Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 165, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

OPTIMAL CONTROL FOR THE MULTI-DIMENSIONAL VISCOUS CAHN-HILLIARD EQUATION

NING DUAN, XIUFANG ZHAO

Abstract. In this article, we study the multi-dimensional viscous Cahn-Hilliard equation. We prove the existence of optimal solutions and establish the optimality system.

1. Introduction

In this article, we consider the viscous Cahn-Hilliard equation

$$u_t - k\Delta u_t + \gamma\Delta^2 u = \Delta\varphi(u), \quad (x,t) \in \Omega \times (0,T), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n (n \leq 3)$ is a bounded domain with smooth boundary, the unknown function $u(x,t)$ is the concentration of one of the two phases, $\gamma > 0$ is the interfacial energy parameter, $k > 0$ represents the viscous coefficient, $\varphi(u)$ is the intrinsic chemical potential. The viscous Cahn-Hilliard equation, which was first propounded by Novick-Cohen [12], arises in the dynamics of viscous first order phase transitions in cooling binary solutions such as glasses, alloys and polymer mixtures (see[1, 6]). Note that if we take $k = 0$, the equation becomes the well-known Cahn-Hilliard type equation (see [17, 20]), which is originally proposed for modelling phase separation phenomena in a binary mixture, and it can be used to describe many other physical and biological phenomena, including the growth and dispersal in the population which is sensitive to time-periodic factors.

During the past years, many papers were devoted to the viscous Cahn-Hilliard equation. In [10], Liu and Yin considered the global existence and blow-up of classical solutions for viscous Cahn-Hilliard equation in $\mathbb{R}^n$ ($n \leq 3$). In Grinfeld and Novick-Cohen's paper [7], a Morse decomposition of the stationary solutions of the 1D viscous Cahn-Hilliard equation was established by explicit energy calculations, and the global attractor for the viscous Cahn-Hilliard equation was also considered. Li and Yin [8] investigate the existence, uniqueness and asymptotic behavior of solutions to the 1D viscous Cahn-Hilliard equation with time periodic potentials and sources. We also noticed that some investigations of the viscous Cahn-Hilliard equation were studied, such as in [3, 4, 11, 13].

In past decades, the optimal control of distributed parameter system had been received much more attention in academic field. Many papers have already been

2010 *Mathematics Subject Classification.* 35K55, 49A22.
Key words and phrases. Optimal control; viscous Cahn-Hilliard equation; optimal solution; optimality condition.

Submitted June 26, 2014. Published June 17, 2015.

published to study the control problems of nonlinear parabolic equations, for example [2, 5, 14, 16, 17, 19].

In this article, we consider the distributed optimal control problem

$$\min J(u,w) = \frac{1}{2}\|Cu - z_d\|_S^2 + \frac{\delta}{2}\|w\|_{L^2(Q_0)}^2, \tag{1.2}$$

subject to the initial boundary value problem for the viscous Cahn-Hilliard equation

$$\begin{gathered} u_t - k\Delta u_t + \gamma\Delta^2 u - \Delta\varphi(u) = Bw, \quad (x,t) \in \Omega \times (0,T), \\ u(x,t) = \Delta u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T) \\ u(0) = u_0, \quad x \in \Omega, \end{gathered} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n$ $(n \leq 3)$ is a bounded domain with smooth boundary, $k > 0$ and $\gamma > 0$ are two constants, $\varphi(u)$ is an intrinsic chemical potential with typical example as

$$\varphi(u) = \gamma_2 u^3 + \gamma_1 u^2 - u,$$

for some constants $\gamma_2 > 0$ and γ_1.

Remark 1.1. The main difference between the viscous Cahn-Hilliard equation and the standard Cahn-Hilliard equation is the viscous term $k\Delta u_t$, which describe the viscosity of glasses, alloys and polymer. Note that the viscous term $k\Delta u_t$ is not only dependent on x but also dependent on t. Because of the existence of this term, we can obtain the results on the a prior estimates more directed.

Remark 1.2. In [18], Zhao and Liu studied the optimal control problem for equation (1.1) in 1D case with $\varphi(s) = s^3 - s$. Based on Lions' [9] classical theory, they proved the existence of optimal solution to the equation. Here, we consider the n-D case of equation (1.1), where $n \leq 3$. We also established the optimality system, which was not established in [18]. In fact, for the well-known Cahn-Hilliard equation, using the same method as above, we can also obtain the results on the existence of optimal solutions and the optimality conditions.

The control target is to match the given desired state z_d in L^2-sense by adjusting the body force w in a control volume $Q_0 \subseteq Q = \Omega \times (0,T)$ in the L^2-sense.

In the following, we introduce some notations that will be used throughout the paper. For fixed $T > 0$, $V = H^2(\Omega) \bigcap H_0^1(\Omega)$ and $H = L^2(\Omega)$, let V^*, H^* be dual spaces of V and H. Then, we obtain

$$V \hookrightarrow H = H^* \hookrightarrow V^*.$$

Clearly, each embedding being dense.

The extension operator $B \in \mathcal{L}\left(L^2(Q_0), L^2(0,T;V^*)\right)$ which is called the controller is introduced as

$$Bq = \begin{cases} q, & q \in Q_0, \\ 0, & q \in Q \setminus Q_0. \end{cases} \tag{1.4}$$

We supply H with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$, and define a space $W(0,T;V)$ as

$$W(0,T;V) = \left\{v : v \in L^2(0,T;V),\ \frac{\partial v}{\partial t} \in L^2(0,T;V^*)\right\},$$

which is a Hillbert space endowed with common inner product.

This article is organized as follows. In the next section, we prove the existence and uniqueness of the weak solution to problem (1.3) in a special space and discuss the relation among the norms of weak solution, initial value and control item; In Section 3, we consider the optimal control problem and prove the existence of optimal solution; In the last section, the optimality conditions is showed and the optimality system is derived.

In the following, the letters c, c_i $(i = 1, 2, \cdots)$ will always denote positive constants different in various occurrences.

2. Existence and uniqueness of weak solution

In this section, we study the existence and uniqueness of weak solution for the equation

$$u_t - k\Delta u_t + \gamma\Delta^2 u - \Delta\varphi(u) = Bw, \quad \text{in } \Omega \times (0, T), \tag{2.1}$$

with the boundary value conditions

$$u(x, t) = \Delta u(x, t) = 0, \quad \text{in } \partial\Omega \times (0, T), \tag{2.2}$$

and initial condition

$$u(x, 0) = u_0(x), \quad \text{in } \Omega, \tag{2.3}$$

where $Bw \in L^2(0, T; V^*)$ and a control $w \in L^2(Q_0)$.

Now, we give the definition of the weak solution for problem (2.1)-(2.3) in the space $W(0, T; V)$.

Definition 2.1. For all $\eta \in V$, $t \in (0, T)$, the function $u(x, t) \in W(0, T; V)$ is called a weak solution to problem (2.1)-(2.3), if

$$\frac{d}{dt}(u, \eta) + k\frac{d}{dt}(\nabla u, \nabla\eta) + \gamma(\Delta u, \Delta\eta) + (\nabla\varphi(u), \nabla\eta) = (Bw, \eta)_{V^*,V}. \tag{2.4}$$

We shall give Theorem 2.2 on the existence and uniqueness of weak solution to problem (2.1)-(2.3).

Theorem 2.2. *Suppose $u_0 \in V$, $Bw \in L^2(0, T; V^*)$, then the problem* (2.1)-(2.3) *admits a unique weak solution $u(x, t) \in W(0, T; V)$ in the interval* $[0, T]$.

Proof. Galerkin's method is applied for the proof. Let $\{z_j(x)\}$ $(j = 1, 2, \cdots)$ be the orthonormal base in $L^2(\Omega)$ being composed of the eigenfunctions of the eigenvalue problem

$$\Delta z + \lambda z = 0, \; z(0) = z_0,$$

corresponding to eigenvalues λ_j $(j = 1, 2, \cdots)$.

Suppose that $u_n(x, t) = \sum_{j=1}^{N} u_{nj}(t)z_j(x)$ is the Galerkin approximate solution to the problem (2.1)-(2.3) require $u_n(0, \cdot) \to u_0$ in H holds true, where $u_{nj}(t)$ $(j = 1, 2, \cdots, N)$ are undermined functions, n is a natural number. By analyzing the limiting behavior of sequences of smooth function $\{u_n\}$, we can prove the existence of weak solution to the problem (2.1)-(2.3).

Performing the Galerkin procedure for the problem (2.1)-(2.3), we obtain

$$\begin{gathered} \big(u_{nt} - k\Delta u_{nt} + \gamma\Delta^2 u_n - \Delta\varphi(u_n), z_j\big) = (Bw, z_j), \\ (u_n(\cdot, 0), z_j) = (u_{n0}(\cdot), z_j), \; j = 1, 2, \cdots, N. \end{gathered} \tag{2.5}$$

Obviously, the equation in (2.4) is an ordinary differential equation and according to ODE theory, there exists an unique solution to the equation (2.4) in the interval $[0, t_n)$. what we should do is to show that the solution is uniformly bounded when $t_n \to T$. we need also to show that the times t_n there are not decaying to 0 as $n \to \infty$.

There are four steps for us to prove it.

Step 1. Multiplying both sides of the equation in (2.4) by $u_{nj}(t)$, summing up the products over $j = 1, 2, \dots, N$, we derive that

$$\frac{1}{2}\frac{d}{dt}(\|u_n\|^2 + k\|\nabla u_n\|^2) + \gamma\Delta u_n + \int_\Omega \varphi'(u_n)|\nabla u_n|^2 dx = (Bw, u_n)_{V^*,V}.$$

By Hölder's inequality, we conclude that

$$\begin{aligned}(Bw, u_n)_{V^*,V} &\le \|Bw\|_{V^*}\|u_n\|_V \le c_1\|Bw\|_{V^*}\|\Delta u_n\| \\ &\le \frac{\gamma}{2}\|\Delta u_n\|^2 + \frac{c_1^2}{2\gamma}\|Bw\|_{V^*}^2.\end{aligned}$$

Note that

$$\varphi'(u_n) = 3\gamma_2 u_n^2 + 2\gamma_1 u_n^2 - 1 \ge -\frac{\gamma_1^2}{3\gamma_2} - 1 = -c_2.$$

Summing up,

$$\begin{aligned}&\frac{d}{dt}(\|u_n\|^2 + k\|\nabla u_n\|^2) + \gamma\|\Delta u_n\|^2 \\ &\le \frac{c_1^2}{\gamma}\|Bw\|_{V^*}^2 + 2c_2\|\nabla u_n\|^2 \\ &\le \frac{c_1^2}{\gamma}\|Bw\|_{V^*}^2 + \frac{\gamma}{2}\|\Delta u_n\|^2 + \frac{c_2^2}{\gamma}\|u_n\|^2 \\ &\le \frac{c_1^2}{\gamma}\|Bw\|_{V^*}^2 + \frac{\gamma}{2}\|\Delta u_n\|^2 + \frac{c_2^2}{\gamma}(\|u_n\|^2 + k\|\nabla u_n\|^2).\end{aligned}$$

Since $Bw \in L^2(0, T; V^*)$ is the control item, we can assume that $\|Bw\|_{V^*} \le M$, where M is a positive constant. Then, we have

$$\frac{d}{dt}(\|u_n\|^2 + k\|\nabla u_n\|^2) + \frac{\gamma}{2}\|\Delta u_n\|^2 \le \frac{c_1^2}{\gamma}M^2 + \frac{c_2^2}{\gamma}(\|u_n\|^2 + k\|\nabla u_n\|^2).$$

Using Gronwall's inequality, we obtain

$$\begin{aligned}\|u_n\|^2 + k\|\nabla u_n\|^2 &\le e^{\frac{c_2^2}{\gamma}t}(\|u_n(0)\|^2 + k\|\nabla u_n(0)\|^2) + \frac{c_1^2}{c_2^2}M^2 \\ &\le e^{\frac{c_2^2}{\gamma}T}(\|u_n(0)\|^2 + k\|\nabla u_n(0)\|^2) + \frac{c_1^2}{c_2^2}M^2 = c_3^2.\end{aligned} \tag{2.6}$$

By Sobolev's embedding theorem, we immediately obtain

$$\|u_n(\cdot, t)\|_p \le c_4, \quad p \in (\frac{n}{2}, \frac{2n}{n-2}). \tag{2.7}$$

Step 2. Multiplying both sides of the equation of (2.4) by $\lambda_j u_{nj}(t)$, summing up the products over $j = 1, 2, \cdots, N$, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u_n\|^2 + k\|\Delta u_n\|^2) + \gamma\|\nabla\Delta u_n\|^2 = -(\Delta\varphi(u_n), \Delta u_n) - (Bw, \Delta u_n)_{V^*,V}.$$

Note that

$$\Delta\varphi(u_n) = (3\gamma_2 u_n^2 + 2\gamma_1 u_n - 1)\Delta u_n + (6\gamma_2 u_n + 2\gamma_1)|\nabla u_n|^2.$$

Hence

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u_n\|^2 + k\|\Delta u_n\|^2) + \gamma\|\nabla\Delta u_n\|^2 + \gamma_2\|u_n\Delta u_n\|^2$$

$$
\begin{aligned}
&= -2\gamma_2 \int_\Omega u_n^2 |\Delta u_n|^2 dx - 2\gamma_1 \int_\Omega u_n |\Delta u_n|^2 dx + \|\Delta u_n\|^2 \\
&\quad - 6\gamma_2 \int_\Omega u_n |\nabla u_n|^2 \Delta u_n \, dx - 2\gamma_1 \int_\Omega |\nabla u_n|^2 \Delta u_n \, dx - (Bw, \Delta u_n)_{V^*,V} \\
&\le \gamma_2 \int_\Omega u_n^2 |\Delta u_n|^2 dx + c_5(\|\Delta u_n\|^2 + \|\nabla u_n\|_4^4 + \|Bw\|_{V^*}^2 + \|u_n\|^2) \\
&\quad + \frac{\gamma}{4} \|\nabla \Delta u_n\|^2.
\end{aligned}
$$

Using Nirenberg's inequality, we deduce that

$$
c_5 \|\nabla u_n\|_4^4 \le c_4 (c' \|\nabla \Delta u_n\|^{\frac{n}{8}} \|\nabla u_n\|^{1-\frac{n}{8}} + c'' \|\nabla u_n\|)^4 \le \frac{\gamma}{8} \|\nabla \Delta u_n\|^2 + c_6.
$$

On the other hand, we also have

$$
c_5 \|\Delta u_n\|^2 \le \frac{\gamma}{8} \|\nabla \Delta u_n\|^2 + \frac{2c_5^2}{\gamma} \|\nabla u_n\|^2 \le \frac{\gamma}{8} \|\nabla \Delta u_n\|^2 + \frac{2c_3^2 c_5^2}{\gamma k}.
$$

Summing up, we derive that

$$
\frac{d}{dt}(\|\nabla u_n\|^2 + k\|\Delta u_n\|^2) + \gamma \|\nabla \Delta u_n\|^2 \le 2c_6 + 2c_3^2 c_5 + \frac{4c_3^2 c_5^2}{\gamma k} + 2c_5 \|Bw\|_{V^*}^2, \tag{2.8}
$$

which means

$$
\frac{d}{dt}(\|\nabla u_n\|^2 + k\|\Delta u_n\|^2) + \gamma \|\nabla \Delta u_n\|^2 \le 2c_6 + 2c_3^2 c_5 + \frac{4c_3^2 c_5^2}{\gamma k} + 2c_5 M^2.
$$

Therefore,

$$
\begin{aligned}
&\|\nabla u_n\|^2 + k\|\Delta u_n\|^2 \\
&\le \|\nabla u_n(0)\|^2 + k\|\Delta u_n(0)\|^2 + (2c_6 + 2c_3^2 c_5 + \frac{4c_3^2 c_5^2}{\gamma k} + 2c_5 M^2) T \\
&= (c_6')^2.
\end{aligned} \tag{2.9}
$$

By (2.7), (2.9) and Sobolev's embedding theorem, we conclude that

$$
\|u_n(\cdot, t)\|_\infty \le c_7. \tag{2.10}
$$

Adding (2.7) and (2.9) together gives

$$
\|u_n(x,t)\|_{L^2(0,T;V)}^2 \le c \int_0^T (\|u_n\|^2 + \|\nabla u_n\|^2 + \|\Delta u_n\|^2) dt \le c_8^2. \tag{2.11}
$$

Step 3. We prove a uniform $L^2(0,T;V^*)$ bound on a sequence $\{u_{n,t}\}$. Set $y_n = u_n - k\Delta u_n$, by (2.4) and Sobolev's embedding theorem, we obtain

$$
\begin{aligned}
\|y_{n,t}\|_{V^*} &= \sup_{\|\psi\|_V = 1} (y_{n,t}, \psi)_{V^*,V} \\
&\le \sup_{\|\psi\|_V = 1} \{(Bw, \psi)_{V^*,V} + \gamma |(\Delta u_n, \Delta \psi)| + |(\varphi(u_n), \Delta \psi)|\} \\
&\le c(\|B^* \bar{\omega}\|_{V^*} + \|\Delta u_n\| + \|u_n\|) \\
&\le c(M + \|\Delta u_n\| + \|u_n\|).
\end{aligned} \tag{2.12}
$$

Integrating (2.12) with respect to t on $[0,T]$, we obtain

$$
\|y_{n,t}\|_{L^2(0,T;V^*)}^2 \le c(M^2 T + \|\Delta u_n\|_{L^2(0,T;H)} + \|u_n\|_{L^2(0,T;H)}).
$$

Hence

$$\|u_{n,t}\|^2_{L^2(0,T;V^*)} = \|(I-k\Delta)^{-1}y_{n,t}\|^2_{L^2(0,T;V^*)} \le c_9^2. \tag{2.13}$$

Step 4. Integrating (2.9) with respect to $[0,T]$, combining its result and (2.11) together, we deduce that

$$\|u_n\|_{L^2(0,T;H^3)} \le c_{10}. \tag{2.14}$$

By the compactness of the embedding $L^\infty(0,T;H^2) \hookrightarrow L^\infty(0,T;H^1)$ and of $L^2(0,T;H^3) \hookrightarrow L^2(0,T;H^1)$, we find that there exist $u \in L^\infty(0,T;H^1)$ and $u \in L^2(0,T;H^1)$ such that, up to a subsequence,

$$\begin{gathered} u_n \to u \quad \text{strongly in } L^\infty(0,T;H^1), \\ u_n \to u \quad \text{strongly in } L^2(0,T;H^1). \end{gathered} \tag{2.15}$$

It then follows from (2.14) that

$$\|u_n - u\|_{L^\infty(0,T;H^1)} \to 0, \quad \|\Delta u_n - \Delta u\|_{L^2(0,T;H^2)} \to 0.$$

According to the previous subsequences $\{u_n\}$, we conclude that $\Delta\varphi(u_n)$ weakly converges to $\Delta\varphi(u)$ in $L^2(0,T;V^*)$. In fact, for any $w \in L^2(0,T;V^*)$, we have

$$\begin{aligned} &\Big|\int_0^T (\Delta\varphi(u_n) - \Delta\varphi(u), w)_{V^*,V}\,dt\Big| \\ &\le C\Big|\int_0^T (\varphi(u_n) - \varphi(u))w\,dt\Big| \\ &\le C\Big|\int_0^T \varphi'(\theta u_n + (1-\theta)u)(u_n - u)w\,dt\Big| \\ &\le C\int_0^T \|\varphi'(\theta u_n + (1-\theta)u\|_\infty \|u_n - u\|\|w\|\,dt \\ &\le C\|u_n - u\|_{L^2(0,T;H)}\|w\|_{L^2(0,T;H)}, \end{aligned} \tag{2.16}$$

where $\theta \in (0,1)$. By (2.16), we know that there exists a subsequence $\{u_n(x,t)\}$ such that $\Delta\varphi(u_n)$ converges weakly to $\Delta\varphi(u)$ in $L^2(0,T;V^*)$. On the other hand, the subsequence $\{u_{n,t}\}$ weakly converge to $\{u_t\}$ in $L^2(0,T;V^*)$.

Based on the above discussion, we conclude that there exists a function $u(x,t) \in W(0,T;V)$ which satisfies (2.4). Since the proof of uniqueness is easy, we omit it. Then, Theorem 2.2 has been proved. $\square$

For the relation among the norm of weak solution and initial value and control item, basing on the above discussion, we obtain the following theorem immediately.

Corollary 2.3. *Suppose that $u_0 \in V$, $Bw \in L^2(0,T;V^*)$, then there exists positive constants C' and C'' such that*

$$\|u\|^2_{W(0,T;V)} \le C'(\|u_0\|^2_V + \|w\|^2_{L^2(Q_0)}) + C'', \tag{2.17}$$

3. Optimal control problem

In this section, we consider the optimal control problem associated with the viscous Cahn-Hilliard equation and prove the existence of optimal solution.

In the following, we suppose $L^2(Q_0)$ is a Hilbert space of control variables, we also suppose $B \in \mathcal{L}(L^2(Q_0), L^2(0,T;V^*))$ is the controller and a control $w \in L^2(Q_0)$, consider the following control system

$$\begin{gathered} u_t - k\Delta u_t + \gamma\Delta^2 u - \Delta\varphi(u) = Bw, \quad (x,t) \in \Omega \times (0,T), \\ u(x,t) = \Delta u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T) \\ u(0) = u_0, \quad x \in \Omega. \end{gathered} \tag{3.1}$$

Here it is assume that $u_0 \in V$. By Theorem 2.2, we can define the solution map $w \to u(w)$ of $L^2(Q_0)$ into $W(0,T;V)$. The solution u is called the state of the control system (3.1). The observation of the state is assumed to be given by Cu. Here $C \in \mathcal{L}(W(0,T;V), S)$ is an operator, which is called the observer, S is a real Hilbert space of observations. The cost function associated with the control system (3.1) is given by

$$J(u,w) = \frac{1}{2}\|Cu - z_d\|_S^2 + \frac{\delta}{2}\|w\|^2_{L^2(Q_0)}, \tag{3.2}$$

where $z_d \in S$ is a desired state and $\delta > 0$ is fixed. An optimal control problem about the viscous Cahn-Hilliard equation is

$$\min J(u,w), \tag{3.3}$$

where (u,w) satisfies (3.1).

Let $X = W(0,T;V) \times L^2(Q_0)$ and $Y = L^2(0,T;V) \times H$. We define an operator $e = e(e_1, e_2) : X \to Y$, where

$$\begin{gathered} e_1 = (\Delta^2)^{-1}(u_t - k\Delta u_t + \gamma\Delta^2 u - \Delta\varphi(u) - Bw), \\ e_2 = u(x,0) - u_0. \end{gathered}$$

Here Δ^2 is an operator from V to V^*. Then, we write (3.3) in the form

$$\min J(u,w) \quad \text{subject to } e(y,w) = 0.$$

Theorem 3.1. *Suppose that $u_0 \in V$, $Bw \in L^2(0,T;V^*)$, then there exists an optimal control solution (u^*, w^*) to problem* (3.1).

Proof. Suppose that (u,w) satisfy the equation $e(u,w) = 0$. In view of (3.2), we deduce that

$$J(u,w) \geq \frac{\delta}{2}\|w\|^2_{L^2(Q_0)}.$$

By Corollary 2.3, we obtain that $\|u\|_{W(0,T;V)} \to \infty$ yields $\|w\|_{L^2(Q_0)} \to \infty$. Therefore,

$$J(u,w) \to \infty, \quad \text{when } \|(u,w)\|_X \to \infty. \tag{3.4}$$

As the norm is weakly lower semi-continuous, we achieve that J is weakly lower semi-continuous. Since for all $(u,w) \in X$, $J(u,w) \geq 0$, there exists $\lambda \geq 0$ defined by

$$\lambda = \inf\{J(u,w) : (u,w) \in X,\ e(u,w) = 0\},$$

which means the existence of a minimizing sequence $\{(u^n, w^n)\}_{n\in\mathbb{N}}$ in X such that

$$\lambda = \lim_{n\to\infty} J(u^n, w^n) \quad \text{and} \quad e(u^n, w^n) = 0, \quad \forall n \in \mathbb{N}.$$

From (3.4), there exists an element $(u^*, w^*) \in X$ such that when $n \to \infty$,

$$u^n \to u^*, \quad \text{weakly, } u \in W(0,T;V), \tag{3.5}$$

$$w^n \to w^*, \quad \text{weakly, } w \in L^2(Q_0). \tag{3.6}$$

Since $u_n \in L^\infty(0,T;V)$, $u_{n,t} \in L^2(0,T;V^*)$, we also have $L^\infty(0,T;V)$ is continuously embedded into $L^2(0,T;L^\infty)$. Hence by [15, Lemma 4] we have $u^n \to u^*$ strongly in $L^2(0,T;L^\infty)$, as $n \to \infty$, $u^n \to u^*$ strongly in $C(0,T;H)$, as $n \to \infty$.

As the sequence $\{u^n\}_{n\in\mathbb{N}}$ converges weakly, then $\|u^n\|_{W(0,T;V)}$ is bounded. Based on the embedding theorem, $\|u^n\|_{L^2(0,T;L^\infty)}$ is also bounded.

Because $u^n \to u^*$ in $L^2(0,T;L^\infty)$ as $n \to \infty$, we know that $\|u^*\|_{L^2(0,T;L^\infty)}$ is also bounded.

It then follows from (3.5) that

$$\lim_{n\to\infty} \int_0^T (u_t^n(x,t) - u_t^*, \psi(t))_{V^*,V} dt = 0, \quad \forall \psi \in L^2(0,T;V).$$

and

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^T (\Delta u_t^n(x,t) - \Delta u_t^*, \psi(t))_{V^*,V} dt \\
&= \lim_{n\to\infty} \int_0^T (u_t^n(x,t) - u_t^*, \Delta\psi(t))_{V^*,V} dt = 0, \quad \forall \psi \in L^2(0,T;V).
\end{aligned}$$

Using (3.6) again, we derive that

$$\Big| \int_0^T \int_\Omega (Bw - Bw^*)\eta \, dx\, dt \Big| \to 0, \quad n \to \infty, \ \forall \eta \in L^2(0,T;H).$$

By (3.5) again, we deduce that

$$\begin{aligned}
&| \int_0^T \int_\Omega (\Delta\varphi(u^n) - \Delta\varphi(u^*))\, \eta \, dx\, dt| \\
&= | \int_0^T \int_\Omega (\varphi(u^n) - \varphi(u^*))\, \Delta\eta \, dx\, dt| \\
&= | \int_0^T \int_\Omega [\gamma_2((u^n)^3 - (u^*)^3) + \gamma_1((u^n)^2 - (u^*)^2) - (u^n - u^*)]\Delta\eta \, dx\, dt| \\
&= | \int_0^T \int_\Omega \big[\gamma_2(u^n - u^*)((u^n)^2 + u^n u^* + (u^*)^2) + \gamma_1(u^n - u^*)(u^n + u^*) \\
&\quad - (u^n - u^*)\big]\Delta\eta \, dx\, dt| \\
&\le c| \int_0^T \big(\|(u^n)^2 + u^n u^* + (u^*)^2\|_\infty + \|u^n + u^*\|_\infty + 1\big)\|u^n - u^*\|_H \|\Delta\eta\|_H dt| \\
&\le \big(\|(u^n)^2 + u^n u^* + (u^*)^2\|_{L^2(0,T;L^\infty)} + \|u^n + u^*\|_{L^2(0,T;L^\infty)} + 1\big) \\
&\quad \times \|u^n - u^*\|_{C(0,T;H)} |\|\eta\|_{L^2(0,T;V)} \to 0, \quad n \to \infty, \ \forall \eta \in L^2(0,T;V).
\end{aligned}$$

Hence we have $u = u(\bar\omega)$ and therefore

$$J(u,\bar\omega) \le \lim_{n\to\infty} J(u^n, \bar\omega^n) = \lambda.$$

In view of the above discussions, we obtain

$$e_1(u^*, w^*) = 0, \quad \forall n \in \mathbb{N}.$$

Noticing that $u^* \in W(0,T;V)$, we derive that $u^*(0) \in H$. Since $u^n \to u^*$ weakly in $W(0,T;V)$, we can infer that $u^n(0) \to u^*(0)$ weakly when $n \to \infty$. Thus, we obtain

$$(u^n(0) - u^*(0), \eta) \to 0, \quad n \to \infty, \ \forall \eta \in H,$$

which means $e_2(u^*, w^*) = 0$. Therefore, we obtain

$$e(u^*, w^*) = 0, \quad \text{in } Y.$$

So, there exists an optimal solution (u^*, w^*) to problem (3.1). Then, the proof of Theorem 3.1 is complete. $\square$

4. Optimality conditions

It is well known that the optimality conditions for w are given by the variational inequality

$$J'(u,w)(v-w) \geq 0, \quad \text{for all } v \in L^2(Q_0), \tag{4.1}$$

where $J'(u,w)$ denotes the Gateaux derivative of $J(u,v)$ at $v = w$. The following Lemma 4.1 is essential in deriving necessary optimality conditions.

Lemma 4.1. *The map $v \to u(v)$ of $L^2(Q_0)$ into $W(0,T;V)$ is weakly Gateaux differentiable at $v = w$ and such the Gateaux derivative of $u(v)$ at $v = w$ in the direction $v - w \in L^2(Q_0)$, say $z = \mathcal{D}u(w)(v-w)$, is a unique weak solution of the problem*

$$\begin{gathered} z_t - k\Delta z_t + \gamma\Delta^2 z - \Delta(\varphi'(u(w))z) = B(v-w), \quad (x,t) \in Q, \\ z(x,t) = \Delta z(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ z(0) = 0, \quad x \in \Omega. \end{gathered} \tag{4.2}$$

Proof. Let $0 \leq h \leq 1$, u_h and u be the solutions of (3.1) corresponding to $w + h(v - w)$ and w, respectively. Then we prove the lemma in the following two steps:

Step 1. We prove that $u_h \to u$ strongly in $C(0,T;H_0^1)$ as $h \to 0$. Let $q = u_h - u$, then

$$\begin{gathered} \frac{dq}{dt} - k\frac{d\Delta q}{dt} + \gamma\Delta^2 q - \Delta(\varphi(u_h) - \varphi(u)) = hB(v-w), \quad 0 < t \leq T, \\ q(x,t) = \Delta q(x,t) = 0, \quad x \in \partial\Omega, \\ q(0) = 0, \quad x \in \Omega. \end{gathered} \tag{4.3}$$

Using Corollary 2.3 and Sobolev's embedding,

$$\|u\|_\infty \leq c_1', \quad \|u_h\|_\infty \leq c_2'.$$

Taking the scalar product of (4.3) with q, we have

$$\frac{1}{2}\frac{d}{dt}(\|q\|^2 + k\|\nabla q\|^2) + \gamma\|\Delta q\|^2 = (hB(v-w), q) + (\Delta(\varphi(u_h) - \varphi(u)), q).$$

Noticing that

$$\begin{aligned} (\Delta(\varphi(u_h) - \varphi(u)), q) &= (\gamma_2(u_h^3 - u^3) + \gamma_1(u_h^2 - u^2) - (u_h - u), \Delta q) \\ &= ([\gamma_2(u_h^2 + u^2 + u_h u) + \gamma_1(u_h + u) - 1]q, \Delta q) \\ &\leq \|\gamma_2(u_h^2 + u^2 + u_h u) + \gamma_1(u_h + u) - 1\|_\infty \|q\|\|\Delta q\| \\ &\leq c_3'\|q\|\|\Delta q\| \leq \frac{\gamma}{2}\|\Delta q\|^2 + \frac{(c_3')^2}{2\gamma}\|q\|^2. \end{aligned}$$

Hence

$$\frac{d}{dt}(\|q\|^2 + k\|\nabla q\|^2) + \gamma\|\Delta q\|^2 \leq \frac{(c_3')^2}{\gamma}\|q\|^2 + 2h\|B(v-w)\|\|q\|$$

$$\leq \Big(\frac{(c_3')^2}{\gamma}+1\Big)\|q\|^2+h^2\|B(v-w)\|^2,$$

Using Gronwall's inequality, it is easy to see that $\|q\|^2 \to 0$ as $h \to 0$. Then, $u_h \to u$ strongly in $C(0,T;H_0^1)$ as $h \to 0$.

Step 2. We prove that $\frac{u_h-u}{h} \to z$ strongly in $W(0,T;V)$. Rewrite (4.3) in the following form

$$\begin{gathered}
\frac{d}{dt}\Big(\frac{u_h-u}{h}\Big)-k\frac{d}{dt}\Delta\Big(\frac{u_h-u}{h}\Big)+\gamma\Delta^2\Big(\frac{u_h-u}{h}\Big)-\Delta\Big(\frac{\varphi(u_h)-\varphi(u)}{h}\Big)\\
=B(v-w), \quad 0<t\leq T,\\
\frac{u_h-u}{h}(x,t)=\Delta\Big(\frac{u_h-u}{h}\Big)(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,T),\\
\frac{u_h-u}{h}(0)=0, \quad x\in\Omega.
\end{gathered}$$

We can easily verify that the above problem possesses a unique weak solution in $W(0,T;V)$. On the other hand, it is easy to check that the linear problem (4.2) possesses a unique weak solution $z \in W(0,T;V)$. Let $r=\frac{u_h-u}{h}-z$, thus r satisfies

$$\begin{gathered}
\frac{d}{dt}r+k\frac{d}{dt}\Delta r+\gamma\Delta^2 r-\Delta\Big(\frac{\varphi(u_h)-\varphi(u)}{h}-\varphi'(u)z\Big)=0, \quad 0<t\leq T,\\
r(x,t)=\Delta r(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,T),\\
r(0)=0, \quad x\in\Omega.
\end{gathered} \tag{4.4}$$

Taking the scalar product of (4.4) with r, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|r\|^2+k\|\nabla r\|^2)+\gamma\|\Delta r\|^2=\Big(\Delta(\frac{\varphi(u_h)-\varphi(u)}{h}-\varphi'(u)z),r\Big).$$

Noticing that

$$\begin{aligned}
&\Big(\Delta(\frac{\varphi(u_h)-\varphi(u)}{h}-\varphi'(u)z),r\Big)\\
&=\Big(\frac{\varphi(u_h)-\varphi(u)}{h}-\varphi'(u)z,\Delta r\Big)\\
&\leq\|\frac{\varphi(u_h)-\varphi(u)}{h}-\varphi'(u)z\|\|\Delta r\|\\
&=\|\varphi'(u+\theta(u_h-u))\frac{u_h-u}{h}-\varphi'(u)z\|\|\Delta r\|\\
&\leq\frac{\gamma}{2}\|\Delta r\|^2+c_4'\|\varphi'(u+\theta(u_h-u))\frac{u_h-u}{h}-\varphi'(u)z\|^2,
\end{aligned}$$

where $\theta\in(0,1)$. We have $u_h \to u$ strongly in $C(0,T;H_0^1)$ as $h \to 0$, then

$$\begin{aligned}
&\|\varphi'(u+\theta(u_h-u))\frac{u_h-u}{h}-\varphi'(u)z\|^2\\
&\to\|\varphi'(u)(\frac{u_h-u}{h}-z)\|^2\\
&\leq c_5'\|r\|^2 \quad \text{as } h\to 0.
\end{aligned}$$

Therefore,

$$\Big(\Delta(\frac{\varphi(u_h)-\varphi(u)}{h}-\varphi'(u)z),r\Big)\leq\frac{\gamma}{2}\|\Delta r\|^2+c_4'c_5'\|r\|^2.$$

Summing up, we obtain

$$\frac{d}{dt}(\|r\|^2 + k\|\nabla r\|^2) + \gamma\|\Delta r\|^2 \le 2c_4'c_5'(\|r\|^2 + k\|\nabla r\|^2).$$

Using Gronwall's inequality, it is easy to check that $\frac{u_h-u}{h}$ is strongly convergent to z in $W(0,T;V)$. Then, Lemma 4.1 is proved. $\square$

As in [9], we denote the Λ the canonical isomorphism of S onto S^*, where S^* is the dual spaces of S. By calculating the Gateaux derivative of (3.2) via Lemma 4.1, we see that the cost $J(v)$ is weakly Gateaux differentiable at w in the direction $v-w$. Therefore, (4.1) can be rewritten as

$$(C^*\Lambda\,(Cu(w)-z_d), z)_{W(V)^*,W(V)} + \frac{\delta}{2}(w, v-w)_{L^2(Q_0)} \ge 0, \quad \forall v \in L^2(Q_0), \tag{4.5}$$

where z is the solution of (4.2).

Now, we study the necessary conditions of optimality. To avoid the complexity of observation states, we consider the two types of distributive and terminal value observations.

Case 1. $C \in \mathcal{L}(L^2(0,T;V);S)$. In this case, $C^* \in \mathcal{L}(S^*;L^2(0,T;V^*))$, (4.5) may be written as

$$\int_0^T (C^*\Lambda(Cu(w)-z_d), z)_{V^*,V}\,dt + \frac{\delta}{2}(w, v-w)_{L^2(Q_0)} \ge 0, \quad \forall v \in L^2(Q_0). \tag{4.6}$$

We introduce the adjoint state $p(v)$ by

$$\begin{gathered}
-\frac{d}{dt}[p(v) + k\Delta p(v)] + \gamma\Delta^2 p(v) - \varphi'(u(v))\Delta p(v) = C^*\Lambda(Cu(v)-z_d),\ (x,t) \in Q,\\
p(v) = \Delta p(v) = 0, \quad x \in \partial\Omega,\\
p(x,T;v) = 0.
\end{gathered} \tag{4.7}$$

According to Theorem 2.2, the above problem admits a unique solution (after changing t into $T-t$).

Multiplying both sides of (4.7) (with $v = w$) by z, using Lemma 4.1, we obtain

$$\begin{gathered}
\int_0^T \Big(-\frac{d}{dt}p(w), z\Big)_{V^*,V} dt = \int_0^T \Big(p(w), \frac{d}{dt}z\Big)dt,\\
\int_0^T \Big(-\frac{d}{dt}\Delta p(w), z\Big)_{V^*,V} dt = \int_0^T \Big(p(w), \frac{d}{dt}\Delta z\Big)dt,\\
\int_0^T \Big(\Delta^2 p(w), z\Big)_{V^*,V} dt = \int_0^T (p(w), \Delta^2 z)dt,\\
\int_0^T (\varphi'(u(w))\Delta p(w), z)_{V^*,V}\,dt = \int_0^T \big(p(w), \Delta(\varphi'(u(w))z)\big)\,dt
\end{gathered}$$

Thus, we obtain

$$\begin{aligned}
&\int_0^T (C^*\Lambda(Cu(w)-z_d), z)_{V^*,V}\,dt\\
&= \int_0^T \Big(p(w), \frac{d}{dt}(z + k\Delta z) + \gamma\Delta^2 z - \Delta(\varphi'(u)z)x\Big)dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T (p(w), Bv - Bw)dt \\
&= (B^* p(w), v - w).
\end{aligned}
$$

Therefore, (4.6) may be written as

$$
\int_0^T \int_0^1 B^* p(w)(v - w)\,dx\,dt + \frac{\delta}{2}(w, v - w)_{L^2(Q_0)} \geq 0, \quad \forall v \in L^2(Q_0). \tag{4.8}
$$

Then, we have proved the following theorem.

Theorem 4.2. *Assume that $C \in \mathcal{L}(L^2(0,T;V);S)$ and all conditions of Theorem 3.1 hold. Then, the optimal control w is characterized by the system of two PDEs and an inequality:* (3.1), (4.7) *and* (4.8).

Case 2. $C \in \mathcal{L}(H;S)$. In this case, we observe $Cu(v) = Du(T;v)$, $D \in \mathcal{L}(H;H)$. The associated cost function is

$$
J(u, v) = \|Du(T;v) - z\|_S^2 + \frac{\delta}{2}\|v\|_{L^2(Q_0)}^2, \quad \forall v \in L^2(Q_0). \tag{4.9}
$$

Then, for all $v \in L^2(Q_0)$, the optimal control w for (4.9) is characterized by

$$
(Du(T;w) - z, Du(T;v) - Du(T;w))_{V^*,V} + \frac{\delta}{2}(w, v - w)_{L^2(Q_0)} \geq 0. \tag{4.10}
$$

We introduce the adjoint state $p(v)$ by

$$
\begin{gathered}
-\frac{d}{dt}[p(v) + k\Delta p(v)] + \gamma\Delta^2 p(v) - \varphi'(u(v))\Delta p(v)x = 0, \quad (x,t) \in Q, \\
p(v) = \Delta p(v) = 0, \quad x \in \partial\Omega, \\
p(T;v) = D^*(Du(T;v) - z_d).
\end{gathered} \tag{4.11}
$$

According to Theorem 2.2, the above problem admits a unique solution (after changing t into $T - t$).

Set $v = w$ in the above equations and scalar multiply both side of the first equation of (4.11) by $u(v) - u(w)$ and integrate from 0 to T. A simple calculation shows that (4.10) is equivalent to

$$
\int_0^T \int_0^1 B^* p(w))(v - w)\,dx\,dt + \frac{\delta}{2} w, v - w)_{L^2(Q_0)} \geq 0, \quad \forall v \in L^2(Q_0). \tag{4.12}
$$

We obtain the following result.

Theorem 4.3. *Assume that $D \in \mathcal{L}(H;H)$ and all conditions of Theorem 3.1 hold. Then, the optimal control w is characterized by the system of two PDEs and an inequality:* (3.1), (4.11) *and* (4.12).

Acknowledgements. The authors would like to thank the anonymous referees and Dr. Xiaopeng Zhao for their valuable comments and suggestions about this paper.

References

[1] F. Bai, C. M. Elliott, A. Gardiner, A. M. Stuart; *The viscous Cahn-Hilliard equation. I. Computations*, Nonlinearity, 8 (1995), 131-160.

[2] R. Becker, B. Vexler; *Optimal control of the convection-diffusion equation using stabilized finite element methods*, Numer. Math., 106 (2007), 349-367.

[3] A. S. Bonfoh; *Exponential attractors for the viscous Cahn-Hilliard equation in an unbounded domain*, Int. J. Evol. Equ., 4 (2009), 113-119.

[4] A. N. Carvalho, T. Dlotko; *Dynamics of the Viscous Cahn-Hilliard equation*, J. Math. Anal. Appl., 344 (2008), 703-725.

[5] Ning Duan, Wenjie Gao; *The optimal control of a modified Swift-Hohenberg equation*, Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 155, pp. 1-12.

[6] C. M. Elliott, A. M. Stuart; *Viscous Cahn-Hilliard equation. II. Analysis*, J. Differential Equations, 128 (1996), 387-414.

[7] M. Grinfeld, A. Novick-Cohen; *The viscous Cahn-Hilliard equation: Morse decomposition and structure of the global attractor*, Trans. Amer. Math. Soc., 351(6) (1999), 2375-2406.

[8] Y. Li, J. Yin; *The viscous Cahn-Hilliard equation with periodic potentials and sources*, J. Fixed Point Theory Appl., 9 (2011), 63-84.

[9] J. L. Lions; *Optimal control of systems governed by parttial differential equations*, Springer, Berlin, 1971.

[10] C. Liu, J. Yin; *Some properties of solutions for viscous Cahn-Hilliard equation*, Northeast Math. J., 14 (1998), 455-466.

[11] C. Liu, J. Zhou, J. Yin; *A note on large time behaviour of solutions for viscous Cahn-Hilliard equation*, Acta Math. Scientia, 29B(5) (2009), 1216-1224.

[12] A. Novick-Cohen; *On the viscous Cahn-Hilliard equation.* In: Material Instabilities in Continuum Mechanics, Oxford Univ. Press, New York, 1988, 329-342.

[13] R. Rossi; *On two classes of generalized viscous Cahn-Hilliard equations*, Comm. Pure Appl. Anal., 4 (2005), 405-430.

[14] S.-U. Ryu, A. Yagi; *Optimal control of Keller-Segel equations*, J. Math. Anal. Appl., 256 (2001), 45-66.

[15] J. Simon; *Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure*, SIAM J. Math. Anal., 21(5) (1990), 1093-1117.

[16] L. Tian, C. Shen; *Optimal control of the viscous Degasperis-Procesi equation*, J. Math. Phys., 48(11) (2007), 113513.

[17] J. Yong, S. Zheng; *Feedback stabilization and optimal control for the Cahn-Hilliard equation*, Nonlinear Anal. TMA, 17 (1991), 431-444.

[18] X. Zhao, C. Liu; *Optimal control problem for viscous Cahn-Hilliard equation*, Nonlinear Anal., 74 (2011), 6348-6357.

[19] X. Zhao, C. Liu; *Optimal control for the convective Cahn-Hilliard equation in 2D case*, Appl. Math. Optim., 70 (2014), 61-82.

[20] S. M. Zheng, A. Milani; *Global attractors for singular perturbations of the Cahn-Hilliard equations*, J. Differential Equations, 209(2005), 101-139.

Ning Duan

School of Science, Jiangnan University, Wuxi 214122, China

E-mail address: `123332453@qq.com`

Xiufang Zhao

School of Science, Qiqihar University, Qiqihar 161006, China

E-mail address: `17815358@qq.com`