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A MATHEMATICAL MODEL FOR SUSPENSION BRIDGES WITH ENERGY DEPENDENT BOUNDARY CONDITIONS

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Abstract. We suggest a new mathematical model for dynamical suspension bridge with energy dependent boundary conditions. The roadway of the bridge is viewed as a long-narrow thin rectangular plate. After reducing the evolution problem corresponding to the model to a variational problem, we show that the original evolution problem admits a unique solution. Moreover, the unique solution is explicitly represented.

1. Introduction

In a recent paper, Ferrero-Gazzola [3] suggested a rectangular plate model for the description of dynamical suspension bridges. The two short edges of the plate are hinged and the remaining two edges are assumed to be free with no physical constraints. Under these boundary conditions, the problem they derived has a global variational structure. Then according to variational principles, they showed that there exists a unique solution to the evolution problem with suitable initial data. Subsequently, the same problem but not coercive was investigated by Wang in [12], where global existence and finite time blow-up of solutions for different initial data are obtained.

Several decades ago, Pugsley in his monograph [10] analyzed the theory on the oscillations of suspension bridges under lateral winds. He stated that

> in a complete bridge the wind forces are resisted partly by the elastic flexure of the deck structure in a horizontal plane and partly by gravity action induced by the cables.

In order to understand the nature and origin of various oscillations like purely vertical type and purely torsional type, Pugsley gave an explanation by regarding the deck as a flat plate and by considering the action of a lateral airstream upon a section of its length, see [10, pp. 123-125]. As the lateral wind blows harder in speed, the magnitude and frequency of the fluctuating air force on the deck will increase, and the bridge deck will gradually start to oscillate in vertical type. The behavior is more sensitive to the wind speed, and oscillations tend to change at some critical wind speed, which is usually called in literature *flutter speed*.

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About 10 years later than Pugsley [10] and Scanlan [11] pointed out that

> With such bridges natural structural vibration modes tend no longer to be simply into "bending" and "torsion" but to be fully three-dimensional in character, with components of vertical, torsional and lateral sway motion ...

The term "lateral sway motion" implies that the unsteady lateral force acting on the plate has a crucial effect on the oscillation of the bridge. For this reason, Gazzola [4] recently suggested to study models with dynamical boundary conditions in order to display the oscillations appearing in the actual suspension bridges. And indeed, it appears necessary to combine variational methods arising from energy balance with aerodynamics effects due to the lateral wind.

For a different model of suspension bridges, Bleich-McCullough-Rosecrans-Vincent [2] referred to the energies involved in the structure. They made a careful quantitative analysis of the energies, such as the kinetic energy, the potential energy and the elastic energy. Concerning the qualitative analysis on the energies, Gazzola [4] claims that the *flutter speed* should be seen as a *critical energy threshold* which, if exceeded, gives rise to uncontrolled phenomenon. Moreover, Arioli-Gazzola [1] recently pointed out that

> if the total energy increases over the critical energy threshold, then tiny torsional oscillations may suddenly become, without intermediate stages, wider oscillations.

This implies that "the total energy of the system" and "the critical energy threshold" have a direct influence on the oscillations of the plate. Therefore, it is necessary to assume that the total energy of the system and the critical energy threshold appear explicitly in the model.

Motivated by these remarks, in this paper we suggest a new mathematical model for dynamical suspension bridges. In this model, the differential equation is deduced by applying a variational principle to the energy functional $\mathscr{A}(u)$ given in [3]. However, since we mainly focus on the effects of the unsteady lateral wind acting on the edges of the bridge, the boundary conditions we established are dynamical and depend on the energy of the system and on the critical energy threshold. This paper could be considered as a first step in order to obtain more interesting and challenging results.

This article is organized as follows. In Section 2 a new mathematical model is suggested for dynamical suspension bridges. The boundary conditions we established depends on the energy and then we derive the corresponding evolution problem to be solved. An auxiliary problem is introduced in Section 3. The auxiliary problem, which admits a unique solution, plays a crucial role in solving our original problem. In Section 4, we first transfer the original problem to the type of the auxiliary problem. Then we show that there exists a unique solution to the original problem, see Theorem 4.1. Moreover, the unique solution is explicitly represented, see Theorem 4.2. Finally, we list several possible future developments about the model in the last section.

2. The new mathematical model

Assume that an open rectangular domain $\Omega = (0, \pi) \times (-l, l) \subset \mathbb{R}^2$ represents the roadway of a suspension bridge, where π is the length of the roadway and $2l$ is its width. A realistic assumption is that $2l \ll \pi$. We recall the energy functional $\mathscr{A}(u)$ given in [3] (see also [5, 7]):

$$\mathscr{A}(u) = \int_0^\infty \int_\Omega \Big(\frac{1}{2}u_t^2 - \Big(\frac{1}{2}(\Delta u)^2 + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy}) + H - \varphi u\Big)\Big)\, dx\, dy\, dt,$$

where $u = u(x, y, t)$ denotes the displacement of the plate in the vertical direction at the point (x, y) and at time $t > 0$, $\sigma \in (0, 1/2)$ is the Poisson ratio that depends on the material of the roadway of the bridges, $\int_\Omega H\, dx\, dy$ is a potential energy due to the restoring force h produced by the hangers of the suspension bridge, $\varphi = \varphi(x, y, t)$ represents an external source, such as the wind and the weight of the plate.

By applying variational principles to the energy functional $\mathscr{A}(u)$ in the function space $C_c^\infty(\Omega)$, we obtain the following fourth order equation by adding a damping term $\mu u_t (\mu \geq 0)$ representing the positive structural damping of the structure, including internal frictions

$$u_{tt} + \Delta^2 u + \mu u_t + h = \varphi, \quad \text{in } \Omega \times (0, +\infty). \tag{2.1}$$

Since the plate is very narrow (compared to its length), we may assume that the restoring elastic force due to the hangers acts on every point of the plate and has the linear form $h = ku$ with an elasticity constant $k > 0$. Then we have an initial value problem

$$\begin{gathered} u_{tt} + \Delta^2 u + \mu u_t + ku = \varphi, \quad (x, y) \in \Omega, t > 0, \\ u(x, y, 0) = u_0, \quad (x, y) \in \Omega, \\ u_t(x, y, 0) = u_1, \quad (x, y) \in \Omega, \end{gathered} \tag{2.2}$$

which is to be complemented with some suitable boundary conditions. Here, $u_0 = u_0(x, y)$ is the initial position of the plate, $u_1 = u_1(x, y)$ is the initial vertical velocity of the plate.

As well-known, the external force φ acting on the bridge inserts an energy into the structure. We denote it by

$$\mathcal{E}(t) = \int_\Omega \varphi(x, y, t)^2\, dx\, dy.$$

Assume that $\overline{E}_\mu > 0$ is given and represents the critical energy threshold (see Arioli-Gazzola [1]) above which the bridge displays self-excited oscillations and it increasingly depends on the damping parameter μ: for instance, $\overline{E}_\mu = \overline{E}_0 + c\mu$ for some $c > 0$ and $\overline{E}_0 > 0$ being the threshold of the undamped problem.

Now, we turn to set up the boundary conditions to be associated to the initial value problem (2.2). Due to several physical constraints being present on the sides of the plate, we seek the boundary conditions which describe the physical situation appearing in the actual suspension bridges. On the two short edges $x = 0$ and $x = \pi$, which are connected with the ground, we assume that they are hinged and then

$$u(x, y, t) = u_{xx}(x, y, t) = 0, \quad \text{for } (x, y) \in \{0, \pi\} \times (-l, l),\ t > 0, \tag{2.3}$$

which is the same as any other model we met. While on the other two sides $y = \pm l$, due to the continuous impact from the external forces occurring on them, it is quite delicate to choose a kind of "stationary" boundary conditions. Hence, in the present paper, we try to provide several suitable dynamical boundary conditions which describes the situation more closely.

At first, since the plate is very narrow, it is natural to assume that the cross section of the plate tends to remain straight. But due to the appearance of the torsional oscillation, the cross section cannot always be in a horizontal position. Therefore, we assume that these two boundaries satisfy

$$\begin{aligned} &u_y(x,-l,t) - u_y(x,l,t) = 0, \quad x \in (0,\pi), t > 0, \\ &u_y(x,-l,t) + u_y(x,l,t) = \alpha, \quad x \in (0,\pi),\ t > 0, \end{aligned} \tag{2.4}$$

where $\alpha = \alpha(x,t)$ is a real function, which depends on φ and u_0 for $y = \pm l$. Furthermore, for any cross section of the bridge, the vertical displacements at the two endpoints also depend on the initial position and the external forces acting on these two points. Thus, it is reasonable to assume that the sum of the two displacements fulfills

$$u(x,-l,t) + u(x,l,t) = \beta, \quad \text{for } x \in (0,\pi), t > 0, \tag{2.5}$$

where $\beta = \beta(x,t)$ is a real function, which depends on $\overline{\varphi}(x,t)$ and $\overline{u}_0(x)$ with

$$\overline{\varphi}(x,t) = \frac{\varphi(x,l,t) + \varphi(x,-l,t)}{2}, \quad \overline{u}_0(x) = \frac{u_0(x,l) + u_0(x,-l)}{2}.$$

Moreover, the vibration of the cross section will switch to some different kind of oscillations, such as the torsional oscillation, when the inserted energy $\mathcal{E}(t)$ exceeds $\overline{E}_\mu$. Therefore, as long as $\mathcal{E}(t) \le \overline{E}_\mu$, we assume that there is only vertical vibration appearing in the motion, then the difference between the two displacements should be zero, that is,

$$u(x,-l,t) - u(x,l,t) = 0, \quad \text{for } x \in (0,\pi). \tag{2.6}$$

While if $\mathcal{E}(t) > \overline{E}_\mu$, the torsional oscillation begins to arise and its amplitude increases as the energy $\mathcal{E}(t) \to \infty$, which means that the difference between the two displacements is related to the energy $\mathcal{E}(t)$. Concretely, if $\mathcal{E}(t) \downarrow \overline{E}_\mu$, then the motion tends to vertical-type so that

$$u(x,-l,t) - u(x,l,t) \to 0,$$

while if $\mathcal{E}(t) \to \infty$, then the difference also increases. But it cannot increase to infinity because the bridge will collapse earlier. Therefore, as long as $\mathcal{E}(t) > \overline{E}_\mu$, the boundaries are assumed to satisfy the equation

$$\frac{d}{dt}\Big\{[u(x,-l,t) - u(x,l,t)]\frac{\mathcal{E}(0) - \overline{E}_\mu}{\mathcal{E}(t) - \overline{E}_\mu} \exp\Big(\frac{\mathcal{E}(0)}{\overline{E}_\mu} - \frac{\mathcal{E}(t)}{\overline{E}_\mu}\Big)\Big\} = \gamma, \quad \text{for } x \in (0,\pi), \tag{2.7}$$

here we assume $\mathcal{E}(0) > \overline{E}_\mu$ and $\gamma = \gamma(x,t)$ is a real function satisfying

$$\gamma \text{ has the same sign as } (u_0(x,-l) - u_0(x,l)).$$

In this case, the conditions (2.6)-(2.7) can be combined into

$$u_t(x,-l,t) - u_t(x,l,t) - \eta(t)(u(x,-l,t) - u(x,l,t)) = \theta(t)\gamma, \quad \text{for } x \in (0,\pi), t > 0, \tag{2.8}$$

where

$$\theta(t) = \frac{(\mathcal{E}(t) - \overline{E}_\mu)^+}{\mathcal{E}(0) - \overline{E}_\mu} \exp\Big(\frac{\mathcal{E}(t)}{\overline{E}_\mu} - \frac{\mathcal{E}(0)}{\overline{E}_\mu}\Big), \tag{2.9}$$

$$\eta(t) = \mathcal{E}'(t)E(t)\Big(\frac{1}{\overline{E}_\mu} + \frac{1}{(\mathcal{E}(t) - \overline{E}_\mu)^+}\Big) \tag{2.10}$$

with $(\mathcal{E}(t)-\overline{E}_\mu)^+ = \max\{\mathcal{E}(t)-\overline{E}_\mu, 0\}$, $\mathcal{E}'(t)=\frac{d}{dt}\mathcal{E}(t)$ and

$$E(t)=\begin{cases}-1, & \mathcal{E}(t)\le \overline{E}_\mu,\\ 1, & \mathcal{E}(t)>\overline{E}_\mu.\end{cases} \tag{2.11}$$

Summarizing, the boundary conditions for a rectangular plate Ω modeling dynamical suspension bridges are (2.3)-(2.5) and (2.8), which together with (2.2) yield an evolution problem

$$\begin{gathered} u_{tt}+\Delta^2 u+\mu u_t+ku=\varphi, \quad (x,y)\in\Omega, t>0,\\ u(x,y,0)=u_0, \quad (x,y)\in\Omega,\\ u_t(x,y,0)=u_1, \quad (x,y)\in\Omega, \end{gathered} \tag{2.12}$$

with the boundary conditions: for every $t>0$,

$$\begin{gathered} u(x,y,t)=u_{xx}(x,y,t)=0, \quad (x,y)\in\{0,\pi\}\times(-l,l),\\ u_y(x,-l,t)-u_y(x,l,t)=0, \quad x\in(0,\pi),\\ u_y(x,-l,t)+u_y(x,l,t)=\alpha, \quad x\in(0,\pi),\\ u(x,-l,t)+u(x,l,t)=\beta, \quad x\in(0,\pi),\\ u_t(x,-l,t)-u_t(x,l,t)-\eta(t)(u(x,-l,t)-u(x,l,t))=\theta(t)\gamma,\ x\in(0,\pi). \end{gathered} \tag{2.13}$$

The dynamical boundary conditions (2.13) are energy dependent boundary conditions since the energy $\mathcal{E}(t)$ appears explicitly.

3. An Auxiliary Problem

In this section we give an auxiliary problem which plays a crucial role in solving the original problem (2.12)-(2.13). We first introduce a subspace of $H^2(\Omega)$ denoted by

$$V:=\{u\in H^2(\Omega): u=0 \text{ on } \partial\Omega \text{ and } u_y=0 \text{ on } (0,\pi)\times\{-l,l\}\}.$$

Clearly, $H_0^2(\Omega)\subset V\subset H_0^1(\Omega)\cap H^2(\Omega)$. Hence, one may define a scalar product on the space V by

$$(u,v)_V=\int_\Omega \Delta u\Delta v\,dx\,dy, \quad \text{for any } u,v\in V,$$

which induces the norm

$$\|u\|_V=\Big(\int_\Omega |\Delta u|^2\,dx\,dy\Big)^{1/2}, \quad \text{for all } u\in V.$$

Now we consider the nonhomogeneous linear problem

$$\begin{gathered} v_{tt}+\Delta^2 v+\mu v_t+kv=f, \quad (x,y)\in\Omega, t>0,\\ v(x,y,t)=v_{xx}(x,y,t)=0, \quad (x,y)\in\{0,\pi\}\times(-l,l), t>0,\\ v(x,y,t)=v_y(x,y,t)=0, \quad (x,y)\in(0,\pi)\times\{-l,l\}, t>0,\\ v(x,y,0)=v_0, \quad (x,y)\in\Omega,\\ v_t(x,y,0)=v_1, \quad (x,y)\in\Omega, \end{gathered} \tag{3.1}$$

where $f=f(x,y,t)\in C^0([0,\infty);L^2(\Omega))$, $v_0=v_0(x,y)\in V$ and $v_1=v_1(x,y)\in L^2(\Omega)$.

Claim. Problem (3.1) admits a unique weak solution

$$v\in C^0([0,\infty);V)\cap C^1([0,\infty);L^2(\Omega)).$$

Actually, the variational problem (3.1) is very analogous to [3, problem (22)] if we take the nonlinear term $h = kv$. The only difference lies in the boundary conditions on $(0, \pi) \times \{-l, l\}$, but this has no influence when we solve it by following the procedure in [3]. Hence, we only list the key steps, for details see [3, Section 8].

Step 1. Consider the approximated problems for $m \geq 1$

$$\begin{gathered} v_m'' + Lv_m + \mu v_m' + P_m(kv_m) = P_m(f), \quad t \in [0, \tau_m) \\ v(0) = v_0^m, \quad v'(0) = v_1^m \end{gathered} \tag{3.2}$$

where L is defined by $\langle Lu, v \rangle := (u, v)_V$ for any $u, v \in V$, $P_m : V \to W_m$ is the orthogonal projection onto W_m, which is spanned by the eigenfunctions $\{w_m\}_{m \geq 1}$ of the problem

$$\begin{gathered} \Delta^2 w = \lambda w, \quad (x, y) \in \Omega, \\ w(x, y) = w_{xx}(x, y) = 0, \quad (x, y) \in \{0, \pi\} \times (-l, l), \\ w(x, y) = w_y(x, y) = 0, \quad (x, y) \in (0, \pi) \times \{-l, l\}. \end{gathered}$$

By Galerkin-type procedure, we obtain that problem (3.2) has a unique local solution $v_m \in C^2([0, \tau_m); V)$, where $[0, \tau_m)$ is the maximal interval of the continuation of v_m.

Step 2. The solution sequence $\{v_m\}$ is uniform bounded.

Indeed, testing (3.2) with v_m' and integrating over $(0, t)$, we have by several estimates

$$\|v_m\|_V^2 + \|v_m'\|_{L^2}^2 \leq C \quad \text{for any } t \in [0, \tau_m) \text{ and } m \geq 1,$$

where C is independent of m and t. Hence, the solution v_m is globally defined in $[0, \infty)$ and the sequence $\{v_m\}$ is bounded in $C^0([0, \infty); V) \cap C^1([0, \infty); L^2(\Omega))$.

Step 3. $\{v_m\}$ admits a strongly convergent subsequence in the set $C^0([0, \infty); V) \cap C^1([0, \infty); L^2(\Omega))$.

By Ascoli-Arzelà Theorem, we deduce that, up to subsequences, there exists a function $v \in C^0([0, \infty); L^2(\Omega))$ such that $v_m \to v$ strongly in $C^0([0, \infty); L^2(\Omega))$.

On the other hand, the sequence $\{v_m\}$ is a Cauchy sequence in the space $C^0([0, \infty); V) \cap C^1([0, \infty); L^2(\Omega))$. Hence, up to subsequences,

$$v_m \to v \quad \text{in } C^0([0, \infty); V) \cap C^1([0, \infty); L^2(\Omega)) \quad \text{as } m \to \infty.$$

Step 4. Take the limit in (3.2) and then we prove the existence of solution to (3.1).

Step 5. The solution to (3.1) is unique.

Assume that v_1, v_2 are two solutions of (3.1), denote $v = v_1 - v_2$. Then using v' as a test function, we obtain after integration over $(0, t)$

$$\|v'\|_{L^2}^2 + \|v\|_V^2 = -2\mu \int_0^t \|v'(s)\|_{L^2}^2 ds \leq 0,$$

from which it immediately follows that $v = 0$. Therefore, the problem (3.1) admits a unique weak solution $v \in C^0([0, \infty); V) \cap C^1([0, \infty); L^2(\Omega))$.

4. Main Results

Due to the dynamical boundary conditions (2.13), it is not easy to solve the original problem (2.12)-(2.13) directly. Hence, we first transfer it to a simpler case, which uses the auxiliary problem. Recalling the boundary conditions (2.4), we have

$$u_y(x,l,t) = u_y(x,-l,t) = \alpha/2, \quad \text{for any } x \in (0,\pi),\ t > 0.$$

Moreover, let β be a C^1 function in t, then we obtain by (2.5) and (2.8) for any $x \in (0,\pi)$

$$\begin{gathered} u_t(x,l,t) - \eta(t)u(x,l,t) = g_1(x,t), \quad t > 0, \\ u(x,l,0) = u_0(x,l), \end{gathered}$$

and

$$\begin{gathered} u_t(x,-l,t) - \eta(t)u(x,-l,t) = g_2(x,t), \quad t > 0, \\ u(x,-l,0) = u_0(x,-l), \end{gathered}$$

where $g_1(x,t) = \frac{1}{2}(\beta_t(x,t) - \eta(t)\beta(x,t)) - \frac{1}{2}\theta(t)\gamma(x,t)$ and $g_2(x,t) = \frac{1}{2}(\beta_t(x,t) - \eta(t)\beta(x,t)) + \frac{1}{2}\theta(t)\gamma(x,t)$. For any fixed $x \in (0,\pi)$, they are first order ordinary differential problems with respect to t, and then one gets the explicit representation of the values of u on $(0,\pi) \times \{-l,l\}$, which we denote by $(h_1,h_2) = (u(x,l,t), u(x,-l,t))$. Then the problem (2.12)-(2.13) reduces to an evolution linear problem

$$\begin{gathered} u_{tt} + \Delta^2 u + \mu u_t + ku = \varphi, \quad (x,y) \in \Omega,\ t > 0, \\ u(x,y,0) = u_0, \quad (x,y) \in \Omega, \\ u_t(x,y,0) = u_1, \quad (x,y) \in \Omega \end{gathered} \tag{4.1}$$

with nonhomogeneous boundary conditions

$$\begin{gathered} u(x,y,t) = u_{xx}(x,y,t) = 0, \quad (x,y) \in \{0,\pi\} \times (-l,l),\ t > 0, \\ u(x,y,t) = h_1, \quad u_y(x,y,t) = \alpha/2, \quad (x,y) \in (0,\pi) \times \{l\},\ t > 0, \\ u(x,y,t) = h_2, \quad u_y(x,y,t) = \alpha/2, \quad (x,y) \in (0,\pi) \times \{-l\},\ t > 0. \end{gathered} \tag{4.2}$$

Now we want to reduce the boundary conditions (4.2) to the homogeneous case. To this end, it is necessary to construct a suitable inverse trace operator. Note that $\Omega \subset \mathbb{R}^2$ is a rectangular domain. Let us define the space

$$H^4_*(\Omega) := \{u \in H^4(\Omega) : u = u_{xx} = 0 \text{ on } \{0,\pi\} \times (-l,l)\}$$

and a continuous map

$$\begin{gathered} T : H^4_*(\Omega) \to \prod_{i=1}^{2} \Big(H^{7/2}(\Sigma_i) \times H^{5/2}(\Sigma_i) \times H^{3/2}(\Sigma_i) \times H^{1/2}(\Sigma_i) \Big) \\ \phi \mapsto \prod_{i=1}^{2} \big(\phi_{|\Sigma_i}, (\phi_y)_{|\Sigma_i}, (\phi_{yy})_{|\Sigma_i}, (\phi_{yyy})_{|\Sigma_i} \big) \end{gathered}$$

where $\Sigma_1 := (0,\pi) \times \{l\}$, $\Sigma_2 := (0,\pi) \times \{-l\}$. For more details, see [8, Section 2.5] or [6, Chapter 1].

Denote the range of T by $R(T) := T(H^4_*(\Omega))$ and define the following norm on $R(T)$,

$$\|v\|_{R(T)} := \inf\{\|w\|_{H^4(\Omega)} : w \in H^4_*(\Omega), T(w) = v\} \quad \text{for any } v \in R(T),$$

then $R(T)$ is a Banach space with the norm $\|\cdot\|_{R(T)}$. Therefore, the restriction of the previous map T to $(\ker(T))^{\perp}$, i.e, $T_{|(\ker(T))^{\perp}} : (\ker(T))^{\perp} \to R(T)$ is an isometric isomorphism.

Since $R(T) \subset \prod_{i=1}^{2} \big(H^{7/2}(\Sigma_i) \times H^{5/2}(\Sigma_i) \times H^{3/2}(\Sigma_i) \times H^{1/2}(\Sigma_i)\big)$, one may represent T as (T_1, T_2, T_3, T_4) with

$$
\begin{gathered}
T_1 : H^4_*(\Omega) \to \prod_{i=1}^{2} H^{7/2}(\Sigma_i), \quad T_2 : H^4_*(\Omega) \to \prod_{i=1}^{2} H^{5/2}(\Sigma_i), \\
T_3 : H^4_*(\Omega) \to \prod_{i=1}^{2} H^{3/2}(\Sigma_i), \quad T_4 : H^4_*(\Omega) \to \prod_{i=1}^{2} H^{1/2}(\Sigma_i),
\end{gathered}
$$

which are continuous maps from $H^4_*(\Omega)$ to the respective spaces each of them endowed with its normal norm. Then one may define the four subspaces

$$
\begin{aligned}
V_1 &:= \{v \in R(T) : v = (T_1(w), 0, 0, 0), w \in H^4_*(\Omega)\}, \\
V_2 &:= \{v \in R(T) : v = (0, T_2(w), 0, 0), w \in H^4_*(\Omega)\}, \\
V_3 &:= \{v \in R(T) : v = (0, 0, T_3(w), 0), w \in H^4_*(\Omega)\}, \\
V_4 &:= \{v \in R(T) : v = (0, 0, 0, T_4(w)), w \in H^4_*(\Omega)\}.
\end{aligned}
$$

Concerning the fact that $T_{|(\ker(T))^{\perp}}$ is an isometric isomorphism and the continuity of the map T, one can show that V_i $(i = 1, 2, 3, 4)$ are closed in $R(T)$ endowed with $\|\cdot\|_{R(T)}$.

Now, let $\alpha \in C^2([0,\infty); V_2)$, $\beta \in C^1([0,\infty); V_1)$, $\gamma \in C^0([0,\infty); V_1)$ and h_1, h_2 be in $C^2([0,\infty); V_1)$. Then one may define the map

$$
w := T^{-1}_{|(\ker T)^{\perp}} \left((h_1, \alpha/2, 0, 0) \times (h_2, \alpha/2, 0, 0)\right)
$$

with $w = h_i$, $w_y = \alpha/2$, $w_{yy} = 0$ and $w_{yyy} = 0$ on the boundary Σ_i. In this way we have that $w \in C^2([0,\infty); H^4_*(\Omega))$. Then putting $u = v + w$ (v to be fixed) into the problem (4.1)-(4.2), we obtain the following variational problem

$$
\begin{gathered}
v_{tt} + \Delta^2 v + \mu v_t + k v = \widetilde{\varphi}, \quad (x,y) \in \Omega,\ t > 0, \\
v(x,y,t) = v_{xx}(x,y,t) = 0, \quad (x,y) \in \{0,\pi\} \times (-l,l),\ t > 0, \\
v(x,y,t) = v_y(x,y,t) = 0, \quad (x,y) \in (0,\pi) \times \{-l,l\},\ t > 0, \\
v(x,y,0) = v_0, \quad (x,y) \in \Omega, \\
v_t(x,y,0) = v_1, \quad (x,y) \in \Omega,
\end{gathered} \tag{4.3}
$$

where $\widetilde{\varphi} = \varphi - w_{tt} - \Delta^2 w - \mu w_t - k w$, $v_0 = u_0(x,y) - w(x,y,0)$ and $v_1 = u_1(x,y) - w_t(x,y,0)$.

Recalling the function space

$$
H^2_*(\Omega) := \{u \in H^2(\Omega) : u = 0 \text{ on } \{0,\pi\} \times (-l,l)\},
$$

which is defined in [3], we have the uniqueness result.

Theorem 4.1. *Assume that $\varphi \in C^0([0,\infty); L^2(\Omega))$, $u_0 \in H^2_*(\Omega)$ and $u_1 \in L^2(\Omega)$. Then there exists a unique solution to the problem* (2.12)-(2.13).

Proof. It is easy to see that the functions $\widetilde{\varphi}$, v_0 and v_1 satisfy

$$
\widetilde{\varphi} \in C^0([0,\infty); L^2(\Omega)), \quad v_0 \in V, \quad v_1 \in L^2(\Omega).
$$

Hence, we obtain from Section 3 that the problem (4.3) has a unique solution

$$v \in C^0([0,\infty);V) \cap C^1([0,\infty);L^2(\Omega)).$$

Then, according to the arguments above,

$$u = v + w \in C^0([0,\infty);H^2_*(\Omega)) \cap C^1([0,\infty);L^2(\Omega))$$

is the unique solution of the original problem (2.12)-(2.13) with the initial conditions $u(x,y,0) = u_0$, $u_t(x,y,0) = u_1$ and the boundary conditions (2.3), (2.4), (2.5), (2.8). □

Since (2.12)-(2.13) is a linear problem, it is possible to find an explicit form of the unique solution. We first consider an initial value problem for $S = S(t)$

$$\begin{gathered} S'' + \mu S' + aS = f(t), \quad t > 0, \\ S(0) = A, \quad S'(0) = B \end{gathered} \tag{4.4}$$

where $f(t)$ is a given function, μ and a are positive constants, A, B are constants.

We know that (4.4) is a second-order ordinary differential problem. According to the eigenvalue method, we have the following three cases if we denote $\delta = \mu^2 - 4a$:

Case 1. If $\delta < 0$, then the two eigenvalues are $\lambda_1 = -\frac{\mu}{2} + \frac{\sqrt{-\delta}}{2} i$ and $\lambda_2 = -\frac{\mu}{2} - \frac{\sqrt{-\delta}}{2} i$. Hence, we have

$$\begin{aligned} S(t) &= \exp\big(-\frac{\mu}{2}t\big)\cos\Big(\frac{\sqrt{-\delta}}{2}t\Big)\Big(A + \frac{2}{\sqrt{-\delta}}\int_0^t \exp\big(\frac{\mu}{2}s\big)\cos\Big(\frac{\sqrt{-\delta}}{2}s\Big)f(s)ds\Big) \\ &\quad - \frac{2}{\sqrt{-\delta}}\exp\big(-\frac{\mu}{2}t\big)\sin\Big(\frac{\sqrt{-\delta}}{2}t\Big)\Big(B + \frac{\mu}{2}A \\ &\quad + \int_0^t \exp\big(\frac{\mu}{2}s\big)\sin\Big(\frac{\sqrt{-\delta}}{2}s\Big)f(s)ds\Big); \end{aligned}$$

Case 2. If $\delta = 0$, then the two eigenvalues are $\lambda_1 = \lambda_2 = -\frac{\mu}{2}$. Therefore,

$$S(t) = \Big(A + \Big(B + \frac{\mu}{2}A\Big)t\Big)\exp\big(-\frac{\mu}{2}t\big) - \exp\big(-\frac{\mu}{2}t\big)\int_0^t (s-t)\exp\big(\frac{\mu}{2}s\big)f(s)ds;$$

Case 3. If $\delta > 0$, then the two eigenvalues are $\lambda_1 = -\frac{\mu}{2} + \frac{\sqrt{\delta}}{2}$ and $\lambda_2 = -\frac{\mu}{2} - \frac{\sqrt{\delta}}{2}$. Therefore,

$$\begin{aligned} S(t) &= \exp\Big(\Big(-\frac{\mu}{2} + \frac{\sqrt{\delta}}{2}\Big)t\Big)\Big(\frac{\mu + \sqrt{\delta}}{2\sqrt{\delta}}A + \frac{1}{\sqrt{\delta}}B \\ &\quad + \frac{1}{\sqrt{\delta}}\int_0^t \exp\Big(-\Big(-\frac{\mu}{2} + \frac{\sqrt{\delta}}{2}\Big)s\Big)f(s)ds\Big) \\ &\quad + \exp\Big(\Big(-\frac{\mu}{2} - \frac{\sqrt{\delta}}{2}\Big)t\Big)\Big(\frac{\sqrt{\delta} - \mu}{2\sqrt{\delta}}A \\ &\quad - \frac{1}{\sqrt{\delta}}B - \frac{1}{\sqrt{\delta}}\int_0^t \exp\Big(-\Big(-\frac{\mu}{2} - \frac{\sqrt{\delta}}{2}\Big)s\Big)f(s)ds\Big). \end{aligned}$$

To obtain a Fourier series, we introduce the following

$$\begin{aligned} u_{0m}(y) &= \frac{2}{\pi}\int_0^\pi u_0(x,y)\sin(mx)dx, \quad u_{1m}(y) = \frac{2}{\pi}\int_0^\pi u_1(x,y)\sin(mx)dx, \\ \alpha_m(t) &= \frac{2}{\pi}\int_0^\pi \alpha(x,t)\sin(mx)dx, \quad \beta_m(t) = \frac{2}{\pi}\int_0^\pi \beta(x,t)\sin(mx)dx \\ \gamma_m(t) &= \frac{2}{\pi}\int_0^\pi \gamma(x,t)\sin(mx)dx, \quad \varphi_m(y,t) = \frac{2}{\pi}\int_0^\pi \varphi(x,y,t)\sin(mx)dx \end{aligned} \tag{4.5}$$

and define a function by

$$R_m(t) = \theta(t)\Big(\frac{u_{0m}(l) - u_{0m}(-l)}{2l} - \int_0^t \frac{\gamma_m(s)}{2l}ds\Big), \quad t > 0, \tag{4.6}$$

here $\theta(t)$ is as in (2.9).

Then the unique solution to the original problem (2.12)-(2.13) can be explicitly represented.

Theorem 4.2. *Assume that the functions $\varphi \in C^0([0,\infty); L^2(\Omega))$, $u_0 \in H_*^2(\Omega)$ and $u_1 \in L^2(\Omega)$. Then the unique solution to (2.12)-(2.13) is given by*

$$u(x,y,t) = \sum_{m=1}^\infty U_m(y,t)\sin(mx) \tag{4.7}$$

with

$$U_m(y,t) = \sum_{n=1}^\infty (T_m)_n(t)\sin\Big(\frac{n\pi}{l}y\Big) + \sum_{n=1}^\infty (S_m)_n(t)\cos\Big(\frac{n\pi}{l}y\Big) + C_m(t)y + D_m(t), \tag{4.8}$$

where $(T_m)_n(t)$, $(S_m)_n(t)$ and $D_m(t)$ are in the form of $S(t)$, the solution of (4.4), $C_m(t)$ is as in (4.6).

Proof. According to the boundary conditions for $x = 0, \pi$, we seek the solution u of the problem (2.12)-(2.13) in the form

$$u(x,y,t) = \sum_{m=1}^\infty U_m(y,t)\sin(mx). \tag{4.9}$$

Inserting (4.9) in (2.12)-(2.13) and recalling (4.5), for every $m \geq 1$, we have

$$\begin{gathered} (U_m)_{tt} + (U_m)_{yyyy} - 2m^2(U_m)_{yy} + (m^4+k)U_m + \mu(U_m)_t \\ = \varphi_m(y,t), \quad y \in (-l,l),\ t > 0, \\ U_m(y,0) = u_{0m}(y), \quad y \in (-l,l), \\ (U_m)_t(y,0) = u_{1m}(y), \quad y \in (-l,l), \end{gathered} \tag{4.10}$$

with the boundary conditions

$$\begin{gathered} (U_m)_y(-l,t) - (U_m)_y(l,t) = 0, \quad t > 0, \\ (U_m)_y(-l,t) + (U_m)_y(l,t) = \alpha_m(t), \quad t > 0, \\ U_m(-l,t) + U_m(l,t) = \beta_m(t), \quad t > 0, \\ (U_m)_t(-l,t) - (U_m)_t(l,t) - \eta(t)(U_m(-l,t) - U_m(l,t)) = \gamma_m(t)\theta(t), \quad t > 0. \end{gathered} \tag{4.11}$$

Now, we look for the solution to (4.10)-(4.11) in the form of (4.8); that is,

$$U_m(y,t)=\sum_{n=1}^{\infty}(T_m)_n(t)\sin\Big(\frac{n\pi}{l}y\Big)+\sum_{n=1}^{\infty}(S_m)_n(t)\cos\Big(\frac{n\pi}{l}y\Big)+C_m(t)y+D_m(t).$$

Putting it into the equation in (4.10), we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}\Big((T_m)_n''(t)+\mu(T_m)_n'(t)+\Big(\Big(m^2+\Big(\frac{n\pi}{l}\Big)^2\Big)^2+k\Big)(T_m)_n(t)\Big)\sin\big(\frac{n\pi}{l}y\big)\\
&+\sum_{n=1}^{\infty}\Big((S_m)_n''(t)+\mu(S_m)_n'(t)+\Big(\Big(m^2+\big(\frac{n\pi}{l}\big)^2\Big)^2+k\Big)(S_m)_n(t)\Big)\cos\big(\frac{n\pi}{l}y\big)\\
&+C_m''(t)y+\mu C_m'(t)y+(m^4+k)C_m(t)y+D_m''(t)+\mu D_m'(t)+(m^4+k)D_m(t)\\
&=\varphi_m(y,t).
\end{aligned}$$

For $C_m(t)$ being as in the form in (4.6), we define the function

$$\phi_m(y,t)=C_m''(t)y+(m^4+k+\mu\eta(t))C_m(t)y-\frac{\mu\gamma_m(t)\theta(t)}{2l}y$$

and let $\varphi_m(y,t)=\varphi_m(y,t)-\phi_m(y,t)+\phi_m(y,t)$, then it follows that

$$C_m'(t)-\eta(t)C_m(t)=-\frac{\gamma_m(t)\theta(t)}{2l}, \tag{4.12}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty}\Big((T_m)_n''(t)+\mu(T_m)_n'(t)+\Big(\Big(m^2+\big(\frac{n\pi}{l}\big)^2\Big)^2+k\Big)(T_m)_n(t)\Big)\sin\big(\frac{n\pi}{l}y\big)\\
&+\sum_{n=1}^{\infty}\left((S_m)_n''(t)+\mu(S_m)_n'(t)+\left(\left(m^2+\Big(\frac{n\pi}{l}\Big)^2\right)^2+k\right)(S_m)_n(t)\right)\cos\Big(\frac{n\pi}{l}y\Big)\\
&\quad+D_m''(t)+\mu D_m'(t)+(m^4+k)D_m(t)\\
&=\sum_{n=1}^{\infty}\left(\varphi_{1m}\right)_n(t)\sin\Big(\frac{n\pi}{l}y\Big)+\sum_{n=1}^{\infty}\left(\varphi_{2m}\right)_n(t)\cos\Big(\frac{n\pi}{l}y\Big)+\varphi_m(t),
\end{aligned}\tag{4.13}$$

where

$$\begin{gathered}
\left(\varphi_{1m}\right)_n(t)=\frac{1}{l}\int_{-l}^{l}(\varphi_m(y,t)-\phi_m(y,t))\sin\Big(\frac{n\pi}{l}y\Big)dy,\\
\left(\varphi_{2m}\right)_n(t)=\frac{1}{l}\int_{-l}^{l}(\varphi_m(y,t)-\phi_m(y,t))\cos\Big(\frac{n\pi}{l}y\Big)dy,\\
\varphi_m(t)=\frac{1}{2l}\int_{-l}^{l}\varphi_m(y,t)dy.
\end{gathered}$$

The initial conditions yield

$$
\begin{aligned}
&\sum_{n=1}^{\infty}(T_m)_n(0)\sin\Big(\frac{n\pi}{l}y\Big)+\sum_{n=1}^{\infty}(S_m)_n(0)\cos\Big(\frac{n\pi}{l}y\Big)+D_m(0)\\
&=\sum_{n=1}^{\infty}(A_{1m})_n\sin\Big(\frac{n\pi}{l}y\Big)+\sum_{n=1}^{\infty}(A_{2m})_n\cos\Big(\frac{n\pi}{l}y\Big)+A_m,\quad y\in(-l,l),\\
&\sum_{n=1}^{\infty}(T_m)'_n(0)\sin\Big(\frac{n\pi}{l}y\Big)+\sum_{n=1}^{\infty}(S_m)'_n(0)\cos\Big(\frac{n\pi}{l}y\Big)+D'_m(0)\\
&=\sum_{n=1}^{\infty}(B_{1m})_n\sin\Big(\frac{n\pi}{l}y\Big)+\sum_{n=1}^{\infty}(B_{2m})_n\cos\Big(\frac{n\pi}{l}y\Big)+B_m,\quad y\in(-l,l),
\end{aligned}
\tag{4.14}
$$

where

$$
\begin{gathered}
(A_{1m})_n=\frac{1}{l}\int_{-l}^{l}(u_{0m}(y)-C_m(0)y)\sin\Big(\frac{n\pi}{l}y\Big)dy,\\
(A_{2m})_n=\frac{1}{l}\int_{-l}^{l}(u_{0m}(y))\cos\Big(\frac{n\pi}{l}y\Big)dy,\\
(B_{1m})_n=\frac{1}{l}\int_{-l}^{l}(u_{1m}(y)-C'_m(0)y)\sin\Big(\frac{n\pi}{l}y\Big)dy,\\
(B_{2m})_n=\frac{1}{l}\int_{-l}^{l}(u_{1m}(y))\cos\Big(\frac{n\pi}{l}y\Big)dy,\\
A_m=\frac{1}{2l}\int_{-l}^{l}u_{0m}(y)dy,\quad B_m=\frac{1}{2l}\int_{-l}^{l}u_{1m}(y)dy.
\end{gathered}
$$

Assume that the functions $\alpha_m(t)$, $\beta_m(t)$ and $\gamma_m(t)$ are given by

$$
\begin{gathered}
\alpha_m(t)=2\sum_{n=1}^{\infty}\frac{n\pi}{l}(T_m)_n(t)\cos(n\pi),\quad t>0,\\
\beta_m(t)=2\sum_{n=1}^{\infty}(S_m)_n(t)\cos(n\pi)+2D_m(t),\quad t>0,\\
\gamma_m(t)=-2l(C'_m(t)-\eta(t)C_m(t))/\theta(t),\quad t>0,
\end{gathered}
\tag{4.15}
$$

then the boundary conditions are satisfied. Moreover, we are led to the several ordinary differential problems

$$
\begin{gathered}
C'_m(t)-\eta(t)C_m(t)=-\frac{\gamma_m(t)\theta(t)}{2l},\quad t>0,\\
C_m(t)=\frac{u_{0m}(l)-u_{0m}(-l)}{2l},\quad t=0\,;
\end{gathered}
\tag{4.16}
$$

$$
\begin{gathered}
D''_m(t)+\mu D'_m(t)+(m^4+k)D_m(t)=\varphi_m(t),\quad t>0,\\
D_m(t)=A_m,\quad t=0,\\
D'_m(t)=B_m,\quad t=0\,;
\end{gathered}
\tag{4.17}
$$

$$\begin{gathered}(T_m)''_n(t)+\mu(T_m)'_n(t)+\Big((m^2+\Big(\frac{n\pi}{l}\Big)^2)^2+k\Big)(T_m)_n(t)=(\varphi_{1m})_n\,(t),\quad t>0,\\ (T_m)_n(t)=(A_{1m})_n,\quad t=0,\\ (T_m)'_n(t)=(B_{1m})_n,\quad t=0\,;\end{gathered}\tag{4.18}$$

$$\begin{gathered}(S_m)''_n(t)+\mu(S_m)'_n(t)+\Big((m^2+\Big(\frac{n\pi}{l}\Big)^2)^2+k\Big)(S_m)_n(t)=(\varphi_{2m})_n\,(t),\quad t>0,\\ (S_m)_n(t)=(A_{2m})_n,\quad t=0,\\ (S_m)'_n(t)=(B_{2m})_n,\quad t=0.\end{gathered}\tag{4.19}$$

To complete the proof, we need to check that problem (4.16) admits a solution in the form $R_m(t)$ and that the problems (4.17)-(4.19) have solutions in the form of $S(t)$. Since the last three problems depend on $C_m(t)$, we first consider the problem (4.16). In fact, this is a first order ordinary differential problem and we know that the solution is

$$\begin{aligned}C_m(t)&=C_m(0)\exp\Big(\int_0^t\eta(s)ds\Big)\\ &\quad-\exp\Big(\int_0^t\eta(s)ds\Big)\int_0^t\frac{\gamma_m(s)\theta(s)}{2l}\exp\Big(-\int_0^s\eta(\tau)d\tau\Big)ds.\end{aligned}$$

Since $\eta(t)$ is not a continuous function, see (2.10), to compute conveniently, we denote $E(t)$ given in (2.11) here by

$$E(t)=\begin{cases}-1, & \mathcal{E}(t)\le\overline{E}_\mu-\varepsilon,\\ \frac{1}{\varepsilon}\left(\mathcal{E}(t)-\overline{E}_\mu\right), & \overline{E}_\mu-\varepsilon<\mathcal{E}(t)\le\overline{E}_\mu+\varepsilon,\\ 1, & \mathcal{E}(t)>\overline{E}_\mu+\varepsilon.\end{cases}\tag{4.20}$$

Then we consider the exponential function $\exp\big(\int_0^t\eta(s)ds\big)$. There are three cases:

(1) If $\mathcal{E}(t)>\overline{E}_\mu+\varepsilon$, then

$$\exp\Big(\int_0^t\eta(s)ds\Big)=\exp\Big(\frac{\mathcal{E}(t)}{\overline{E}_\mu}-\frac{\mathcal{E}(0)}{\overline{E}_\mu}+\ln\frac{\mathcal{E}(t)-\overline{E}_\mu}{\mathcal{E}(0)-\overline{E}_\mu}\Big);$$

(2) If $\overline{E}_\mu<\mathcal{E}(t)\le\overline{E}_\mu+\varepsilon$, then

$$\begin{aligned}&\exp\Big(\int_0^t\eta(s)ds\Big)\\ &=\exp\Big(\frac{(\mathcal{E}(t)-\overline{E}_\mu)^2}{2\varepsilon\overline{E}_\mu}+\frac{\varepsilon-2\mathcal{E}(0)}{2\overline{E}_\mu}+\ln\frac{\varepsilon}{\mathcal{E}(0)-\overline{E}_\mu}+\frac{1}{\varepsilon}(\mathcal{E}(t)-\overline{E}_\mu)\Big);\end{aligned}$$

(3) If $\mathcal{E}(t)\le\overline{E}_\mu$, then

$$\exp\Big(\int_0^t\eta(s)ds\Big)=\exp\Big(\frac{\varepsilon-2\mathcal{E}(0)}{2\overline{E}_\mu}+\ln\frac{\varepsilon}{\mathcal{E}(0)-\overline{E}_\mu}+\int_{t_\mu}^t\eta(s)ds\Big),$$

where t_μ satisfies $\mathcal{E}(t_\mu)=\overline{E}_\mu$.

Let $\varepsilon \to 0$, we have

$$\exp\Big(\int_0^t \eta(s)ds\Big) = \begin{cases} 0, & \mathcal{E}(t) \le \overline{E}_\mu, \\ \frac{\mathcal{E}(t)-\overline{E}_\mu}{\mathcal{E}(0)-\overline{E}_\mu} \exp\Big(\frac{\mathcal{E}(t)}{\overline{E}_\mu} - \frac{\mathcal{E}(0)}{\overline{E}_\mu}\Big), & \mathcal{E}(t) > \overline{E}_\mu. \end{cases}$$

Therefore,

$$C_m(t) = \begin{cases} 0, & \mathcal{E}(t) \le \overline{E}_\mu, \\ \frac{\mathcal{E}(t)-\overline{E}_\mu}{\mathcal{E}(0)-\overline{E}_\mu} \exp\Big(\frac{\mathcal{E}(t)}{\overline{E}_\mu} - \frac{\mathcal{E}(0)}{\overline{E}_\mu}\Big)\Big(\frac{u_{0m}(l)-u_{0m}(-l)}{2l} - \int_0^t \frac{\gamma_m(s)}{2l}ds\Big), & \mathcal{E}(t) > \overline{E}_\mu, \end{cases}$$

which is as in (4.6) if one recalls that (2.9).

Next, for every $n \ge 1$ and $m \ge 1$, we solve the other three ordinary differential problems (4.17), (4.18) and (4.19). Recalling the initial value problem (4.4) and let $a = m^4 + k$ for (4.17), $a = \Big(m^2 + \big(\frac{n\pi}{l}\big)^2\Big)^2 + k$ for (4.18) and (4.19), we obtain that there exists a unique solution for (4.17), (4.18) and (4.19) separately in the form of $S(t)$ with the constants A, B replaced by $A_m, B_m, (A_{1m})_n, (B_{1m})_n, (A_{2m})_n, (B_{2m})_n$. That is, if we denote $\delta = \mu^2 - 4(m^4 + k)$, then:

Case 1. If $\delta < 0$, then we have

$$\begin{aligned} D_m(t) &= \exp\big(-\frac{\mu}{2}t\big)\cos\Big(\frac{\sqrt{-\delta}}{2}t\Big)\Big(A_m + \frac{2}{\sqrt{-\delta}}\int_0^t \exp\big(\frac{\mu}{2}s\big)\cos\Big(\frac{\sqrt{-\delta}}{2}s\Big)\varphi_m(s)ds\Big) \\ &\quad - \frac{2}{\sqrt{-\delta}}\exp\big(-\frac{\mu}{2}t\big)\sin\Big(\frac{\sqrt{-\delta}}{2}t\Big)\Big(\frac{B_m + \mu A_m}{2} \\ &\quad + \int_0^t \exp\big(\frac{\mu}{2}s\big)\sin\Big(\frac{\sqrt{-\delta}}{2}s\Big)\varphi_m(s)ds\Big); \end{aligned}$$

Case 2. If $\delta = 0$, then

$$D_m(t) = \exp\big(-\frac{\mu}{2}t\big)\Big(A_m + \Big(B_m + \frac{\mu}{2}A_m\Big)t - \int_0^t (s-t)\exp\big(\frac{\mu}{2}s\big)\varphi_m(s)ds\Big);$$

Case 3. If $\delta > 0$, the two eigenvalues are $\lambda_1 = -\frac{\mu}{2} + \frac{\sqrt{\delta}}{2}$ and $\lambda_2 = -\frac{\mu}{2} - \frac{\sqrt{\delta}}{2}$, then

$$\begin{aligned} D_m(t) &= \exp(\lambda_1 t)\Big(\frac{B_m - \lambda_2 A_m}{\sqrt{\delta}} + \frac{1}{\sqrt{\delta}}\int_0^t \exp(-\lambda_1 s)\varphi_m(s)ds\Big) \\ &\quad + \exp(\lambda_2 t)\Big(\frac{\lambda_1 A_m - B_m}{\sqrt{\delta}} - \frac{1}{\sqrt{\delta}}\int_0^t \exp(-\lambda_2 s)\varphi_m(s)ds\Big). \end{aligned}$$

For (4.18) and (4.19), denote $\gamma = \mu^2 - 4\Big(\big(m^2 + \big(\frac{n\pi}{l}\big)^2\big)^2 + k\Big)$, then we have:

Case 1. If $\gamma < 0$, then

$$\begin{aligned} &(T_m)_n(t) \\ &= (A_{1m})_n \exp\big(-\frac{\mu}{2}t\big)\cos\Big(\frac{\sqrt{-\gamma}}{2}t\Big) \\ &\quad - \frac{2(B_{1m})_n + \mu(A_{1m})_n}{\sqrt{-\gamma}}\exp\big(-\frac{\mu}{2}t\big)\sin\Big(\frac{\sqrt{-\gamma}}{2}t\Big) \\ &\quad + \frac{2}{\sqrt{-\gamma}}\exp\big(-\frac{\mu}{2}t\big)\cos\Big(\frac{\sqrt{-\gamma}}{2}t\Big)\int_0^t \exp\big(\frac{\mu}{2}s\big)\cos\Big(\frac{\sqrt{-\gamma}}{2}s\Big)(\varphi_{1m})_n(s)ds \end{aligned}$$

$$
\begin{aligned}
&- \frac{2}{\sqrt{-\gamma}} \exp\big(-\frac{\mu}{2}t\big) \sin\Big(\frac{\sqrt{-\gamma}}{2}t\Big) \int_0^t \exp\big(\frac{\mu}{2}s\big) \sin\Big(\frac{\sqrt{-\gamma}}{2}s\Big) \left(\varphi_{1m}\right)_n (s)ds;
\end{aligned}
$$

$$
\begin{aligned}
&(S_m)_n(t) \\
&= (A_{2m})_n \exp\big(-\frac{\mu}{2}t\big) \cos\Big(\frac{\sqrt{-\gamma}}{2}t\Big) \\
&\quad - \frac{2(B_{2m})_n + \mu(A_{2m})_n}{\sqrt{-\gamma}} \exp\big(-\frac{\mu}{2}t\big) \sin\Big(\frac{\sqrt{-\gamma}}{2}t\Big) \\
&\quad + \frac{2}{\sqrt{-\gamma}} \exp\big(-\frac{\mu}{2}t\big) \cos\Big(\frac{\sqrt{-\gamma}}{2}t\Big) \int_0^t \exp\big(\frac{\mu}{2}s\big) \cos\Big(\frac{\sqrt{-\gamma}}{2}s\Big) \left(\varphi_{2m}\right)_n (s)ds \\
&- \frac{2}{\sqrt{-\gamma}} \exp\big(-\frac{\mu}{2}t\big) \sin\Big(\frac{\sqrt{-\gamma}}{2}t\Big) \int_0^t \exp\big(\frac{\mu}{2}s\big) \sin\Big(\frac{\sqrt{-\gamma}}{2}s\Big) \left(\varphi_{2m}\right)_n (s)ds;
\end{aligned}
$$

Case 2. If $\gamma = 0$, then

$$
\begin{aligned}
&(T_m)_n(t) \\
&= \Big((A_{1m})_n + \Big((B_{1m})_n + \frac{\mu}{2}(A_{1m})_n\Big) t\Big) \exp\big(-\frac{\mu}{2}t\big) \\
&\quad - \exp\big(-\frac{\mu}{2}t\big)\Big(\int_0^t s \exp\big(\frac{\mu}{2}s\big) \left(\varphi_{1m}\right)_n (s)ds - t\int_0^t \exp\big(\frac{\mu}{2}s\big) \left(\varphi_{1m}\right)_n (s)ds\Big);
\end{aligned}
$$

$$
\begin{aligned}
&(S_m)_n(t) \\
&= \Big((A_{2m})_n + \Big((B_{2m})_n + \frac{\mu}{2}(A_{2m})_n\Big) t\Big) \exp\big(-\frac{\mu}{2}t\big) \\
&\quad - \exp\big(-\frac{\mu}{2}t\big)\Big(\int_0^t s \exp\big(\frac{\mu}{2}s\big) \left(\varphi_{2m}\right)_n (s)ds - t\int_0^t \exp\big(\frac{\mu}{2}s\big) \left(\varphi_{2m}\right)_n (s)ds\Big);
\end{aligned}
$$

Case 3. If $\gamma > 0$, the two eigenvalues are $\lambda_1 = -\frac{\mu}{2} + \frac{\sqrt{\gamma}}{2}$ and $\lambda_2 = -\frac{\mu}{2} - \frac{\sqrt{\gamma}}{2}$, then

$$
\begin{aligned}
&(T_m)_n(t) \\
&= \exp(\lambda_1 t)\Big(\frac{(B_{1m})_n - \lambda_2(A_{1m})_n}{\sqrt{\gamma}} + \frac{1}{\sqrt{\gamma}}\int_0^t \exp(-\lambda_1 s) \left(\varphi_{1m}\right)_n (s)ds\Big) \\
&\quad + \exp(\lambda_2 t)\left(\frac{\lambda_1(A_{1m})_n - (B_{1m})_n}{\sqrt{\gamma}} - \frac{1}{\sqrt{\gamma}}\int_0^t \exp(-\lambda_2 s) \left(\varphi_{1m}\right)_n (s)ds\right);
\end{aligned}
$$

$$
\begin{aligned}
&(S_m)_n(t) \\
&= \exp(\lambda_1 t)\Big(\frac{(B_{2m})_n - \lambda_2(A_{2m})_n}{\sqrt{\gamma}} + \frac{1}{\sqrt{\gamma}}\int_0^t \exp(-\lambda_1 s) \left(\varphi_{2m}\right)_n (s)ds\Big) \\
&\quad + \exp(\lambda_2 t)\Big(\frac{\lambda_1(A_{2m})_n - (B_{2m})_n}{\sqrt{\gamma}} - \frac{1}{\sqrt{\gamma}}\int_0^t \exp(-\lambda_2 s) \left(\varphi_{2m}\right)_n (s)ds\Big).
\end{aligned}
$$

Therefore, the solution to (2.12)-(2.13) has the form of (4.7) with $U_m(y,t)$ being given in (4.8). And then the proof is finished. □

Finally, by using the explicit form of the solution to (2.12)-(2.13), we are able to analyze the amplitude of the torsional oscillation appearing in suspension bridges.

Corollary 4.3. *Assume that u is the unique solution to the original problem* (2.12)-(2.13). *Then the amplitude of the torsional oscillation on the two sides $y = \pm l$ reads*

$$|u(x,-l,t) - u(x,l,t)| = \theta(t)\big|u_0(x,-l) - u_0(x,l) + 2l\int_0^t \gamma(x,s)ds\big|,$$

where $\theta(t)$ is given in (2.9).

By (2.9), if $\mathcal{E}(t) \le \overline{E}_\mu$, then $\theta(t) = 0$, which yields that $|u(x,l,t) - u(x,-l,t)| = 0$ by Corollary 4.3. That is, when $\mathcal{E}(t) \le \overline{E}_\mu$, there is no torsional oscillation appearing in the bridge structure. However, once the energy $\mathcal{E}(t)$ exceeds $\overline{E}_\mu$, then $\theta(t) > 0$ and $|u(x,l,t) - u(x,-l,t)| \ne 0$, which show that the torsional oscillation appears. Since $\theta(t)$ is an increasing function with respect to $\mathcal{E}(t)$ and by Corollary 4.3, if the energy $\mathcal{E}(t)$ ($\mathcal{E}(t) > \overline{E}_\mu$) increases, then the amplitude of the torsional oscillation will go up till the bridges collapse.

5. Future developments

In this article, a new mathematical model for dynamical suspension bridges is suggested. We solve the evolution problem corresponding to the model and obtain a unique explicit solution which is to display the behavior appearing in actual suspension bridges. In this section, we indicate two possible future developments.

5.1. **Quantitative results.** In this article, we suggest a dynamical model for suspension bridges and obtain an explicit solution from a theoretical point of view. In this model, $\varphi(x,y,t)$, $u_0(x,y)$, $u_1(x,y)$, $\alpha(x,t)$, $\beta(x,t)$ and $\gamma(x,t)$ are assumed to be real functions, but we gave no hint on their explicit forms. Therefore, one can carry out several numerical experiments to estimate them approximately. Once they are given explicitly, an exact solution is to be expected and one can even calculate the blow-up time of the solution, namely, the time when the bridge collapses.

5.2. **Nonlinear restoring force.** The model we suggested in this paper is a linear model. According to the opinion of McKenna [9]

> We doubt that a bridge oscillating up and down by about 10 meters every 4 seconds obeys Hooke's law.

it may be not suitable to assume the restoring force h due to the hangers to be in a linear case, see also [4]. Therefore, it could be better to consider the model with nonlinear restoring force $h = h(x,y,u)$

$$u_{tt} + \Delta^2 u + \mu u_t + h = \varphi, \quad \text{in } \Omega \times (0,T).$$

However, this problem appears significantly more difficult.

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