C_+(\overline{\Omega})$, we define

$$h^+=\max\{h(x);\ x\in\overline{\Omega}\},\quad h^-=\min\{h(x);\ x\in\overline{\Omega}\};$$

For any $p\in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space*

$$\begin{aligned} L^{p(x)}(\Omega)=\Big\{u;u \text{ is a measurable real-valued function such that} \\ \int_\Omega |u(x)|^{p(x)}dx<\infty\Big\}, \end{aligned}$$

endowed with the so-called *Luxemburg norm*

$$|u|_{p(x)}=\inf\Big\{\mu>0;\ \int_\Omega\big|\frac{u(x)}{\mu}\big|^{p(x)}dx\leq 1\Big\}.$$

Then $(L^{p(x)}(\Omega),|\cdot|_{p(x)})$ becomes a Banach space.

Proposition 2.1 ([6]). *The space $(L^{p(x)}(\Omega),|\cdot|_{p(x)})$ is separable, uniformly convex, reflexive and its conjugate space is $L^{q(x)}(\Omega)$ where $q(x)$ is the conjugate function of $p(x)$, i.e.,*

$$\frac{1}{p(x)}+\frac{1}{q(x)}=1,$$

for all $x\in\Omega$. For $u\in L^{p(x)}(\Omega)$ and $v\in L^{q(x)}(\Omega)$, we have

$$\Big|\int_\Omega uv\,dx\Big|\leq\Big(\frac{1}{p^-}+\frac{1}{q^-}\Big)|u|_{p(x)}|v|_{q(x)}\leq 2|u|_{p(x)}|v|_{q(x)}.$$

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined as

$$W^{k,p(x)}(\Omega)=\{u\in L^{p(x)}(\Omega):D^\alpha u\in L^{p(x)}(\Omega),|\alpha|\leq k\},$$

where $D^\alpha u=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\dots\partial x_N^{\alpha_N}}u$, with $\alpha=(\alpha_1,\dots,\alpha_N)$ is a multi-index and $|\alpha|=\sum_{i=1}^N\alpha_i$. The space $W^{k,p(x)}(\Omega)$ equipped with the norm

$$\|u\|_{k,p(x)}=\sum_{|\alpha|\leq k}|D^\alpha u|_{p(x)},$$

also becomes a separable and reflexive Banach space. For more details, we refer the reader to [3, 6, 10, 13]. Denote

$$p_k^*(x) = \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } kp(x) < N, \\ +\infty & \text{if } kp(x) \geq N \end{cases}$$

for any $x \in \overline{\Omega}$, $k \geq 1$.

Proposition 2.2 ([6]). *For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

If we replace $\leq$ with $<$, the embedding is compact.

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$. Note that the weak solutions of problem (1.1) are considered in the generalized Sobolev space

$$X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$$

equipped with the norm

$$\|u\| = \inf \Big\{ \mu > 0 : \int_\Omega \Big(\Big| \frac{\Delta u(x)}{\mu} \Big|^{p(x)} + \lambda \Big| \frac{u(x)}{\mu} \Big|^{p(x)} \Big) dx \leq 1 \Big\}.$$

Remark 2.3. According to [14], the norm $\|\cdot\|_{2,p(x)}$ is equivalent to the norm $|\Delta \cdot|_{p(x)}$ in the space X. Consequently, the norms $\|\cdot\|_{2,p(x)}$, $\|\cdot\|$ and $|\Delta \cdot|_{p(x)}$ are equivalent.

Proposition 2.4 ([1]). *If we denote $\rho(u) = \int_\Omega (|\Delta u|^{p(x)} + |u|^{p(x)}) dx$, then for $u, u_n \in X$, we have*

1. *$\|u\| < 1$ (respectively=1; > 1) $\iff \rho(u) < 1$ (respectively $= 1$; > 1);*
2. *$\|u\| \leq 1 \Rightarrow \|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$;*
3. *$\|u\| \geq 1 \Rightarrow \|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$;*
4. *$\|u\| \to 0$ (respectively $\to \infty$) $\iff \rho(u) \to 0$ (respectively $\to \infty$).*

It is clear that the energy functional associated with (1.1) is defined by

$$I_{\lambda,\mu}(u) = \int_\Omega \frac{1}{p(x)} (|\Delta u|^{p(x)} + |u|^{p(x)}) dx - \lambda \int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx - \mu \int_\Omega \frac{1}{\gamma(x)} |u|^{\gamma(x)} dx.$$

Let us define the functional

$$J(u) = \int_\Omega \frac{1}{p(x)} (|\Delta u|^{p(x)} + |u|^{p(x)}) dx.$$

It is well known that J is well defined, even and C^1 in X. Moreover, the operator $L = J' : X \to X^*$ defined as

$$\langle L(u), v \rangle = \int_\Omega (|\Delta u|^{p(x)-2} \Delta u \Delta v + |u|^{p(x)-2} uv) dx$$

for all $u, v \in X$ satisfies the following assertions.

Proposition 2.5 ([1]).

1. *L is continuous, bounded and strictly monotone.*
2. *L is a mapping of (S_+) type, namely: $u_n \rightharpoonup u$, and $\limsup_{n\to+\infty} L(u_n)(u_n - u) \leq 0$ implies $u_n \to u$.*
3. *L is a homeomorphism.*

3. Main results and proofs

In this section, we study the existence and non-existence of weak solutions for problem (1.1). We use the letter c_i in order to denote a positive constant.

Theorem 3.1. *Assume that $q(x),\gamma(x)\in C_+(\overline{\Omega})$ and $p^+<q^-\le q(x)<p_2^*(x)$, $\gamma^+<p^-$ for any $x\in\overline{\Omega}$. Then we have*

- (i) *For every $\lambda>0$, $\mu\in\mathbb{R}$, (1.1) has a sequence of weak solutions $(\pm u_k)$ such that $I_{\lambda,\mu}(\pm u_k)\to+\infty$ as $k\to+\infty$.*
- (ii) *For every $\mu>0$, $\lambda\in\mathbb{R}$, (1.1) has a sequence of weak solutions $(\pm v_k)$ such that $I_{\lambda,\mu}(\pm v_k)<0$ and $I_{\lambda,\mu}(\pm v_k)\to 0$ as $k\to+\infty$.*
- (iii) *For every $\lambda<0$, $\mu<0$, (1.1) has no nontrivial weak solution.*

We will use the following Fountain theorem to prove (i) and the Dual of the Fountain theorem to prove (ii).

Lemma 3.2 ([15])**.** *Let X be a reflexive and separable Banach space, then there exist $\{e_j\}\subset X$ and $\{e_j^*\}\subset X^*$ such that*

$$X=\overline{\operatorname{span}\{e_j:j=1,2,\dots\}},\quad X^*=\overline{\operatorname{span}\{e_j^*:j=1,2,\dots\}},$$

and

$$\langle e_i,e_j^*\rangle=\begin{cases}1 & \text{if } i=j,\\ 0 & \text{if } i\neq j,\end{cases}$$

We define

$$X_j=\operatorname{span}\{e_j\},\quad Y_k=\oplus_{j=1}^{k}X_j,\quad Z_k=\overline{\oplus_{j=k}^{\infty}X_j}. \tag{3.1}$$

Then we have the following Lemma.

Lemma 3.3 ([1])**.** *If $q(x),\gamma(x)\in C_+(\overline{\Omega})$, $q(x)<p_2^*(x)$, and $\gamma(x)<p_2^*(x)$ for all $x\in\overline{\Omega}$, denote*

$$\beta_k=\sup\{|u|_{q(x)};\|u\|=1,\,u\in Z_k\}$$
$$\theta_k=\sup\{|u|_{\gamma(x)};\|u\|=1,\,u\in Z_k\},$$

then $\lim_{k\to\infty}\beta_k=0$, $\lim_{k\to\infty}\theta_k=0$.

Lemma 3.4 (Fountain Theorem [12])**.** *Let*

- (A1) *$I\in C^1(X,\mathbb{R})$ be an even functional, where $(X,\|\cdot\|)$ is a separable and reflexive Banach space, the subspaces X_k, Y_k and Z_k are defined by (3.1). If for each $k\in\mathbb{N}$, there exist $\rho_k>r_k>0$ such that*
- (A2) $\inf\{I(u)\ :\ u\in Z_k,\ \|u\|=r_k\}\to+\infty$ *as $k\to+\infty$.*
- (A3) $\max\{I(u)\ :\ u\in Y_k,\ \|u\|=\rho_k\}\le 0$.
- (A4) *I satisfies the (PS) condition for every $c>0$.*

Then I has an unbounded sequence of critical points.

Lemma 3.5 (Dual Fountain Theorem [12])**.** *Assume* (A1) *is satisfied and there is $k_0>0$ such that, for each $k\ge k_0$, there exist $\rho_k>r_k>0$ such that*

- (B1) $a_k=\inf\{I(u):u\in Z_k,\,\|u\|=\rho_k\}\ge 0$.
- (B2) $b_k=\max\{I(u):u\in Y_k,\,\|u\|=r_k\}<0$.
- (B3) $d_k=\inf\{I(u):u\in Z_k,\,\|u\|\le\rho_k\}\to 0$ *as $k\to+\infty$.*
- (B4) *I satisfies the $(PS)_c^*$ condition for every $c\in[d_{k_0},0)$.*

Then I has a sequence of negative critical values converging to 0.

Definition 3.6. We say that $I_{\lambda,\mu}$ satisfies the $(PS)_c^*$ condition (with respect to (Y_n)), if any sequence $\{u_{n_j}\} \subset X$ such that $n_j \to +\infty$, $u_{n_j} \in Y_{n_j}$, $I_{\lambda,\mu}(u_{n_j}) \to c$ and $(I_{\lambda,\mu}|_{Y_{n_j}})'(u_{n_j}) \to 0$, contains a subsequence converging to a critical point of $I_{\lambda,\mu}$.

Proof of Theorem 3.1. (i) First we verify $I_{\lambda,\mu}$ satisfies the (PS) condition. Suppose that $(u_n) \subset X$ is (PS) sequence, i.e.,

$$|I_{\lambda,\mu}(u_n)| \le c_9, \quad I'_{\lambda,\mu}(u_n) \to 0 \quad \text{as } n \to \infty.$$

By Propositions 2.2 and 2.1, we know that if we denote

$$\phi(u) = -\lambda \int_\Omega \frac{1}{q(x)} |u|^{q(x)}, dx, \quad \psi(u) = -\mu \int_\Omega \frac{1}{\gamma(x)} |u|^{\gamma(x)}, dx,$$

then they are both weakly continuous and their derivative operators are compact. By Proposition 2.5, we deduce that $I'_{\lambda,\mu} = L + \phi' + \psi'$ is also of type (S_+). Thus it is sufficient to verify that (u_n) is bounded. Assume $\|u_n\| > 1$ for convenience. For n large enough, we have

$$\begin{aligned}
&c_9 + 1 + \|u_n\| \\
&\ge I_{\lambda,\mu}(u_n) - \frac{1}{q^-}\langle I'_{\lambda,\mu}(u_n), u_n\rangle \\
&= \Big[\int_\Omega \frac{1}{p(x)}(|\Delta u_n|^{p(x)} + |u_n|^{p(x)})dx - \lambda \int_\Omega \frac{1}{q(x)}|u_n|^{q(x)}dx - \mu \int_\Omega \frac{1}{\gamma(x)}|u_n|^{\gamma(x)}dx\Big] \\
&\quad - \frac{1}{q^-}\Big[\int_\Omega (|\Delta u_n|^{p(x)} + |u_n|^{p(x)})dx - \lambda \int_\Omega |u_n|^{q(x)}dx - \mu \int_\Omega |u_n|^{\gamma(x)}dx\Big] \\
&\ge \Big(\frac{1}{p^+} - \frac{1}{q^-}\Big)||u_n||^{p^-} - c_{10}\|u_n\|^{\gamma^+}.
\end{aligned} \tag{3.2}$$

Since $q^- > p^+$ and $p^- > \gamma^+$, we know that $\{u_n\}$ is bounded in X. In the following we will prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that (A2) and (A3) hold.

(A2) For any $u \in Z_k$, $\|u\| = r_k > 1$ (r_k will be specified below), we have

$$\begin{aligned}
I_{\lambda,\mu}(u) &= \int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} + |u|^{p(x)})dx - \lambda \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx - \mu \int_\Omega \frac{1}{\gamma(x)}|u|^{\gamma(x)}dx \\
&\ge \frac{1}{p^+}\|u\|^{p^-} - \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}dx - \frac{c_{11}|\mu|}{\gamma^-}\|u\|^{\gamma^+}.
\end{aligned}$$

Since $p^- > \gamma^+$, there exists $r_0 > 0$ large enough such that $\frac{c_{11}|\mu|}{\gamma^-}\|u\|^{\gamma^+} \le \frac{1}{2p^+}\|u\|^{p^-}$ as $r = \|u\| \ge r_0$. If $|u|_{q(x)} \le 1$ then $\int_\Omega |u|^{q(x)}dx \le |u|_{q(x)}^{q^-} \le 1$. However, if $|u|_{q(x)} > 1$ then $\int_\Omega |u|^{q(x)}dx \le |u|_{q(x)}^{q^+} \le (\beta_k\|u\|)^{q^+}$. So, we conclude that

$$\begin{aligned}
I_{\lambda,\mu}(u) &\ge \begin{cases} \frac{1}{2p^+}\|u\|^{p^-} - \frac{\lambda c_{12}}{q^-} & \text{if } |u|_{q(x)} \le 1, \\ \frac{1}{2p^+}\|u\|^{p^-} - \frac{\lambda}{q^-}(\beta_k\|u\|)^{q^+} & \text{if } |u|_{q(x)} > 1. \end{cases} \\
&\ge \frac{1}{p^+}\|u\|^{p^-} - \frac{\lambda}{q^-}(\beta_k\|u\|)^{q^+} - c_{13},
\end{aligned}$$

choose $r_k = \left(\frac{2\lambda}{q^-} q^+ \beta_k^{q^+}\right)^{\frac{1}{p^- - q^+}}$, we have

$$I_{\lambda,\mu}(u) = \frac{1}{2}\Big(\frac{1}{p^+} - \frac{1}{q^+}\Big) r_k^{p^-} - c_{13} \to \infty \quad \text{as} k \to \infty,$$

because of $p^+ < q^- \leq q^+$ and $\beta_k \to 0$.

(A3) Let $u \in Y_k$ such that $\|u\| = \rho_k > r_k > 1$. Then

$$\begin{aligned} I_{\lambda,\mu}(u) &= \int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} + |u|^{p(x)})dx - \lambda \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx - \mu \int_\Omega \frac{1}{\gamma(x)}|u|^{\gamma(x)}dx \\ &\leq \frac{1}{p^-}\|u\|^{p^+} - \frac{\lambda}{q^+}\int_\Omega |u|^{q(x)}dx + \frac{|\mu|}{\gamma^-}\int_\Omega |u|^{\gamma(x)}dx. \end{aligned}$$

Since $\dim Y_k < \infty$, all norms are equivalent in Y_k, we obtain

$$I_{\lambda,\mu}(u) \leq \frac{1}{p^-}\|u\|^{p^+} - \frac{\lambda}{q^+}\|u\|^{q^-} + \frac{|\mu|}{\gamma^-}\|u\|^{\gamma^+}.$$

We get that: $I_{\lambda,\mu}(u) \to -\infty$ as $\|u\| \to +\infty$ since $q^- > p^+$ and $\gamma^+ < p^-$. So (A2) holds. From the proof of (A2) and (A3), we can choose $\rho_k > r_k > 0$. Obviously $I_{\lambda,\mu}$ is even and the proof of (i) is complete.

(ii) We use the Dual Fountain theorem to prove conclusion (ii). Now we prove that there exist $\rho_k > r_k > 0$ such that if k is large enough (B1), (B2) and (B3) are satisfied.

(B1) For any $u \in Z_k$ we have

$$\begin{aligned} I_{\lambda,\mu}(u) &= \int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} + |u|^{p(x)})d - \lambda \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx - \mu \int_\Omega \frac{1}{\gamma(x)}|u|^{\gamma(x)}dx \\ &\geq \frac{1}{p^+}\|u\|^{p^+} - \frac{c_{14}|\lambda|}{q^-}\|u\|^{q^-} - \frac{\mu}{\gamma^-}\int_\Omega |u|^{\gamma(x)}dx. \end{aligned}$$

Since $q^- > p^+$, there exists $\rho_0 > 0$ small enough such that $\frac{c_{14}|\lambda|}{q^-}\|u\|^{q^-} \leq \frac{1}{2p^+}\|u\|^{p^+}$ as $0 < \rho = \|u\| \leq \rho_0$. Then from the proof above, we have

$$I_{\lambda,\mu}(u) \geq \begin{cases} \frac{1}{2p^+}\|u\|^{p^+} - \frac{\mu c_{15}}{\gamma^-} & \text{if } |u|_{\gamma(x)} \leq 1, \\ \frac{1}{2p^+}\|u\|^{p^+} - \frac{\mu}{\gamma^-}(\theta_k\|u\|)^{\gamma^+} & \text{if } |u|_{\gamma(x)} > 1. \end{cases} \tag{3.3}$$

Choose $\rho_k = \big(\frac{2p^+\mu\theta_k^{\gamma^+}}{\gamma^-}\big)^{\frac{1}{p^+ - \gamma^+}}$, then

$$I_{\lambda,\mu}(u) = \frac{1}{2p^+}(\rho_k)^{p^+} - \frac{1}{2p^+}(\rho_k)^{p^+} = 0.$$

Since $p^- > \gamma^+$, $\theta_k \to 0$, we know $\rho_k \to 0$ as $k \to \infty$.

(B2) For $u \in Y_k$ with $\|u\| \leq 1$, we have

$$\begin{aligned} I_{\lambda,\mu}(u) &= \int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} + |u|^{p(x)})dx - \lambda \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx - \mu \int_\Omega \frac{1}{\gamma(x)}|u|^{\gamma(x)}dx \\ &\leq \frac{1}{p^-}\|u\|^{p^-} + \frac{|\lambda|}{q^-}\int_\Omega |u|^{q(x)}dx - \frac{\mu}{\gamma^+}\int_\Omega |u|^{\gamma(x)}dx. \end{aligned}$$

Since $\dim Y_k = k$, conditions $\gamma^+ < p^-$ and $p^+ < q^-$ imply that there exists a $r_k \in (0, \rho_k)$ such that $I_{\lambda,\mu}(u_n) < 0$ when $\|u\| = r_k$. So we obtain

$$\max_{u \in Y_k, \|u\| = r_k} I_{\lambda,\mu}(u) < 0,$$

i.e., (B2) is satisfied.

(B3) Because $Y_k \cap Z_k \neq \emptyset$ and $r_k < \rho_k$, we have

$$d_k = \inf_{u \in Z_k, \|u\| \leq \rho_k} I_{\lambda,\mu}(u) \leq b_k = \max_{u \in Y_k, \|u\| = r_k} I_{\lambda,\mu}(u) < 0.$$

From (3.3) , for $u \in Z_k$, $\|u\| \leq \rho_k$ small enough we can write

$$I_{\lambda,\mu}(u) \geq \frac{1}{2p^+}\|u\|^{p^+} - \frac{\lambda}{\gamma^-}\theta_k^{\gamma^+}\|u\|^{\gamma^+} \geq -\frac{\lambda}{\gamma^-}\theta_k^{\gamma^+}\|u\|^{\gamma^+},$$

Since $\theta_k \to 0$ and $\rho_k \to 0$ as $k \to \infty$, (B3) holds. Finally we verify the $(PS)_c^*$ condition. Suppose $\{u_{n_j}\} \subset X$ such that

$$n_j \to +\infty, \quad u_{n_j} \in Y_{n_j}, \quad I_{\lambda,\mu}(u_{n_j}) \to c_{16}, \quad (I_{\lambda,\mu}|_{Y_{n_j}})'(u_{n_j}) \to 0.$$

If $\lambda \geq 0$, similar to (3.2), we can get the boundedness of $\|u_{n_j}\|$. Assume $\|u_{n_j}\| \geq 1$ for convenience. If $\lambda < 0$, for $n > 0$ large enough, we have

$$\begin{aligned}
c_{16} + 1 + \|u_{n_j}\| &\geq I_{\lambda,\mu}(u_{n_j}) - \frac{1}{q^+}\langle I'_{\lambda,\mu}(u_{n_j}), u_{n_j}\rangle \\
&= \Big[\int_\Omega \frac{1}{p(x)}(|\Delta u_{n_j}|^{p(x)} + |u_{n_j}|^{p(x)})dx - \lambda\int_\Omega \frac{1}{q(x)}|u_{n_j}|^{q(x)}dx \\
&\quad - \mu\int_\Omega \frac{1}{\gamma(x)}|u_{n_j}|^{\gamma(x)}dx\Big] - \frac{1}{q^+}\Big[\int_\Omega(|\Delta u_{n_j}|^{p(x)} + |u_{n_j}|^{p(x)})dx \\
&\quad - \lambda\int_\Omega |u_{n_j}|^{q(x)}dx - \mu\int_\Omega |u_{n_j}|^{\gamma(x)}dx\Big] \\
&\geq \Big(\frac{1}{p^+} - \frac{1}{q^+}\Big)\|u_{n_j}\|^{p^-} - c_{17}\|u_{n_j}\|^{\gamma^+}.
\end{aligned}$$

Since $p^- > \gamma^+$ and $q^+ > p^+$, we know that $\{u_{n_j}\}$ is bounded in X. Hence there exists $u \in X$ such that $u_{n_j} \rightharpoonup u$ in x. Observe now that $X = \overline{\cup_{n_j} Y_{n_j}}$, then we can find $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \to u$. We have

$$\langle I'_{\lambda,\mu}(u_{n_j}), u_{n_j} - u\rangle = \langle I'_{\lambda,\mu}(u_{n_j}), u_{n_j} - v_{n_j}\rangle + \langle I'_{\lambda,\mu}(u_{n_j}), v_{n_j} - u\rangle.$$

Having in mind that $(u_{n_j} - v_{n_j}) \in Y_{n_j}$, it yields

$$\langle I'_{\lambda,\mu}(u_{n_j}), u_{n_j} - u\rangle = \langle (I_{\lambda,\mu}|_{Y_{n_j}})'(u_{n_j}), u_{n_j} - v_{n_j}\rangle + \langle I'_{\lambda,\mu}(u_{n_j}), v_{n_j} - u\rangle \to 0 \quad (3.4)$$

as $n \to \infty$. By Proposition 2.5, the operator $I'_{\lambda,\mu}$ is obviously of (S_+) type. Using this fact with (3.4), we deduce that $u_{n_j} \to u$ in X, furthermore $I'_{\lambda,\mu}(u_{n_j}) \to I'_{\lambda,\mu}(u)$.

We claim now that u is in fact a critical point of $I_{\lambda,\mu}$. Taking $\omega_k \in Y_k$, notice that when $n_j \geq k$ we have

$$\begin{aligned}
\langle I'_{\lambda,\mu}(u), \omega_k\rangle &= \langle I'_{\lambda,\mu}(u) - I'_{\lambda,\mu}(u_{n_j}), \omega_k\rangle + \langle I'_{\lambda,\mu}(u_{n_j}), \omega_k\rangle \\
&= \langle I'_{\lambda,\mu}(u) - I'_{\lambda,\mu}(u_{n_j}), \omega_k\rangle + \Big\langle (I_{\lambda,\mu}|_{Y_{n_j}})'(u_{n_j}), \omega_k\Big\rangle.
\end{aligned}$$

Going to the limit on the right side of the above equation reaches

$$\langle I'_{\lambda,\mu}(u), \omega_k\rangle = 0, \quad \forall \omega_k \in Y_k,$$

so $I'_{\lambda,\mu}(u) = 0$, this show that $I_{\lambda,\mu}$ satisfies the $(PS)_c^*$ condition for every $c \in \mathbb{R}$.

(iii) Assume for the sake of contradiction, $u \in X\backslash\{0\}$ is a weak solution of problem (1.1). Then multiplying the equation in (1.1) by u, integrating by parts we obtain

$$\int_\Omega (|\Delta u|^{p(x)} + |u|^{p(x)})dx = \lambda \int_\Omega |u|^{q(x)} + \mu \int_\Omega |u|^{\gamma(x)}.$$

This leads to contradiction and the proof is complete.

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