Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 188, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

NUMERICAL SOLUTION TO INVERSE ELLIPTIC PROBLEM WITH NEUMANN TYPE OVERDETERMINATION AND MIXED BOUNDARY CONDITIONS

CHARYYAR ASHYRALYYEV, YASAR AKKAN

Abstract. This article studies the numerical solution of inverse problems for the multidimensional elliptic equation with Dirichlet-Neumann boundary conditions and Neumann type overdetermination. We present first and second order accuracy difference schemes. The stability and almost coercive stability inequalities for the solution are obtained. Numerical examples with explanation on the implementation illustrate the theoretical results.

1. Introduction

Inverse problems arise in many branches of science and mathematics (see [21, 28, 31] and the bibliography therein). In recent years, the subject of the inverse problems for partial differential equations is of significant and quickly growing interest for many scientists and engineers. Especially, theory and methods of solutions of inverse problems of determining unknown parameter of partial differential equations have been comprehensively studied by a few researchers (see [1], [2], [4]–[7], [9]–[22], [24]–[31], [34]–[36], and references therein).

Existence, uniqueness, and Fredholm property theorems for the inverse problem of finding the source in an abstract second-order elliptic equation on a finite interval are established in [24].

In [34]–[36], the author investigated source determination for the elliptic equation in plane, rectangle and cylinder. Sufficient conditions for the unique solvability of the inverse coefficient problems with overdetermination on the boundary, where the Dirichlet conditions are supplemented with the vanishing condition for the normal derivative on part of the boundary were given.

Approximation of inverse Bitzadze-Samarsky problem for abstract elliptic differential equations with Neumann type overdetermination which is based on semigroup theory and a functional analysis approach are described in [26].

Simultaneous reconstruction of coefficients and source parameters in elliptic systems modelled with many boundary value problems was discussed in [30]. It was proposed in [18] and [29] that the determination of the problem of an unknown boundary condition in the boundary value problem in the regularization procedures

2010 *Mathematics Subject Classification.* 35N25, 39A14, 39A30, 65J22.
Key words and phrases. Inverse elliptic problem; stability; almost coercive stability; overdetermination; difference scheme.

Submitted June 23, 2015. Published July 13, 2015.

can be performed with the help of an extra measurement at an internal point. Well-posedness of inverse problems for elliptic differential and difference equations were investigated in [10]–[15].

These works are devoted to identification problems of an elliptic differential and difference equations with Dirichlet type overdetermination.

The exact estimates for the solution of the boundary value problem of determining the parameter of an elliptic equation with a positive operator in an Banach space are obtained in [4]. The papers [4, 10], [11], [13]–[15] are devoted to getting the stability and coercive stability inequalities for the solutions of various inverse problems with Dirichlet type overdetermination for elliptic differential and difference equations.

In [15], the inverse problem for the multi-dimensional elliptic equation with Dirichlet type overdetermination and mixed boundary conditions, and also its first and second order accuracy approximations presented. Moreover, the stability, almost coercive stability and coercive stability inequalities for the solution of these difference schemes are showed.

The third and fourth order of accuracy stable difference schemes for the solution of the inverse problem with Dirichlet type overdetermination and Dirichlet boundary condition are presented in [10]. By using the result of established abstract results, well-posedness of high order accuracy difference schemes of the inverse problem for a multidimensional elliptic equation were obtained. High order stable difference schemes for the approximately solution of inverse problem for the multidimensional elliptic equation with Dirichlet-Neumann boundary conditions and the stability estimates for their solutions were disscussed in [11].

In [12], the inverse problem for the multidimensional elliptic equation with Neumann type overdetermination and Dirichlet boundary condition was considered.

Our aim in this work is investigation of the inverse problem for the multidimensional elliptic equation with Neumann type overdetermination and mixed boundary conditions. We construct the first and second order of accuracy difference schemes and give stability estimates for their solutions. Numerical example with explanation on the realization on computer will be done to illustrate theoretical results.

Let $\Omega = (0,\ell) \times (0,\ell) \times \cdots \times (0,\ell)$ be the open cube in the n -dimensional Euclidean space with boundary $S = S_1 \cup S_2$, $\overline{\Omega} = \Omega \cup S$, where

$$\begin{gathered}
S = \big\{x = (x_1, \dots, x_n) : x_i = 0 \text{ or } x_i = \ell,\ 0 \le x_k \le \ell,\ k \ne i,\ 1 \le i \le n\big\}, \\
S_1 = \big\{x = (x_1, \dots, x_n) : x_i = 0,\ 0 \le x_k \le \ell,\ k \ne i,\ 1 \le i \le n\big\}, \\
S_2 = \big\{x = (x_1, \dots, x_n) : x_i = \ell,\ 0 < x_k \le \ell,\ k \ne i,\ 1 \le i \le n\big\}.
\end{gathered}$$

We consider the inverse problem of finding pair functions $u(t,x)$ and $p(x)$ for the multidimensional elliptic equation with Dirichlet-Neumann boundary conditions and Neumann type overdetermination

$$\begin{gathered}
-u_{tt}(t,x) - \sum_{i=1}^{n} (a_i(x) u_{x_i})_{x_i} + \delta u(t,x) = f(t,x) + tp(x), \quad x \in \Omega,\ 0 < t < 1, \\
u_t(0,x) = \varphi(x), \quad u_t(1,x) = \psi(x), \quad u_t(\lambda,x) = \xi(x), \quad x \in \overline{\Omega}, \\
u(t,x) = 0,\ x \in S_1, \quad \frac{\partial u}{\partial \vec{n}}(t,x) = 0,\ x \in S_2,\ 0 \le t \le 1.
\end{gathered} \tag{1.1}$$

Here, $0 < \lambda < 1$ and $\delta > 0$ are known numbers, $a_i(x)$ $(i = 1, \dots, n; x \in \Omega)$, $\varphi(x), \psi(x), \xi(x)$ $(x \in \overline{\Omega})$, and $f(t, x)$ $(t \in (0, T), x \in \Omega)$ are given smooth functions, $a_i(x) \geq a > 0$ $(x \in \Omega)$.

In this article, the first and second order of accuracy difference schemes for approximate solution of the inverse problem are constructed (1.1) and stability, almost coercive stability estimates for the solution of these difference schemes are established.

This paper is organized as follows: In Section 2, we present the first and second order accuracy difference schemes for the inverse problem (1.1). Section 3 is devoted to the stability and almost coercive stability estimates for the solution of these difference schemes. In Section 4, we present numerical results for two dimensional elliptic equation. The conclusion is given in the final Section 5.

2. Difference schemes

The differential expression ([8, 23])

$$Au(x) = -\sum_{i=1}^{n} (a_i\ (x) u_{x_i}(x))_{x_i} + \delta u(x) \tag{2.1}$$

defines a self-adjoint positive definite operator A acting on space $L_2(\overline{\Omega})$ with the domain $D(A) = \{u(x) \in W_2^2(\overline{\Omega}) : u = 0 \text{ on } S_1 \text{ and } \frac{\partial u}{\partial \vec{n}} = 0 \text{ on } S_2\}$.

By using the substitution

$$u(t, x) = v(t, x) + A^{-1}(pt), \tag{2.2}$$

problem (1.1) can be reduced to auxiliary nonlocal problem for $v(t, x)$ function:

$$\begin{gathered} -v_{tt}(t, x) + Av(t, x) = f(t, x), \quad 0 < t < 1, x \in \Omega, \\ v_t(0, x) - v_t(\lambda, x) = \varphi(x) - \xi(x), \\ v_t(1, x) - v_t(\lambda, x) = \psi(x) - \xi(x), \quad x \in \Omega \end{gathered} \tag{2.3}$$

Then, the unknown function $p(x)$ will be defined by

$$p(x) = A\xi(x) - Av_t(\lambda, x). \tag{2.4}$$

Now, we describe approximation of inverse problem (1.1). Define the set of grid points in space variables.

$$\begin{gathered} \widetilde{\Omega}_h = \{x = (h_1 m_1, \dots, h_n m_n) : m_i = 0, \dots, M_i,\ h_i M_i = \ell,\ i = 1, \dots, n\}, \\ \Omega_h = \widetilde{\Omega}_h \cap \Omega, \quad S_h^1 = \widetilde{\Omega}_h \cap S_1, \quad S_h^2 = \widetilde{\Omega}_h \cap S_2. \end{gathered}$$

We introduce the Hilbert spaces $L_{2h} = L_2(\widetilde{\Omega}_h)$ and $W_{2h}^2 = W_2^2(\widetilde{\Omega}_h)$ of grid functions $g^h(x) = \{g(h_1 m_1, \dots, h_n m_n) : m_i = 0, \dots, M_i,\ i = 1, \dots, n\}$ defined on $\widetilde{\Omega}_h$, equipped with the norms

$$\|g^h\|_{L_{2h}} = \Big(\sum_{x \in \widetilde{\Omega}_h} |g^h(x)|^2 h_1 \dots h_n \Big)^{1/2},$$

$$\begin{aligned} \|g^h\|_{W_{2h}^2} = \|g^h\|_{L_{2h}} &+ \Big(\sum_{x \in \widetilde{\Omega}_h} \sum_{i=1}^{n} |(g^h(x))_{x_i, m_i}|^2 h_1 \dots h_n \Big)^{1/2} \\ &+ \Big(\sum_{x \in \widetilde{\Omega}_h} \sum_{i=1}^{n} |(g^h(x))_{x_i \overline{x_i},\ m_i}|^2 h_1 \dots h_n \Big)^{1/2}, \end{aligned}$$

respectively.

To the differential operator A in (2.1),we assign the difference operator A_h^x defined by

$$A_h^x u^h = -\sum_{i=1}^{n} (a_i(x) u^h_{\bar{x}_i})_{x_i, m_i} + \delta u^h \tag{2.5}$$

acting in the space of grid functions $u^h(x)$ satisfying the conditions $u^h(x) = 0$, for all $x \in S_h^1$ and $D^h u^h(x) = 0$, for all $x \in S_h^2$. Here, $D^h u^h(x)$ is an approximation of $\frac{\partial u}{\partial \vec{n}}$ Note that ([8, 23]) A_h^x is a self-adjoint positive define operator in $L_2(\widetilde{\Omega}_h)$. Denote

$$D = \frac{1}{2}(\tau A_h^x + \sqrt{4A_h^x + \tau^2 (A_h^x)^2}), \quad R = (I + \tau D)^{-1},$$

First, by using A_h^x, for obtaining $u^h(t, x)$ functions, we arrive at problem

$$\begin{gathered} -\frac{d^2 u^h(t,x)}{dt^2} + A_h^x u^h(t,x) = f^h(t,x) + p^h(x), \quad 0 < t < 1, \ x \in \Omega_h, \\ \frac{du^h(0,x)}{dt} = \varphi^h(x), \quad \frac{du^h(\lambda,x)}{dt} = \xi^h(x), \quad \frac{du^h(T,x)}{dt} = \psi^h(x), \quad x \in \widetilde{\Omega}_h. \end{gathered} \tag{2.6}$$

Second, applying the approximate formula

$$\frac{du^h(\lambda,x)}{dt} = \frac{du^h([\frac{\lambda}{\tau}]\tau, x)}{dt} + o(\tau) \tag{2.7}$$

for $\frac{du^h(\lambda,x)}{dt} = \xi^h(x)$, we replace problem (1.1) with the first order of accuracy difference scheme in t,

$$\begin{gathered} -\frac{u^h_{k+1}(x) - 2u^h_k(x) + u^h_{k-1}(x)}{\tau^2} + A_h^x u^h_k(x) = \theta^h_k(x) + p^h(x), \\ \theta^h_k(x) = f^h(t_k, x), \quad t_k = k\tau, \ 1 \le k \le N-1, \ x \in \Omega_h, \ N\tau = 1, \\ \frac{u^h_1(x) - u^h_0(x)}{\tau} = \varphi^h(x), \quad \frac{u^h_N(x) - u^h_{N-1}(x)}{\tau} = \psi^h(x), \\ \frac{u^h_{l+1}(x) - u^h_l(x)}{\tau} = \xi^h(x), \quad x \in \widetilde{\Omega}_h. \end{gathered} \tag{2.8}$$

Here, $l = [\frac{\lambda}{\tau}]$, $[\cdot]$ is a notation for the greatest integer function.

In a similar manner, the auxiliary nonlocal problem can be changed by the first order of accuracy difference scheme in t,

$$\begin{gathered} -\frac{v^h_{k+1}(x) - 2v^h_k(x) + v^h_{k-1}(x)}{\tau^2} + A_h^x v^h_k(x) = \theta^h_k(x), \\ \theta^h_k(x) = f^h(t_k, x), \quad t_k = k\tau, \ 1 \le k \le N-1, \ x \in \Omega_h, \ N\tau = 1 \\ \frac{v^h_1(x) - v^h_0(x)}{\tau} - \frac{v^h_{l+1}(x) - v^h_l(x)}{\tau} = \varphi^h(x) - \xi^h(x), \\ \frac{v^h_N(x) - v^h_{N-1}(x)}{\tau} - \frac{v^h_{l+1}(x) - v^h_l(x)}{\tau} = \psi^h(x) - \xi^h(x), \quad x \in \widetilde{\Omega}_h. \end{gathered} \tag{2.9}$$

In this step of approximation, by using the approximate formula

$$\frac{du^h}{dt}(\lambda, x) = \frac{du^h}{dt}(l\tau, x) + (\frac{\lambda}{\tau} - l)(\frac{du^h}{dt}(l\tau + \tau, x) - \frac{du^h}{dt}(l\tau, x)) + o(\tau^2) \tag{2.10}$$

for condition $\frac{du^h}{dt}(\lambda,x)=\frac{d\xi^h}{dt}(x)$, we can get the following second order of accuracy difference schemes

$$\begin{gathered}
-\frac{u^h_{k+1}(x)-2u^h_k(x)+u^h_{k-1}(x)}{\tau^2}+A^x_h u^h_k(x)=\theta^h_k(x)+p^h(x),\\
\theta^h_k(x)=f^h(t_k,x),\quad t_k=k\tau,\ 1\le k\le N-1,\ x\in\widetilde{\Omega}_h,\\
\frac{-3u^h_0(x)+4u^h_1(x)-u^h_2(x)}{2\tau}=\varphi^h(x),\quad \frac{3u^h_N(x)-4u^h_{N-1}(x)+u^h_{N-2}(x)}{2\tau}=\psi^h(x),\\
\frac{3u^h_l(x)-4u^h_{l+1}(x)+u^h_{l+2}(x)}{2\tau}+(\frac{\lambda}{\tau}-l)\Big[\frac{3u^h_{l+1}(x)-4u^h_{l+2}(x)+u^h_{l+3}(x)}{2\tau}\\
-\frac{3u^h_l(x)-4u^h_{l+1}(x)+u^h_{l+2}(x)}{2\tau}\Big]=\xi^h(x),\quad x\in\widetilde{\Omega}_h,\ N\tau=1,
\end{gathered}\tag{2.11}$$

and

$$\begin{gathered}
-\frac{v^h_{k+1}(x)-2v^h_k(x)+v^h_{k-1}(x)}{\tau^2}+A^x_h v^h_k(x)=\theta^h_k(x),\quad \theta^h_k(x)=f^h(t_k,x),\\
t_k=k\tau,\quad 1\le k\le N-1,\ x\in\widetilde{\Omega}_h,\\
\frac{-3v^h_0(x)+4v^h_1(x)-v^h_2(x)}{2\tau}-\frac{-3v^h_l(x)+4v^h_{l+1}(x)-v^h_{l+2}(x)}{2\tau}\\
-(\frac{\lambda}{\tau}-l)\Big(\frac{-3v^h_{l+1}(x)+4v^h_{l+2}(x)-v^h_{l+3}(x)}{2\tau}-\frac{-3v^h_l(x)+4v^h_{l+1}(x)-v^h_{l+2}(x)}{2\tau}\Big)\\
=\varphi^h(x)-\xi^h(x),\\
\frac{3v^h_N(x)-4v^h_{N-1}(x)+v^h_{N-2}(x)}{2\tau}-\frac{-3v^h_l(x)+4v^h_{l+1}(x)-v^h_{l+2}(x)}{2\tau}\\
-(\frac{\lambda}{\tau}-l)\Big(\frac{-3v^h_{l+1}(x)+4v^h_{l+2}(x)-v^h_{l+3}(x)}{2\tau}-\frac{-3v^h_l(x)+4v^h_{l+1}(x)-v^h_{l+2}(x)}{2\tau}\Big)\\
=\psi^h(x)-\xi^h(x),\quad x\in\widetilde{\Omega}_h,\ N\tau=1
\end{gathered}\tag{2.12}$$

for approximate solutions of inverse problem (1.1) and auxialary nonlocal problem (2.3), respectively.

3. Stability estimates

Now, we consider the linear spaces of mesh functions $\theta^\tau=\{\theta_k\}_1^{N-1}$ with values in the Hilbert space H. Denote by $C([0,1]_\tau,H)$ and $\mathcal{C}^{\alpha,\alpha}_{01}([0,1]_\tau,H)$ normed spaces with the norms

$$\|\{\theta_k\}_1^{N-1}\|_{C([0,1]_\tau,H)}=\max_{1\le k\le N-1}\|\theta_k\|_H,$$

$$\begin{aligned}
&\|\{\theta_k\}_1^{N-1}\|_{\mathcal{C}^{\alpha,\alpha}_{0T}([0,1]_\tau,H)}\\
&=\|\{\theta_k\}_1^{N-1}\|_{C([0,1]_\tau,H)}+\sup_{1\le k<k+s\le N-1}\frac{(k\tau+s\tau)^\alpha(1-k\tau)^\alpha\|\theta_{k+s}-\theta_k\|_H}{(s\tau)^\alpha},
\end{aligned}$$

respectively. Let τ and $|h|=\sqrt{h_1^2+\dots+h_n^2}$ be sufficiently small positive numbers.

Theorem 3.1. *The solution $(\{u_k^h\}_1^{N-1}, p^h)$ of difference scheme (2.8) obeys the following stability estimates:*

$$
\begin{aligned}
&\|\{u_k^h\}_1^{N-1}\|_{C([0,1]_\tau, L_{2h})}\\
&\le M\big[\|\varphi^h\|_{L_{2h}} + \|\psi^h\|_{L_{2h}} + \|\xi^h\|_{L_{2h}} + \|\{f_k^h\}_1^{N-1}\|_{C([0,1]_\tau, L_{2h})}\big],
\end{aligned}
$$

$$
\begin{aligned}
&\|p^h\|_{L_{2h}}\\
&\le M\big[\|\varphi^h\|_{W_{2h}^2} + \|\psi^h\|_{W_{2h}^2} + \|\xi^h\|_{W_{2h}^2} + \frac{1}{\alpha(1-\alpha)}\|\{f_k^h\}_1^{N-1}\|_{\mathcal{C}_{0T}^{\alpha,\alpha}([0,1]_\tau, L_{2h})}\big],
\end{aligned}
$$

where M is independent of φ^h, ψ^h, ξ^h, τ, α, h, and $\{f_k^h\}_1^{N-1}$.

Theorem 3.2. *The solution $(\{u_k^h\}_1^{N-1}, p^h)$ of difference scheme (2.8) obeys the almost coercive stability estimate:*

$$
\begin{aligned}
&\|\{\frac{u_{k+1}^h - 2u_k^h + u_{k-1}^h}{\tau^2})\}_1^{N-1}\|_{C([0,1]_\tau, L_{2h})} + \|\{A_h u_k^h\}_1^{N-1}\|_{C([0,1]_\tau, L_{2h})} + \|p^h\|_{L_{2h}}\\
&\le M\Big(\|\varphi^h\|_{W_{2h}^2} + \|\psi^h\|_{W_{2h}^2} + \|\xi^h\|_{W_{2h}^2} + \ln(\frac{1}{\tau+h})\|\{f_k^h\}_1^{N-1}\|_{C([0,1]_\tau, L_{2h})}\Big),
\end{aligned}
$$

where M does not depend on φ^h, ψ^h, ξ^h, τ, α, h, and $\{f_k^h\}_1^{N-1}$.

The proofs of Theorems 3.1–3.2 are based on the symmetry property of operator A_h^x in L_{2h}, and the following formulas

$$
\begin{aligned}
u_k^h &= P(R^k - R^{2N-k} - R^{N-k} + R^{N+k})v_0^h + (I - R^{2N})^{-1}(R^{N-k} - R^{N+k})\\
&\quad\times\Big\{-(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}(R^{N-1-i} - R^{N-1+i})\theta_i^h\tau\\
&\quad + G_1(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\big[(R^{|1-i|-1} - R^i) - (R^{|N-i|-1} - R^{N+i-1})\\
&\quad + (R^{|N-1-i|-1} - R^{N+i-2})\big]\theta_i^h\tau + \tau G_1(\varphi^h - \psi^h)\Big\} - (I - R^{2N})^{-1}\\
&\quad\times(R^{N-k} - R^{N+k})(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}(R^{N-1-i} - R^{N-1+i})\theta_i^h\tau\\
&\quad + (2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\big[(R^{|k-i|-1} - R^{k+i-1})\big]\theta_i^h\tau + t_k p^h,
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
p^h &= -\frac{1}{\tau}P\Big(R^l - R^{2N-l} - R^{N-l} + R^{N+l} - R^{l+1} - R^{2N-l-1} - R^{N-l-1}\\
&\quad + R^{N+l+1}\Big)A_h v_0^h + (I - R^{2N})^{-1}\big(R^{N-l} - R^{N+l} - R^{N-l-1} + R^{N+l+1}\big)\\
&\quad\times\Big\{-(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}(R^{N-1-i} - R^{N-1+i})\theta_i^h\tau + G_1(2I+\tau D)^{-1}\\
&\quad D^{-1}\sum_{i=1}^{N-1}\big[R^{|1-i|-1} - R^i - R^{|N-i|-1} + R^{N+i-1} + R^{|N-1-i|-1} - R^{N+i-2}\big]A_h\theta_i^h\\
&\quad + G_1(A_h\varphi^h - A_h\psi^h)\Big\} - (I - R^{2N})^{-1}(R^{N-l} - R^{N+l} - R^{N-l-1} + R^{N+l+1})
\end{aligned}
$$

$$
\begin{aligned}
&\times (2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}(R^{N-1-i}-R^{N-1+i})A_h\theta_i^h+(2I+\tau D)^{-1}\\
&\times D^{-1}\sum_{i=1}^{N-1}(R^{|l-i|-1}-R^{l+i-1}-R^{|l+1-i|-1}+R^{l+i})A_h\theta_i^h+A_h\xi^h,
\end{aligned}
$$

$$
\begin{aligned}
v_0^h &= Q_1P(R^{N-1}-R^{N+1}-R^{N-l-1}+R^{N+l+1}+R^{N-l}-R^{N+l})\\
&\quad\times\Big\{(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}(R^{N-1-i}-R^{N-1+i})\theta_i\tau-G_1(2I+\tau D)^{-1}\\
&\quad\times D^{-1}\sum_{i=1}^{N-1}(R^{|1-i|-1}-R^{i}+R^{|l+1-i|-1}-R^{l+i}-R^{|l-i|-1}+R^{l+i-1})\theta_i\tau\\
&\quad-\tau G_1(\varphi-\xi)\Big\}+Q_1(I-R^{2N})^{-1}\\
&\quad\times\big[\,(R^{N-1}-R^{N+1})-(R^{N-l-1}-R^{N+l+1})+(R^{N-l}-R^{N+l})\big]\\
&\quad\times(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}(R^{N-1-i}-R^{N-1+i})\theta_i^h\tau]-Q_1(2I+\tau D)^{-1}\\
&\quad\times D^{-1}\sum_{i=1}^{N-1}\big[(R^{|1-i|-1}-R^{i})-(R^{|l+1-i|-1}-R^{L+i})\\
&\quad+(R^{|l-i|-1}-R^{l+i-1})\big]\theta_i^h\tau+\tau Q_1(\varphi^h\ -\xi^h),
\end{aligned}
$$

$$
\begin{aligned}
P&=(I-R^{2N})^{-1},\\
G_1&=(I+R^{N-1})^{-1}(I-R)^{-1}(I+R^{N}),\\
Q_1&=-(I-R^{N-l-1})^{-1}(I-R^{l})^{-1}(I-R^{N})(I-R)^{-1}
\end{aligned}
$$

for difference scheme (2.8), and

$$
\begin{aligned}
u_k^h &= P(R^k-R^{2N-k}-R^{N-k}+R^{N+k})v_0^h+P(R^{N-k}-R^{N+k})\\
&\quad\times\Big\{(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}(R^{N-1-i}-R^{N-1+i})\theta_i^h\tau\\
&\quad-G_2(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}\big[4(R^{|1-i|-1}-R^{i})-(R^{|2-i|-1}-R^{i+1})\\
&\quad-3(R^{|N-i|-1}-R^{N+i-1})+4(R^{|N-1-i|-1}-R^{N+i-2})-R^{|N-2-i|-1}\\
&\quad+R^{N+i-3}\big]\theta_i^h\tau+2\tau G_2(\varphi^h(x)-\psi^h(x))\Big\}-(I-R^{2N})^{-1}(R^{N-k}-R^{N+k})\\
&\quad\times(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}(R^{N-1-i}-R^{N-1+i})\theta_i^h\tau\\
&\quad+(2I+\tau D)^{-1}D^{-1}\sum_{i=1}^{N-1}(R^{|k-i|-1}-R^{k+i-1})\theta_i^h\tau+t_kp^h,
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
p^h &= \frac{1}{2\tau} P\big(R^l - R^{2N-l} - R^{N-l} + R^{N+l} - R^{l+2} + R^{2N-l-2} + R^{N-l-2} \\
&\quad - R^{N+l+2}\big) A_h v_0^h + \frac{1}{2\tau}(I - R^{2N})^{-1}(R^{N-l} - R^{N+l} - R^{N-l-2} + R^{N+l+2}) \\
&\quad \times \Big\{ (2I + \tau D)^{-1} D^{-1} \sum_{i=1}^{N-1} (R^{N-1-i} - R^{N-1+i}) A_h \theta_i^h \tau \\
&\quad - G_2 (2I + \tau D)^{-1} D^{-1} \sum_{i=1}^{N-1} \Big[4(R^{|1-i|-1} - R^i) - (R^{|2-i|-1} - R^{i+1}) \\
&\quad - 3(R^{|N-i|-1} - R^{N+i-1}) + 4(R^{|N-1-i|-1} - R^{N+i-2}) \\
&\quad - (R^{|N-2-i|-1} - R^{N+i-3}) \Big] A_h \theta_i^h \tau + 2\tau G_2 (A_h \varphi^h - A_h \psi^h) \Big\} \\
&\quad - \frac{1}{2}(I - R^{2N})^{-1}(R^{N-l} - R^{N+l} - R^{N-l-2} + R^{N+l+2}) \\
&\quad \times (2I + \tau D)^{-1} D^{-1} \sum_{i=1}^{N-1} (R^{N-1-i} - R^{N-1+i}) A_h \theta_i^h + \frac{1}{2}(2I + \tau D)^{-1} \\
&\quad \times D^{-1} \sum_{i=1}^{N-1} (R^{|l-i|-1} - R^{l+i-1} - R^{|l+2-i|-1} + R^{l+i+1}) A_h \theta_i^h + A_h \xi^h, \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
v_0^h &= -Q_2 P[4(R^{N-1} - R^{N+1}) - (R^{N-2} - R^{N+2})] \\
&\quad \times \Big\{ (2I + \tau D)^{-1} D^{-1} \sum_{i=1}^{N-1} (R^{N-1-i} - R^{N-1+i}) \theta_i^h \tau] - G_2 (2I + \tau D)^{-1} \\
&\quad \times D^{-1} \sum_{i=1}^{N-1} \Big[4(R^{|1-i|-1} - R^i) - (R^{|2-i|-1} - R^{i+1}) - 3(R^{|N-i|-1} - R^{N+i-1}) \\
&\quad + 4(R^{|N-1-i|-1} - R^{N+i-2}) - (R^{|N-2-i|-1} - R^{N+i-3})] \; \theta_i^h \tau + 2\tau G_2 (\varphi^h(x) \\
&\quad - \psi^h(x)) \Big\} + Q_2 P \Big\{ 4(R^{N-1} - R^{N+1}) - (R^{N-2} - R^{N+2}) + (\frac{\lambda}{\tau} - l - 1) \\
&\quad \times \big[(-3(R^l - R^{2N-l}) + 4(R^{l+1} - R^{2N-l-1}) - (R^{l+2} - R^{2N-l-2})) \big] \\
&\quad - (\frac{\lambda}{\tau} - l) \big[(-3(R^{l+1} - R^{2N-l-1}) + 4(R^{l+2} - R^{2N-l-2}) \\
&\quad - (R^{l+3} - R^{2N-l-3})) \big] \Big\} (2I + \tau D)^{-1} D^{-1} \sum_{i=1}^{N-1} (R^{N-1-i} - R^{N-1+i}) \theta_i^h \tau \\
&\quad - Q_2 (2I + \tau D)^{-1} D^{-1} \sum_{i=1}^{N-1} \Big\{ 4(R^{|1-i|-1} - R^i) - (R^{|2-i|-1} - R^{i+1}) \\
&\quad + (\frac{\lambda}{\tau} - l - 1) \big[-3(R^{|l-i|-1} - R^{l+i-1}) + 4(R^{|l+1-i|-1} - R^{l+i}) \\
&\quad - (R^{|l+2-i|-1} - R^{l+i+1}) \big] - (\frac{\lambda}{\tau} - l) \Big[-3(R^{|l+1-i|-1} - R^{l+i}\ ) \\
&\quad + 4(R^{|l+2-i|-1} - R^{l+i+1}) - (R^{|l+3-i|-1} - R^{l+i+2}) \Big] \Big\} \theta_i^h \tau + 2\tau Q_2 (\varphi^h(x) \\
&\quad - \xi^h(x)), \qquad (3.4)
\end{aligned}
$$

$$\begin{aligned}
G_2 &= \big[(I+R^N)^{-1}(I-R)[(R-3I)-R^{N-2}(I-3R)]\big]^{-1},\\
Q_2 &= \big[(I-R^N)^{-1}(I-R)[(R-3I)+R^{N-2}\ (I-3R)]\big]^{-1}
\end{aligned}$$

for difference scheme (2.11), and the following theorem on well-posedness of the elliptic difference problem.

Theorem 3.3 ([33]). *For the solution of the elliptic difference problem*

$$\begin{gathered}
A_h^x u^h(x) = \omega^h(x), \quad x \in \widetilde{\Omega}_h,\\
u^h(x) = 0,\ x \in S_h^1, \quad D^h u^h(x) = 0,\ x \in S_h^2,
\end{gathered}$$

the following coercivity inequality holds:

$$\sum_{q=1}^{n} \|(u^h)_{\overline{x}_q x_q, j_q}\|_{L_{2h}} \le M \|\omega^h\|_{L_{2h}},$$

here M does not depend on h and ω^h.

4. Numerical results

For the numerical result, consider the inverse elliptic problem

$$\begin{gathered}
-\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial}{\partial x}((1+x)\frac{\partial u(t,x)}{\partial x}) + u(t,x) = f(t,x) + tp(x),\\
f(t,x) = \exp(-t)[-2x - x^2 + 2tx] - 2x + x^2 + t^2[3x - \frac{x^2}{2}],\\
0 < x < 1,\ 0 < t < 1,\\
u(0,x) = x^2 - 2x, \quad 0 \le x \le 1,\\
u(1,x) = [e^{-1}+1](\frac{x^2}{2} - x), \quad 0 \le x \le 1,\\
u(\lambda,x) = [\exp(-\lambda)+\lambda](\lambda x - \frac{\lambda x^2}{2} + x^2 - 2x), \quad 0 \le x \le 1,\\
u(t,0) = 0, \quad u_x(t,1) = 0, \quad 0 \le t \le 1, \quad \lambda = \frac{3}{5}.
\end{gathered} \tag{4.1}$$

It can be easily seen that $u(x,t) = [\exp(-t)+t](tx - \frac{tx^2}{2} + x^2 - 2x)$ and $p(x) = -6x + x^2$ are the exact solutions of (4.1).

We introduce the set $[0,1]_\tau \times [0,1]_h$ of grid points

$$\begin{aligned}
[0,1]_\tau \times [0,1]_h = \big\{&(t_k, x_n) : t_k = k\tau,\ k = 1,\dots,N-1,\ N\tau = 1,\\
&x_n = nh,\ n = 1,\dots,M-1,\ Mh = 1\big\}.
\end{aligned}$$

Applying (2.7) and (2.10), respectively, we get the first order of accuracy difference scheme, in t and in x,

$$\begin{gathered}
\frac{v_n^{k+1}-2v_n^k+v_n^{k-1}}{\tau^2}+(1+x_n)\frac{v_{n+1}^k-2v_n^k+v_{n-1}^k}{h^2}+\frac{v_{n+1}^k-v_{n-1}^k}{2h}-v_n^k \\
=-f(t_k,x_n),\quad k=1,\dots,N-1,\ n=1,\dots,M-1, \\
v_0^k=0,\quad v_M^k-v_{M-1}^k=0,\quad k=0,\dots,N, \\
v_n^1-v_n^0-v_n^{l+1}+v_n^l=\tau(\varphi_n-\xi_n), \\
v_n^N-v_n^{N-1}-v_n^{l+1}+v_n^l=\tau(\psi_n-\xi_n), \\
\varphi_n=\varphi(x_n),\quad \psi_n=\psi(x_n),\quad \xi_n=\xi(x_n),\quad n=0,\dots,M,\ l=[\frac{\lambda}{\tau}],
\end{gathered}\tag{4.2}$$

and the second order of accuracy difference scheme, in t and in x,

$$\begin{gathered}
\frac{v_n^{k+1}-2v_n^k+v_n^{k-1}}{\tau^2}+(1+x_n)\frac{v_{n+1}^k-2v_n^k+v_{n-1}^k}{h^2}+\frac{v_{n+1}^k-v_{n-1}^k}{2h}-v_n^k \\
=-f(t_k,x_n),\quad k=1,\dots,N-1,\ n=1,\dots,M-1, \\
v_0^k=0,\quad -3v_M^k+4v_{M-1}^k-v_{M-2}^k=0, \\
10v_M^k-15v_{M-1}^k+6v_{M-2}^k-v_{M-3}^k=0,\quad k=0,\dots,N, \\
(-3v_n^0+4v_n^1-v_n^2)+(\frac{\lambda}{\tau}-l-1)(3v_n^l-4v_n^{l+1}+v_n^{l+2}) \\
-(\frac{\lambda}{\tau}-l)(3v_n^{l+1}-4v_n^{l+2}+v_n^{l+3})=2\tau(\varphi_n-\xi_n) \\
3v_n^N-4v_n^{N-1}+v_n^{N-2}+(\frac{\lambda}{\tau}-l-1)(3v_n^l-4v_n^{l+1}+v_n^{l+2}) \\
-(\frac{\lambda}{\tau}-l)(3v_n^{l+1}-4v_n^{l+2}+v_n^{l+3})=2\tau(\psi_n-\xi_n), \\
\varphi_n=\varphi(x_n),\quad \psi_n=\psi(x_n),\quad \xi_n=\xi(x_n),\quad k=0,\dots,N,\ l=[\frac{\lambda}{\tau}]
\end{gathered}\tag{4.3}$$

for the approximate solution of the auxiliary nonlocal problem (2.3). Note that both difference schemes have the second order of accuracy in x.

The difference scheme (4.2) can be rewritten in the matrix form

$$\begin{gathered}
A_nv_{n+1}+B_nv_n+C_nv_{n-1}=I\theta_n,\quad n=1,\dots,M-1, \\
v_0=\overrightarrow{0},\quad v_M-v_{M-1}=\overrightarrow{0}.
\end{gathered}\tag{4.4}$$

Here, θ_n, v_{n-1},v_n,v_{n+1} are $(N+1)\times 1$ column vectors

$$\begin{gathered}
\theta_n=\begin{bmatrix}\theta_n^0\\ \vdots\\ \theta_n^N\end{bmatrix}_{(N+1)\times 1},\quad v_i=\begin{bmatrix}v_i^0\\ \vdots\\ v_i^N\end{bmatrix}_{(N+1)\times 1},\quad i=n-1,n,n+1, \\
\theta_n^0=\tau(\varphi_n-\xi_n),\quad \theta_n^N=\tau(\psi_n-\xi_n),\quad n=1,\dots,M-1, \\
\theta_n^k=-f(t_k,x_n),\quad k=1,\dots,N-1,\quad n=1,\dots,M-1,
\end{gathered}$$

and A, B, C are $(N+1)\times(N+1)$ square matrices

$$A_n = \begin{bmatrix} 0 & 0 & & & 0 \\ 0 & a_n & & & \\ & & \ddots & & \\ & & & a_n & 0 \\ 0 & & & 0 & 0 \end{bmatrix}, \tag{4.5}$$

$$B_n = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & -1 & & & 0 \\ d & b_n & d & & & & & & \\ 0 & d & b_n & \ddots & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & d & b_n & d & & & \\ & & & & \ddots & \ddots & \ddots & & \\ & & & & & \ddots & b_n & d & 0 \\ & & & & & & d & b_n & d \\ 0 & & & 0 & 1 & -1 & 0 & -1 & 1 \end{bmatrix},$$

$$C_n = \begin{bmatrix} 0 & 0 & & & 0 \\ 0 & c_n & & & \\ & & \ddots & & \\ & & & c_n & 0 \\ 0 & & & 0 & 0 \end{bmatrix}, \tag{4.6}$$

$$\begin{gathered} a_n = \frac{(1+x_n)}{h^2} + \frac{1}{2h}, \quad b = -\frac{2}{\tau^2} - \frac{2(1+x_n)}{h^2} - 1, \\ c = \frac{(1+x_n)}{h^2} - \frac{1}{2h}, \quad d = \frac{1}{\tau^2}, \end{gathered}$$

and I is the $(N+1)\times(N+1)$ identity matrix.

We search for a solution of (4.4) by using the formula (see [32])

$$v_n = \alpha_{n+1} v_{n+1} + \beta_{n+1}, \quad n = 1, \dots, M-1,$$

where $\alpha_1, \dots, \alpha_{M-1}$ are $(N+1)\times(N+1)$ square matrices and $\beta_1, \dots, \beta_{M-1}$ are $(N+1)\times 1$ column matrices. For the solution of system equations (4.4), we have recurrent formulas for calculation of $\alpha_{n+1}, \beta_{n+1}$:

$$\begin{gathered} \alpha_{n+1} = -(B + C\alpha_n)^{-1} A, \\ \beta_{n+1} = -(B + C\alpha_n)^{-1}(I\theta_n - C\beta_n), \quad n = 1, \dots, M-1, \end{gathered}$$

where α_1 is the zero matrix and β_1 is the zero column vector.

Applying formula (2.4), we have

$$\begin{aligned} p_n &= 2x_n((3-\lambda)\exp(-\lambda) + 2\lambda - 2) \\ &\quad + (1+x_n)\Big[\frac{v_{n+1}^{l+2} - v_{n+1}^{l+1} - 2v_n^{l+2} + 2v_n^{l+1} + v_{n-1}^{l+2} - v_{n-1}^{l+1}}{h^2\tau}\Big] \\ &\quad + \Big[\frac{v_{n+1}^{l+2} - v_{n+1}^{l+1} - v_n^{l+2} + v_n^{l+1}}{\tau h}\Big] - \Big[\frac{v_n^{l+2} - v_n^{l+1}}{\tau}\Big], \quad n = 1, \dots, M-1. \end{aligned}$$

Now, the first order accuracy in t and the second order of accuracy in x an approximate solution of inverse problem will be defined by

$$u_n^k = v_n^k + t_k(\xi_n - \frac{v_n^{l+2} - v_n^{l+1}}{\tau}), \quad n = 0, \dots, M,\ k = 0, \dots, N.$$

The difference scheme (4.3) can be rewritten in the matrix form

$$\begin{gathered} A_n v_{n+1} + B_n v_n + C_n v_{n-1} = I\theta_n, \quad n = 1, \dots, M-1, \\ v_0 = \overrightarrow{0}, \quad -3v_M + 4v_{M-1} - v_{M-2} = \overrightarrow{0}. \end{gathered} \tag{4.7}$$

where matrices A_n and C_n are defined by (4.5), (4.6), respectively, B_n is the matrix

$$B_n = \begin{bmatrix} -3 & 4 & -1 & z_l & z_{l+1} & z_{l+2} & & & 0 \\ d & b_n & d & & & & & & \\ 0 & d & b_n & \ddots & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & d & b_n & d & & & \\ & & & & \ddots & \ddots & \ddots & & \\ & & & & & \ddots & b_n & d & 0 \\ & & & & & & d & b_n & d \\ 0 & & & z_l & z_{l+1} & z_{l+2} & 1 & -4 & 3 \end{bmatrix},$$

where

$$\begin{gathered} z_l = 3(\frac{\lambda}{\tau} - l - 1), \quad z_{l+1} = 4 - 7(\frac{\lambda}{\tau} - l), \\ z_{l+2} = -1 + 5(\frac{\lambda}{\tau} - l), \quad z_{l+3} = -\frac{\lambda}{\tau} + l. \end{gathered}$$

We search of a solution of (4.7) by using the formula

$$v_n = \alpha_n v_{n+1} + \beta_n v_{n+2} + \gamma_n, \quad n = 0, \dots, M-2,$$

where $\alpha_0, \dots, \alpha_{M-2}$ and $\beta_0, \dots, \beta_{M-2}$ are $(N+1) \times (N+1)$ square matrices and $\gamma_0, \dots, \gamma_{M-2}$ are $(N+1) \times 1$ column matrices. From (4.7) follows the next formulas for the coefficients $\alpha_n, \beta_n, \gamma_n$:

$$\begin{gathered} \alpha_n = -(B_n + C_n\alpha_{n-1})^{-1}(A_n + C_n\beta_{n-1}), \quad \beta_n = 0, \\ \gamma_n = (B_n + C_n\alpha_{n-1})^{-1}(I_{N+1}\theta_n - C_n\gamma_{n-1}), \quad n = 1, \dots, M-1, \end{gathered}$$

where

$$\begin{gathered} \alpha_0 = 0, \quad \beta_0 = 0, \quad \gamma_0 = 0, \quad \alpha_1 = \frac{8}{5}I, \quad \beta_1 = -\frac{3}{5}I, \\ \alpha_{M-2} = 4I, \quad \beta_{M-2} = -3I, \quad \alpha_{M-3} = \frac{8}{3}I, \quad \beta_{M-3} = -\frac{5}{3}I, \end{gathered}$$

and $\gamma_0, \gamma_1, \gamma_{M-2}, \gamma_{M-3}$ are zero column vectors. We denote

$$\begin{gathered} Q_{11} = -3A_{M-2} - 8B_{M-2} - 8C_{M-2}\alpha_{M-3} - 3C_{M-2}\beta_{M-3}, \\ Q_{12} = 4A_{M-2} + 9B_{M-2} + 9C_{M-2}\alpha_{M-3} + 4C_{M-2}\beta_{M-3}, \\ Q_{21} = -3B_{M-1} - 8C_{M-1},\ Q_{22} = A_{M-1} + 4B_{M-1} + 9C_{M-1}, \\ G_1 = I\theta_{M-2} - C_{M-2}\gamma_{M-3}, G_2 = I\theta_{M-1}. \end{gathered}$$

Then, v_M and v_{M-1} can be calculated by the formulae

$$v_M = (Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})^{-1}(G_1 - Q_{12}Q_{22}^{-1}G_2),$$
$$v_{M-1} = Q_{22}^{-1}(G_2 - Q_{21}v_M).$$

Applying (2.4), we obtain

$$\begin{aligned} p_n &= 2x_n((3-\lambda)\exp(-\lambda) + 2\lambda - 2) \\ &\quad + (1+x_n)\Big[\frac{v_{n+1}^{l+2} - v_{n+1}^{l} - 2v_n^{l+2} + 2v_n^l + v_{n-1}^{l+2} - v_{n-1}^{l}}{2h^2\tau}\Big] \\ &\quad + \Big[\frac{v_{n+1}^{l+2} - v_{n+1}^{l} - v_n^{l+2} + v_n^l}{2\tau h}\Big] - \Big[\frac{v_n^{l+2} - v_n^l}{2\tau}\Big], \quad n = 1, \dots, M-1. \end{aligned}$$

Finally, the second order of accuracy in t and x of an approximate solution of inverse problem will be defined by

$$u_n^k = v_n^k + t_k\big(\xi_n - \frac{v_n^{l+2} - v_n^l}{2\tau}\big), \quad n = 0, \dots, M,\ k = 0, \dots, N.$$

Now, using MATLAB programs, we give numerical results for problem (4.1). The numerical results are presented for different N and M.

u_n^k represents the numerical solution of the corresponding difference schemes for inverse problem at grid point (t_k, x_n). p_n represents the numerical solution at point x_n for unknown function p. The errors of approximate solutions are computed by the norms

$$Eu_M^N = \max_{1\le k\le N-1}\Big(\sum_{n=1}^{M-1}|u(t_k,x_n) - u_n^k|^2 h\Big)^{1/2},$$
$$Ep_M = \Big(\sum_{n=1}^{M-1}|p(x_n) - p_n|^2 h\Big)^{1/2},$$

respectively. Tables 1–2 give the error of approximate solutions of difference schemes for given exact solution. They show numerical results for $N = M = 20, 40, 80, 160$. Errors for p are in Table 1, and for u in Table 2. The numerical results show that the second order of accuracy difference scheme is more accurate comparing with the first order of accuracy difference scheme.

TABLE 1. Errors for p

accuracy DS	$N=M=20$	$N=M=40$	$N=M=80$	$N=M=160$
1st order	0.13122	0.064479	0.031979	0.015927
2nd order	0.0034378	8.93×10^{-4}	2.277×10^{-4}	5.75×10^{-5}

TABLE 2. Errors for u

accuracy DS	$N=M=20$	$N=M=40$	$N=M=80$	$N=M=160$
1st order	0.024351	0.012054	0.005994	0.0029885
2nd order	0.0011314	2.81×10^{-4}	7.01×10^{-5}	1.75×10^{-5}

Conclusion. In this paper, a numerical solution of inverse problem for the multidimensional elliptic equation with Neumann type overdetermination and Dirichlet-Neumann boundary conditions is considered. The first and second order of accuracy difference schemes for this inverse problem are constructed. We establish the stability and almost coercive stability estimates for the solution of these difference schemes. Numerical example with explanation on the realization is included to illustrate theoretical results. Moreover, applying the results of works [3], [8] the high order of accuracy stable difference schemes for the numerical solution of the inverse elliptic problem with Neumann type overdetermination can be presented.

References

[1] T. S. Aleroev, M. Kirane, Salman A. Malik; *Determination of a source term for a time fractional diffusion equation with an integral type over-determining condition*, Electron. J. Differential Equations **2013** (2013), No. 270, pp. 1–16.

[2] A. Ashyralyev; *On the problem of determining the parameter of a parabolic equation*, Ukr. Math. J. **62** (9) (2011), pp. 1397–1408.

[3] A. Ashyralyev, D. Arjmand; *A note on the Taylor's decomposition on four points for a third-order differential equation*, Applied Mathematics and Computation, **188** (2) (2007), pp. 1483–1490.

[4] A. Ashyralyev, C. Ashyralyyev; *On the problem of determining the parameter of an elliptic equation in a Banach space,* Nonlinear Anal. Model. Control **19** (3) (2014), pp. 350–366.

[5] A. Ashyralyev, A. S. Erdogan; *Well-posedness of the inverse problem of a multidimensional parabolic equation*, Vestn. Odessa Nat. Univ., Math. Mech., **15** (18) (2010), pp. 129–135.

[6] A. Ashyralyev, A. S. Erdogan, O. Demirdag; *On the determination of the right-hand side in a parabolic equation*, Appl. Numer. Math. **62** (11) (2012), pp. 1672–1683.

[7] A. Ashyralyev, A. Sarsenbi; *Well-posedness of a parabolic equation with nonlocal boundary condition*, Bound. Value Probl. **2015**:51 (2015).

[8] A. Ashyralyev, P. E. Sobolevskii; *New Difference Schemes for Partial Differential Equations*, Operator Theory Advances and Applications, Birkhäuser Verlag, Basel, Boston, Berlin, (2004).

[9] A. Ashyralyev, M. Urun; *Determination of a control parameter for the Schrodinger equation*, Contemporary Analysis and Applied Mathematics **1** (2) (2013), pp. 156–166.

[10] C. Ashyralyyev; *High order of accuracy difference schemes for the inverse elliptic problem with Dirichlet condition*, Bound. Value Probl. **2014**:5 (2014), 23 pages.

[11] C. Ashyralyyev; *High order approximation of the inverse elliptic problem with Dirichlet-Neumann Conditions*, Filomat, **28** (5) (2014), pp. 947–962.

[12] C. Ashyralyyev; *Inverse Neumann problem for an equation of elliptic type,* AIP Conference Proceedings, **1611** (2014), pp. 46-52.

[13] C. Ashyralyyev, M. Dedeturk; *Finite difference method for the inverse elliptic problem with Dirichlet condition*, Contemporary Analysis and Applied Mathematics, **1** (2) (2013), pp. 132–155.

[14] C. Ashyralyyev, M. Dedeturk; *Approximate solution of inverse problem for elliptic equation with overdetermination*, Abstr. Appl. Anal. **2013** (2013), Article ID 548017, 11 pages.

[15] C. Ashyralyyev, M. Dedeturk; *Approximation of the inverse elliptic problem with mixed boundary value conditions and overdetermination*, Bound. Value Probl. **2015**:51 (2015), 15 pages.

[16] C. Ashyralyyev, O. Demirdag; *The difference problem of obtaining the parameter of a parabolic equation*, Abstr. Appl. Anal. **2012** (2012), Article ID 603018, 14 pages.

[17] C. Ashyralyyev, A. Dural, Y. Sozen; *Finite difference method for the reverse parabolic problem with Neumann condition*, AIP Conference Proceedings **1470** (2012), pp. 102-105.

[18] A. Bouzitouna, N. Boussetila, F. Rebbani; *Two regularization methods for a class of inverse boundary value problems of elliptic type*, Bound. Value Probl. **2013**:178 (2013).

[19] Y. S. Eidel'man; *An inverse problem for an evolution equation*, Mathematical Notes, **49** (5) (1991), pp. 535–540.

[20] A. S. Erdogan, A. Sazaklioglu; *A note on the numerical solution of an identification problem for observing two-phase flow in capillaries*, Mathematical method in the Aplied Sciences, **37** (16) (2014), pp. 2393–2405.

[21] S. I. Kabanikhin; *Inverse and ill-posed problems: theory and applications,* Walter de Gruyter, Berlin, (2011).

[22] M. Kirane, Salman A. Malik, M. A. Al-Gwaiz; *An inverse source problem for a two dimensional time fractional diffusion equation with nonlocal boundary conditions*, Mathematical Methods in the Applied Sciences, **36** (9) (2013), pp. 056–069.

[23] S. G. Krein; *Linear Differential Equations in Banach Space.* Nauka, Moscow, Russia, (1966).

[24] D. G. Orlovskii; *Inverse Dirichlet Problem for an Equation of Elliptic Type*, Differ. Equ., **44** (1) (2008), pp. 124–134.

[25] D. G. Orlovsky, S. Piskarev; *On approximation of inverse problems for abstract elliptic problems*, J. Inverse Ill-Posed Probl., **17** (8) (2009), pp. 765–782.

[26] D. Orlovsky, S. Piskarev; *The approximation of Bitzadze-Samarsky type inverse problem for elliptic equations with Neumann conditions*, Contemporary Analysis and Applied Mathematics **1** (2) (2013), pp. 118–131.

[27] D. Orlovsky, S. Piskarev; *Approximation of inverse Bitzadze-Samarsky problem for elliptic eqaution with Dirichlet conditions*, Differ. Equ. **49** (7) (2013), pp. 895–907.

[28] A. I. Prilepko, D. G. Orlovsky, I. A. Vasin; *Methods for solving inverse problems in mathematical physics*, Marcel Dekker, New York, (2000).

[29] A. Qian; *Identifying an unknown source in the Poisson equation by a wavelet dual least square method*, Bound. Value Probl. **2013**:267 (2013).

[30] N. C. Roberty; *Simultaneous Reconstruction of Coefficients and Source Parameters in Elliptic Systems Modelled with Many Boundary Value Problems*, Mathematical Problems in Engineering **2013** (2013), Article ID 631950.

[31] A. A. Samarskii, P. N. Vabishchevich; *Numerical methods for solving inverse problems of mathematical physics. Inverse and Ill-Posed Problems Series,* Walter de Gruyter, Berlin, (2007).

[32] A. A. Samarskii, E. S. Nikolaev; *Numerical methods for grid equations, Vol 2,* Birkhäuser, Basel, Switzerland, (1989).

[33] P. E. Sobolevskii; *Difference Methods for the Approximate Solution of Differential Equations.* Voronezh State University Press, Voronezh, Russia, 1975 (in Russian)

[34] V. V. Soloviev; *Inverse problems of source determination for the two-dimensional Poisson equation*, Zh. Vychisl. Mat. Mat. Fiz., **44** (5) (2004), pp. 862–871.

[35] V. V. Soloviev; *Inverse Problems of Determining the Source and Coefficient in an Elliptic Equation in a Rectangle*, Zh. Vychisl. Mat. Mat. Fiz., **47** (8) (2007), pp. 1365–1377.

[36] V. V. Soloviev; *Inverse Coefficient Problems for Elliptic Equations in a Cylinder: I,* Differ. Equ., **49** (7) (2011), pp. 908–916.

Charyyar Ashyralyyev
Department of Mathematical Engineering, Gumushane University, Gumushane, Turkey
E-mail address: `charyyar@gumushane.edu.tr`

Yasar Akkan
Department of Mathematical Engineering, Gumushane University, Gumushane, Turkey
E-mail address: `akkanyasar61@gumushane.edu.tr`