PERIODIC ORBITS IN HYPERCHAOTIC CHEN SYSTEMS

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ABSTRACT. In this work, we show a zero-Hopf bifurcation in a Hyperchaotic
Chen system. Using averaging theory, we prove the existence of two periodic
orbits bifurcating from the zero-Hopf equilibria located at the origin of the
Hyperchaotic Chen system.

1. Introduction and statement of the results

Hyperchaos has been widely investigated since in 1979 when Otto Rössler [20]
proposed one of the first hyperchaotic attractors. A hyperchaotic system is a chaotic
system in which two or more Lyapunov exponents are positive indicating that the
chaotic dynamics are expanded in more than one direction. This feature gives rise
to complex attractors harder to control than chaotic ones. Thus, hyperchaos has
been found to be useful in many fields such as: encryption, secure communications
[4], nonlinear circuits [12], liquid mixing, lasers [9], and many more.

It is worth to say that the minimal dimension of the phase space that embeds
a hyperchaotic attractor must be four, so the typical examples of hyperchaotic
systems have been introduced usually as extensions of known autonomous three-
dimensional chaotic systems. Besides the hyperchaotic attractor of Rössler, the
hyperchaotic Lorenz-Haken system [16], the Chua's circuit [11] and the hyperchaotic
Lü system [13] are well-known examples of hyperchaotic models as extensions of
three-dimensional chaotic systems. The book [22] presents a catalog of systems
exhibiting chaos.

In this work we study the hyperchaotic Chen system

$$\begin{aligned}
\dot{x} &= a(y - x) + w, \\
\dot{y} &= dx + cy - xz, \\
\dot{z} &= xy - bz, \\
\dot{w} &= yz + rw.
\end{aligned} \tag{1.1}$$

This system was introduced in [5], as an extension of a three-dimensional chaotic
Chen system. Its study has generated considerable research, techniques of chaos
synchronization, studies of controlling chaos, secure communications, power system
protection and so on, see [10, 6, 17, 21] and the references therein.

The dynamics of Hyperchaotic systems has been studied mainly numerically and by computer simulations. In this work we perform an analytic analysis on the existence of periodic orbits of the differential system (1.1) by applying *averaging theory of first order*. More precisely, we prove that a *zero-Hopf* bifurcation occurs in system (1.1) bifurcating two limit cycles from the *zero-Hopf* equilibria as parameters vary. As far as I know, this is the first time that an analytic analysis of a zero-Hopf bifurcation in hyperchaotic Chen system is performed.

We recall that an equilibrium of a differential system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{f} : A \to \mathbb{R}^n$ and A an open subset of $\mathbb{R}^n$ is a zero-Hopf equilibrium if it has two pure imaginary conjugated eigenvalues and $n-2$ zero eigenvalues. The bifurcation of periodic orbits from zero-Hopf equilibria of three dimensional differentials systems has been studied via averaging theory in [14]. See also [8] for a study in systems with two slow and one fast variables. In [7] is performed an analysis of the periodic orbits bifurcating in the chaotic prototype-4 system of Rössler. Recently, was published [3], a paper on a zero-Hopf bifurcation in the hyperchaotic lorenz system. The main result of this work reads as follows.

Theorem 1.1. *Consider the hyperchaotic Chen system* (1.1) *with $c = a$ for small values of b and r, and satisfying $b(a+d)$ $r > 0$ and $a(a+d) < 0$. Then a zero-Hopf bifurcation occurs emerging two limit cycles from the origin.*

In section 2 we give a summary of the averaging theory of first order for finding limit cycles and we proof Theorem 1.1 in section 3.

2. Averaging theory: perturbing an isochronous system

We consider the problem of bifurcation of T-periodic solutions for a differential system of the form
$$\dot{\mathbf{x}} = \mathbf{F}_0(t, \mathbf{x}) + \varepsilon \mathbf{F}_1(t, \mathbf{x}, \varepsilon), \tag{2.1}$$
where ε is a small positive parameter, $\mathbf{x} \in A$, where A is an open subset of $\mathbb{R}^n$, $t \geq 0$. Moreover we assume that both $\mathbf{F}_0(t, \mathbf{x})$ and $\mathbf{F}_1(t, \mathbf{x}, \varepsilon)$ are $\mathcal{C}^2$ functions and T-periodic in t.

The classical theory of averaging reduce the problem of finding T-periodic solutions of (2.1) for $\varepsilon > 0$ small to the problem of finding simple zeros of the so-called *bifurcation functions*. Many methods encountered in the literature are based on this idea, see [2, 15, 18].

We study here the particular case in which all the solutions of the unperturbed system
$$\dot{\mathbf{x}} = \mathbf{F}_0(t, \mathbf{x}) \tag{2.2}$$
are T-periodic. Then we assume that the unperturbed system is isochronous and the problem in what we are interested in is the problem of the persistence of the periodic orbits under some perturbation.

We denote the linearization of (2.2) along a periodic solution $\mathbf{x}(t, \mathbf{u})$ of (2.2) such that $\mathbf{x}(0, \mathbf{u}) = \mathbf{u}$ by
$$\dot{\mathbf{y}} = D_{\mathbf{x}} \mathbf{F}_0(t, \mathbf{x}(t, \mathbf{u}))$$
where $D_{\mathbf{x}} \mathbf{F}_0$ is the Jacobian matrix of F_0 with respect to $\mathbf{x}$ and let $\Phi_{\mathbf{u}}(t)$ be some fundamental matrix of (2.2). Assume that there exists an open set V with closure $\bar{V} \subset A$ such that for each $\mathbf{u} \in \bar{V}$, $\mathbf{x}(t, \mathbf{u})$ is T-periodic. The following result gives an answer to the question of bifurcating periodic solutions from the T-periodic solutions $\mathbf{x}(t, \mathbf{u})$.

Theorem 2.1. *We assume that there exists an open set V with $\bar{V} \subset A$ such that for each $\mathbf{u} \in \bar{V}$, $\mathbf{x}(t, \mathbf{u})$ is T-periodic. Consider the function $\mathbf{f} : \bar{V} \mapsto \mathbb{R}^n$ given by*

$$\mathbf{f}(\mathbf{u}) = \frac{1}{T} \int_0^T \Phi_{\mathbf{u}}^{-1} \mathbf{F}_1(t, \mathbf{x}(t, \mathbf{u})) \, dt \,. \tag{2.3}$$

If there exists $\mathbf{p} \in V$ with $\mathbf{f}(\mathbf{p}) = 0$ and $\det(D_{\mathbf{u}} \mathbf{f}(\mathbf{p}))) \neq 0$ then there exists a T-periodic solution $\gamma(t, \varepsilon)$ of system (2.1) such that $\gamma(0, \varepsilon) \to \mathbf{p}$ as $\varepsilon \to 0$. Moreover, if all the eigenvalues of $\det(D_{\mathbf{u}} \mathbf{f}(\mathbf{p})))$ have negative real part, the corresponding periodic orbit $\gamma(t, \varepsilon)$ is asymptotically stable for ε sufficiently small.

For a proof of Theorem 2.1 see [15, 19, 1].

3. Proof of Theorem 1.1

The origin of system (1.1) is always an equilibrium point of it under under any choice of parameters. The characteristic polynomial $p(\lambda)$ of the linearization of system (1.1) at the equilibrium point located at the origin is given by

$$p(\lambda) = (r - \lambda)(b + \lambda)(a(c + d - \lambda) + (c - \lambda)\lambda)$$

The eigenvalues associated at the origin are

$$\lambda_1 = r, \quad \lambda_2 = -b, \quad \lambda_{3,4} = \frac{1}{2}\left(-a + c \pm \sqrt{a^2 + 2ac + c^2 + 4ad}\right).$$

This suggest to consider small values of the parameters r and b introducing the small parameter ε in the following way $(r, b) \mapsto (\varepsilon r, \varepsilon b)$. Thus, taking $c = a$ the eigenvalues of system (1.1) at the origin becomes $\lambda_{1,2} = 0$ and $\lambda_{3,4} = \pm\sqrt{a(a + c)}$ when $\varepsilon \mapsto 0$. It follows that the origin is a zero-Hopf equilibrium when $\varepsilon \mapsto 0$ if $a(a + c) < 0$. So, for this choice of parameters we have system (1.1) written as

$$\begin{aligned}
\dot{x} &= a(y - x) + w, \\
\dot{y} &= dx + ay - xz, \\
\dot{z} &= xy - b\varepsilon z, \\
\dot{w} &= yz + r\varepsilon w.
\end{aligned} \tag{3.1}$$

Re-scaling the variables $(x, y, z, w) \mapsto (\varepsilon x, \varepsilon y, \varepsilon z, \varepsilon w)$, system (3.1) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} a(y - x) + w \\ dx + ay \\ 0 \\ 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ -xz \\ xy - bz \\ yz + rw \end{pmatrix}. \tag{3.2}$$

Now we have the Chen system (1.1) written as differential system of the form (2.1), and we can consider the problem of bifurcating periodic solutions of it by using averaging theory. We have to solve first the unperturbed system of (3.2). The solution $\mathbf{x}(t, \mathbf{u}) = (x(t), y(t), z(t), w(t))$ of

$$\begin{aligned}
\dot{x} &= a(y - x) + w, \\
\dot{y} &= dx + ay, \\
\dot{z} &= 0, \\
\dot{w} &= 0,
\end{aligned} \tag{3.3}$$

satisfying the initial condition $\mathbf{u} = (x(0), y(0), z(0), w(0)) = (x_0, y_0, z_0, w_0) \in \mathbb{R}^4$ is

$$x(t) = \frac{1}{a+d}(w_0 + ((a+d)x_0 - w_0)\cosh(\sqrt{a(a+d)}\ t)$$
$$+ \frac{\sqrt{a+d}}{\sqrt{a}}(w_0 + a(y_0 - x_0))\sinh(\sqrt{a(a+d)}\ t)),$$
$$y(t) = \frac{1}{a(a+d)}((dw_0 + a(a+d)y_0)\cosh(\sqrt{a(a+d)}\ t) - dw_0 \qquad (3.4)$$
$$+ \sqrt{a}\sqrt{a+d}(dx_0 + ay_0)\sinh(\sqrt{a(a+d)}\ t)),$$
$$w(t) = w_0,$$
$$z(t) = z_0.$$

Notice that if $a(a+d) < 0$ then $\cosh(\sqrt{a(a+d)}t) = \cos(\sqrt{-a(a+d)}t)$ and $\sinh(\sqrt{a(a+d)}t) = i\sin(\sqrt{-a(a+d)}t)$ being $i^2 = -1$. Hence, when $a(a+d) < 0$ the solution (3.4) of the unperturbed system (3.3) is

$$x(t) = \frac{1}{a+d}(w_0 + ((a+d)x_0 - w_0)\cos(\Omega t) - \frac{\Omega}{a}(w_0 + a(y_0 - x_0))\sin(\Omega t)),$$
$$y(t) = \frac{1}{a(a+d)}((dw_0 + a(a+d)y_0)\cos(\Omega t) - dw_0 - \Omega(dx_0 + ay_0)\sin(\Omega t)),$$
$$w(t) = w_0,$$
$$z(t) = z_0.$$
$$(3.5)$$

where $\Omega = \sqrt{-a(a+d)}$. We have that any solution (3.5) of the unperturbed system (3.3) is periodic of period $T = \frac{2\pi}{\Omega}$. So system (3.2) is, in fact, a perturbation of an isochronous system when $a(a+c) < 0$ and we can apply Theorem 2.1. The first variational system of (3.3) along the solution (3.5) coincides with the unperturbed system (3.3), so the inverse of fundamental matrix solution $\Phi_{\mathbf{u}}(t)$ of (3.3) is

$$\begin{pmatrix} \cos(\Omega t) + \frac{a}{\Omega}\sin(\Omega t) & -\frac{a}{\Omega}\sin(\Omega t) & 0 & \frac{1}{(a+d)}(1 - \cos(\Omega t) + \frac{\Omega}{a}\sin(\Omega t)) \\ -\frac{d}{\Omega}\sin(\Omega t) & \cos(\Omega t) - \frac{a}{\Omega}\sin(\Omega t) & 0 & \frac{d}{a(a+d)}(\cos(\Omega t) - 1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The bifurcation function (2.3) is given by

$$f(\mathbf{u}) = \frac{\Omega}{2\pi}\int_0^{\frac{2\pi}{\Omega}} \Phi_{\mathbf{u}}^{-1} F_1(t, \mathbf{x_u})\, dt = (f_1, f_2, f_3, f_4),$$

where

$$f_1(\mathbf{u}) = \frac{rw_0}{a+d} + \frac{dz_0((a+d)x_0 - 3w_0)}{2a(a+d)^2} - \frac{(a(y_0 - x_0) + w_0)z_0}{2(a+d)},$$
$$f_2(\mathbf{u}) = \frac{d(3dw_0 + a(a+d)y_0)z_0}{2a^2(a+d)^2} - \frac{drw_0}{a(a+d)} - \frac{(dx_0 + ay_0)z_0}{2(a+d)},$$
$$f_3(\mathbf{u}) = \frac{d^2(x_0^2 + 2bz_0)}{2(a+d)^2} + \frac{a(2abz_0 - y_0(2w_0 + a(y_0 - 2x_0))}{2(a+d)^2}$$
$$- \frac{(3w_0^2 + 2aw_0y_0)d}{2a(a+d)^2} + \frac{ad(x_0^2 + 2x_0y_0 - y_0^2 + 4bz_0))}{2(a+d)^2},$$

$$f_4(\mathbf{u}) = w_0\Big(r - \frac{dz_0}{a(a+d)}\Big).$$

The isolated zeros of the map $\mathbf{u} \mapsto f(\mathbf{u}) = (f_1(\mathbf{u}), f_2(\mathbf{u}), f_3(\mathbf{u}), f_4(\mathbf{u}))$ are

$$\mathbf{p}_{1,2} = \Big(\pm\frac{a\sqrt{b(a+d)r}}{d}, \mp\sqrt{b(a+d)r}, \frac{a(a+d)r}{d}, \pm\frac{a\sqrt{b(a+d)r}(a+d)}{d}\Big)$$

Notice that $\mathbf{p}_{1,2} \in \mathbb{R}$ because of conditions of Theorem 1.1. The determinant of the Jacobian matrix of $\mathbf{f}$ at the points $\mathbf{p}_{1,2}$ is

$$\det(D\mathbf{f}(\mathbf{p}_{1,2})) = \frac{b(a^4 + a^3 d - d^2)r^3}{2d^2}.$$

From condition $a(a+d) < 0$ we have that $d \neq 0$ and $a^4 + a^3 d - d^2 = a^2(a(a+d) - d^2) \neq 0$. Moreover, from $b(a+d)r < 0$ we have $br \neq 0$. Then we get $\det(D\mathbf{f}(\mathbf{p}_{1,2})) \neq 0$ and $\mathbf{p}_{1,2}$ are simples zeroes of $\mathbf{f}$. Hence, the averaging theory stated in Theorem 2.1 predicts the existence of two T-periodic orbits $\gamma_{1,2}(t, \varepsilon)$ of system (3.1) with period $\frac{2\pi}{\Omega}$ such that $\gamma_{1,2}(0, \varepsilon) \to (p_{1,2})$ as $\varepsilon \to 0$.

Since we have performed the re-scaling $(x, y, z, w) \mapsto (\varepsilon x, \varepsilon y, \varepsilon z, \varepsilon w)$ for bringing system (1.1) to system (3.1), the solutions $\gamma_{1,2}(t, \varepsilon)$ of system (3.1) provides the periodic orbits $\varepsilon\gamma_{1,2}(t, \varepsilon)$ of system (1.1) tending to the zero-Hopf equilibrium as $\varepsilon \to 0$. This completes the proof.

Remark 3.1. Regarding the stability of the bifurcated periodic orbits, we know that the eigenvalues of $D\mathbf{f}(\mathbf{p}_{1,2})$ are

$$\lambda_{1,2} = \frac{1}{2}\Big(b \pm \sqrt{b(b+8r)}\Big), \quad \lambda_{3,4} = \frac{r}{2}\Big(1 \pm i\frac{a\Omega}{d}\Big).$$

Therefore, under our parameter restrictions we cannot have all the eigenvalues with negative real part. In consequence we cannot use Theorem 2.1 for the analysis of the periodic orbits $\gamma_{1,2}(t, \varepsilon)$.

References

[1] A. Buică, J. P. Françoise, J. Llibre; *Periodic solutions of nonlinear periodic differential systems with a small parameter*, Comm. on Pure and Appl. Anal., **6**, 103–111, (2007).

[2] A. Buică, J. Llibre, O. Makarenkov; *Asymptotic stability of periodic solutions for nonsmooth differential equations with application to the nonsmooth van der Pol oscillator*, SIAM J. Math. Anal. **40** (2009), 2478–2495.

[3] L. Cid-Montiel, J. Llibre, C. Stoica; *Zero-Hopf bifurcation in a hyperchaotic Lorenz system*, Nonlinear Dynam. **75** (2014), no. 3, 561-566.

[4] G. Chen, X. Dong; *From chaos to order, Methodologies, Perspectives and applications*, World Scientific, Singapore, 1998, 12–27

[5] A. M. Chen, J. A. Lu, J.H. Lü, S. M. Yu; *Generating hyperchaos via state feedback control*, International Journal of Bifurcation and Chaos, **15** (10)(2005), 3367–3375.

[6] S. Effati, H. Saberi Nik, A. Jajarmi; *Hyperchaos control of the hyperchaotic Chen system by optimal control design.* Nonlinear Dynam, **73** (2013), no. 1-2, 499-508.

[7] I. A. García, J. Llibre, S. Maza; *Periodic orbits and their stability in the Rössler prototype-4 system*, Phys. Lett. A, **376** (2012), 2234-2237.

[8] I. A. García, J. Llibre, S. Maza; *On the periodic orbit bifurcating from a zero Hopf bifurcation in systems with two slow and one fast variables*, Appl. Math. Comput., **232** (2014), 8490.

[9] H. Haken; *Analogy between higher instabilities in fluids and lasers*, Physiscs Letters A, **53** (1975), 77–78.

[10] A. S. Hegazi, A. E. Matouk; *Dynamical behaviors and synchronization in the fractional order hyperchaotic Chen system*, Appl. Math. Lett., **24** (2011), no. 11, 1938-1944.

[11] T. Kapitaniak, L. Chua; *Hyperchaotic attractors of unidirectionally-coupled Chu'as circuits*, Int. J. Bifurcat. chaos Appl. Sci. Eng., **364** (2006), 103–110.

[12] S. Li, G. Chen and X. Mou; *On the Security of the Yi-Tan-Siew Chaotic Cipher*, IEEE Transactions on Circuits and Systems-II, **51**, no 12, (2004), 665–669

[13] Y. X. Li, W. K. S. Tang, G. R. Chen; *Generating hyperchaotic Lü attractor via state feedback control*, Physica A, **53** (1975), 77–78.

[14] J. Llibre, C. Valls; *Hopf bifurcation for some analytic differential systems in $\mathbb{R}^3$ via averaging theory*, Discrete Contin. Dyn. Syst., **30** (2011), no. 3, 779790.

[15] I. G. Malkin; *Some problems of the theory of nonlinear oscillations*, (Russian) Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 492 pp.

[16] S. C. Ning, H. Haken; *Detuned lasers and the complex lorenz equation: subcritical and supercritical Hopf bifurcations*, Phys. Rev. A, **41** (1990), 3826–3837.

[17] J. H. Park; *Adaptive synchronization of hyperchaotic Chen system with uncertain parameters*, Chaos Solitons Fractals, **26** (2005), no. 3, 959-964.

[18] M. B. H. Rhouma, C. Chicone; *On the continuation of periodic orbits*, Methods Appl. Anal., **7** (2000), 85104.

[19] M. Roseau; *Vibrations non linéaires et théorie de la stabilité*, Springer Tracts in Natural Philosophy, Vol. 8 Springer-Verlag, Berlin-New York 1966.

[20] O. E. Rössler; *An equation for hyperchaos*, Physiscs Letters A, **71** (2-3)(1979), 155–156.

[21] N. Smaoui, A. Karouma, M. Zribi; *Secure communications based on the synchronization of the hyperchaotic Chen and the unified chaotic systems*, Commun. Nonlinear Sci. Numer. Simul. **41** (2011), no. 8, 32793293.

[22] J. C. Sprott; *Elegant chaos. Algebraically simple chaotic flows.* World Scientific Publishing, 2010.

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