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ENERGY DECAY OF A VARIABLE-COEFFICIENT WAVE EQUATION WITH NONLINEAR TIME-DEPENDENT LOCALIZED DAMPING

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Abstract. We study the energy decay for the Cauchy problem of the wave equation with nonlinear time-dependent and space-dependent damping. The damping is localized in a bounded domain and near infinity, and the principal part of the wave equation has a variable-coefficient. We apply the multiplier method for variable-coefficient equations, and obtain an energy decay that depends on the property of the coefficient of the damping term.

1. Introduction

Let $n \geq 2$ be an integer. We consider the energy decay for the solution to the Cauchy problem of the wave equation with time-dependent and space-dependent damping,

$$u_{tt} - \operatorname{div}(A(x)\nabla u) + u + \sigma(t)\rho(x, u_t) = 0, \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.1}$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $A(x) = (a_{ij}(x))$ is a symmetric, positive matrix for each $x \in \mathbb{R}^n$; $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing function of class C^1. Let Ω and Ω_1 be two bounded domains such that $\Omega_1 \subset \Omega$. We assume that

$$\rho(x, u_t) = \begin{cases} a(x)u_t, & x \in \mathbb{R}^n \setminus \Omega_1 \\ a(x)h(u_t), & x \in \Omega_1 \end{cases} \tag{1.3}$$

where $h : \mathbb{R} \to \mathbb{R}$ is a nonlinear continuous nondecreasing function satisfying $h(s)s > 0$ for all $s \neq 0$ and $a(\cdot) \in L^\infty(\mathbb{R}^n)$ is a nonnegative function satisfying

$$a(x) \geq \varepsilon_0 > 0, \quad x \in \Omega_1 \cup \Omega^c. \tag{1.4}$$

When $x \in \Omega_1$,

$$c_1|u_t|^m \leq |h(u_t)| \leq c_2|u_t|^{1/m}, \quad |u_t| \leq 1, \tag{1.5}$$

$$c_3|u_t| \leq |h(u_t)| \leq c_4|u_t|, \quad |u_t| \geq 1 \tag{1.6}$$

where $m \geq 1$ and $c_i > 0$ $(i = 1, 2, 3, 4)$ are given numbers.

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Since (1.4) implies that the dissipation may vanish in $\Omega\setminus\Omega_1$, we call the damping $\sigma(t)\rho(x,u_t)$ a time-dependent localized damping.

Many studies concerning energy decay of wave equations in the bounded domain are available in the literature. We refer the readers to [3, 8, 9, 16]. The author of [8] derived precise energy decay estimates for the initial-boundary value problem to the wave equation with a localized nonlinear dissipation which depended on the time as well as the space variable. The result of [8] is a generalization of the work [9], where the dissipation was independent of the time. A wave equation with time-dependent but space-independent damping was investigated in [3] and the decay rate was obtained by solving a nonlinear ODE.

For the energy decay of wave equation in the exterior domain or on the whole space, many results have been obtained for the case of constant coefficients, that is $A(x)\equiv I$ (see [1, 2, 4, 10, 11, 12, 17]). The damping in [4, 11, 17] were time-independent and localized in a sub-domain of $\mathbb{R}^n$. Energy decay of the Cauchy problem for the wave equation with time-dependent damping was studied in [2] where the coefficient of the damping did not depend on the space variable. It is interesting to study the case where the damping is time-dependent and exists on a local domain of $\mathbb{R}^n$. [1] addressed the decay for a wave equation on an exterior domain where the damping is time-dependent and effective only in a ball of $\mathbb{R}^n$. The dimension of the space variable was restricted to be odd since the Lax-Phillips' scattering theory was used in [1]. Nakao [10] considered the Cauchy problem of wave equation with a dissipation effective outside of a given ball of $\mathbb{R}^n$.

However, in the case of variable coefficient, few results about the energy decay of wave equation in the exterior domain or on the whole space can be seen (see[5, 13]). Yao [13] studied the energy decay for the Cauchy problem of the variable-coefficient wave equation with a linear damping. The authors of [5] considered the energy decay of variable-coefficient wave equation with nonlinear damping in an exterior domain.

The purpose of this paper is to derive the energy decay for (1.1)-(1.2) with the assumptions (1.3)-(1.6). Our first main goal is to dispense with the restriction $A(x)\equiv I$. We will use the Riemannian geometry method which was introduced by Yao [14] (see also [15]) to deal with controllability for the wave equation with variable-coefficient principal part and has been viewed as an important method for variable-coefficient models.

Our second goal is to generalize the work [2], where the damping exists on the whole space. In this paper, we assume that the damping is localized near the origin point and near infinity and may disappear in a large area. We try to explain how the time-dependence of the damping affect the decay when the dissipation is effective in a localized domain. Energy decay is obtained under some conditions on $A(x)$ and the coefficient of the damping.

We introduce a Riemanian metric on $\mathbb{R}^n$ by

$$g(x)=A^{-1}(x)\quad\text{for}\quad x\in\mathbb{R}^n. \tag{1.7}$$

We denote by ∇f and $\nabla_g f$ the gradients of f in the standard metric of the Euclidean space $\mathbb{R}^n$ and in the metric g, respectively.

The energy of the model (1.1)-(1.2) is defined by

$$E(t)=\frac{1}{2}\int_{\mathbb{R}^n}(u_t^2+|\nabla_g u|_g^2+u^2)dx. \tag{1.8}$$

where $\nabla_g u = A(x)\nabla u$ and

$$|\nabla_g u|_g^2 = \langle \nabla_g u, \nabla_g u\rangle_g = \langle A(x)\nabla u, \nabla u\rangle = \sum_{ij=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}.$$

We refer the readers to [14] and [15] for more information about the metric $g(x)$.

We use the following assumption in this article

(H1) There is a vector field H on $(\mathbb{R}^n, g)$ such that

$$D_g H(X,X) \geq \bar{\sigma}|X|_g^2, \quad X \in \mathbb{R}_x^n,\ x \in \bar{\Omega}, \tag{1.9}$$

where $\bar{\sigma} > 0$ is a constant and D_g is the Levi-Civita connection of the metric g.

Our main result read as follows.

Theorem 1.1. *Let $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. In addition to (H1) assume that $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing C^1 function and $\sigma(t) \geq \sigma_0 > 0$ for all $t \geq 0$. Then* (1.1)-(1.2) *admits a unique solution u satisfying*

$$E(t) \leq CE(0)\Big(\frac{1}{\int_0^t \sigma(s)ds}\Big)^{\frac{2}{m-1}}, \quad \forall t > 0 \quad \text{if } m > 1,$$

and

$$E(t) \leq CE(0)\exp(-\omega\int_0^t \sigma(s)ds), \quad \forall t > 0 \quad \text{if } m = 1,$$

for some $\omega > 0$ and $C > 0$.

2. Proof of main result

To prove our main result we use the following lemma.

Lemma 2.1 ([6])**.** *Let $E : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ a strictly increasing C^1 function such that*

$$\phi(0) = 0, \quad \phi(t) \to +\infty \quad \text{as } t \to +\infty.$$

Assume that there exist $r \geq 0$ and $\omega > 0$ such that

$$\int_S^{+\infty} E^{1+r}(t)\phi'(t)dt \leq \frac{1}{\omega}E^r(0)E(S), \quad 0 \leq S < +\infty.$$

Then

$$E(t) \leq E(0)\Big(\frac{1+r}{1+\omega r\phi(t)}\Big)^{1/r} \quad \text{for all } t \geq 0, \text{ if } r > 0,$$
$$E(t) \leq CE(0)e^{1-\omega\phi(t)} \quad \text{for all } t \geq 0, \text{ if } r = 0.$$

To apply Lemma 2.1, we need some estimates on the energy terms, which are based on the identities below. Multiply (1.1) by u_t and integrate by parts over $\mathbb{R}^n$ with respect to the variable x to obtain

$$\frac{d}{dt}E(t) = -\int_{\mathbb{R}^n} \sigma(t)\rho(x,u_t)u_t\,dx. \tag{2.1}$$

Let H be a vector field on $\mathbb{R}^n$. Multiply (1.1) by $H(u)$ and integrate by parts over $\mathbb{R}^n$ with respect to the variable x to obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^n} u_t H(u)dx + \int_{\mathbb{R}^n} D_g H(\nabla_g u, \nabla_g u)dx \\ &+ \int_{\mathbb{R}^n} H(u)u\,dx + \frac{1}{2}\int_{\mathbb{R}^n} (u_t^2 - |\nabla_g u|_g^2)\mathrm{div} H dx \\ &= -\int_{\mathbb{R}^n} \sigma(t)\rho(x,u_t)H(u)dx. \end{aligned} \tag{2.2}$$

Let $q \in C^1(\mathbb{R}^n)$ be a function. Multiply (1.1) by qu and integrate by parts over $\mathbb{R}^n$ with respect to the variable x to obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^n} quu_t\,dx + \int_{\mathbb{R}^n} q\big(|\nabla_g u|_g^2 - u_t^2\big)\,dx - \int_{\mathbb{R}^n} u^2\,\mathrm{div}\,A\nabla q\,dx + \int_{\mathbb{R}^n} qu^2 dx \\ &= -\int_{\mathbb{R}^n} qu\sigma(t)\rho(x,u_t)dx. \end{aligned} \tag{2.3}$$

Proof of Theorem 1.1. Let the vector field H be such that the assumption (H1) holds. Let $\hat{\Omega} \subset \mathbb{R}^n$ and $\hat{\hat{\Omega}} \subset \mathbb{R}^n$ be two bounded domains such that $\bar{\Omega} \subset \hat{\Omega} \subset \bar{\hat{\Omega}} \subset \hat{\hat{\Omega}}$. Let φ and ψ be two C_0^∞ functions and $0 \le \varphi \le 1$, $0 \le \psi \le 1$, and

$$\varphi = \begin{cases} 1, & x \in \Omega, \\ 0, & x \in \hat{\Omega}^c = \mathbb{R}^n \setminus \hat{\Omega}, \end{cases} \qquad \psi = \begin{cases} 1, & x \in \hat{\Omega}, \\ 0, & x \in \mathbb{R}^n \setminus \hat{\hat{\Omega}}. \end{cases} \tag{2.4}$$

Let $q_0 = \frac{\mathrm{div}(\varphi H)}{2} - \bar{\sigma}\varphi$. Replacing H by φH in (2.2) and replacing q by q_0 in (2.3), respectively, we can obtain two identities. Then we add them up and obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^n} \big(u_t\varphi H(u) + q_0 uu_t\big)dx + \int_{\mathbb{R}^n} D_g(\varphi H)(\nabla_g u, \nabla_g u)dx \\ &+ \int_{\mathbb{R}^n} \varphi H(u)u\,dx + \int_{\mathbb{R}^n} \bar{\sigma}\varphi(u_t^2 - |\nabla_g u|_g^2)dx \\ &- \int_{\mathbb{R}^n} |u|^2\,\mathrm{div}(\nabla_g q_0)dx + \int_{\mathbb{R}^n} q_0 u^2 dx \\ &= -\int_{\mathbb{R}^n} q_0 u\sigma(t)\rho(x,u_t)dx - \int_{\mathbb{R}^n} \sigma(t)\rho(x,u_t)\varphi H(u)dx. \end{aligned} \tag{2.5}$$

Let k be a large constant determined later. Combining (2.1) with (2.5), we have

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^n} \big(u_t\varphi H(u) + q_0 uu_t\big)dx + kE'(t) + \int_{\mathbb{R}^n} D_g(\varphi H)(\nabla_g u, \nabla_g u)dx \\ &+ \int_{\mathbb{R}^n} \varphi H(u)u\,dx + \int_{\mathbb{R}^n} \bar{\sigma}\varphi(u_t^2 - |\nabla_g u|_g^2)dx - \int_{\mathbb{R}^n} |u|^2\,\mathrm{div}(\nabla_g q_0)dx \\ &+ \int_{\mathbb{R}^n} q_0 u^2 dx \\ &= -\int_{\mathbb{R}^n} q_0 u\sigma(t)\rho(x,u_t)dx - \int_{\mathbb{R}^n} \sigma(t)\rho(x,u_t)\varphi H(u)dx \\ &\quad - k\int_{\mathbb{R}^n} \sigma(t)\rho(x,u_t)u_t\,dx. \end{aligned} \tag{2.6}$$

Set

$$Y(t) = \int_{\mathbb{R}^n} D_g(\varphi H)(\nabla_g u, \nabla_g u)dx + \int_{\mathbb{R}^n} \bar{\sigma}\varphi(u_t^2 - |\nabla_g u|_g^2)dx + k\int_{\mathbb{R}^n} \sigma(t)\rho(x, u_t)u_t\, dx.$$

Set $\sigma_1 = \sup_{\bar{\hat{\Omega}}} |D_g(\varphi H)(X, X)|$. Noticing (1.3),(1.4),(1.9) and (2.4), we have the following estimates on $Y(t)$:

$$\begin{aligned}
Y(t) &\geq \int_{\hat{\Omega}\setminus\Omega} D_g(\varphi)H(\nabla_g u, \nabla_g u)dx + \int_{\mathbb{R}^n\setminus\Omega} \bar{\sigma}\varphi u_t^2 dx - \int_{\hat{\Omega}\setminus\Omega} \bar{\sigma}\varphi|\nabla_g u|_g^2\, dx \\
&\quad + \frac{k}{4}\int_{\mathbb{R}^n\setminus\Omega} \epsilon_0\sigma(t)u_t^2 dx + \int_\Omega \bar{\sigma}u_t^2 dx + \frac{3k}{4}\int_{\mathbb{R}^n} \sigma(t)\rho(x, u_t)u_t\, dx \\
&\geq (-\sigma_1 - \bar{\sigma})\int_{\hat{\Omega}\setminus\Omega} |\nabla_g u|_g^2 dx + \int_{\mathbb{R}^n\setminus\Omega} \big(\frac{k}{4}\varepsilon_0\sigma(t) + \bar{\sigma}\varphi\big)u_t^2 dx \\
&\quad + \int_\Omega \bar{\sigma}u_t^2 dx + \frac{3k}{4}\int_{\mathbb{R}^n} \sigma(t)\rho(x, u_t)u_t\, dx\,.
\end{aligned} \tag{2.7}$$

Noticing $\sigma(t) \geq \sigma_0 > 0$, we choose k large enough so that $\frac{k\sigma_0\varepsilon_0}{4} > \bar{\sigma}$. Then we have

$$Y(t) \geq \int_{\mathbb{R}^n} \bar{\sigma}|u_t|^2 dx - \int_{\hat{\Omega}\setminus\Omega} (\sigma_1 + \bar{\sigma})|\nabla_g u|_g^2 + \frac{3k}{4}\int_{\mathbb{R}^n} \sigma(t)\rho(x, u_t)u_t\, dx\,. \tag{2.8}$$

Now we estimate the last term of the right side of (2.8).

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sigma(t)\rho(x, u_t)u_t\, dx \\
&\geq \int_{\mathbb{R}^n} \psi\sigma(t)\rho(x, u_t)u_t\, dx = \int_{\hat{\hat{\Omega}}} \psi\sigma(t)\rho(x, u_t)u_t\, dx \\
&= \int_{\hat{\hat{\Omega}}\setminus\Omega_1} \psi\sigma(t)a(x)u_t^2 dx + \int_{\Omega_1} \psi\sigma(t)\rho(x, u_t)u_t\, dx \\
&= \int_{\hat{\hat{\Omega}}} \psi\sigma(t)a(x)u_t^2\, dx - \int_{\Omega_1} \sigma(t)a(x)u_t^2\, dx + \int_{\Omega_1} \psi\sigma(t)\rho(x, u_t)u_t\, dx\,.
\end{aligned} \tag{2.9}$$

For the first term on the right of (2.9), noticing $\sigma(t) \geq \sigma_0$ and using (1.4), (2.4) and (2.3) for $q = \sigma_0\psi a(x)$, we have

$$\begin{aligned}
\int_{\hat{\hat{\Omega}}} \psi\sigma(t)a(x)u_t^2\, dx &\geq \sigma_0\int_{\hat{\hat{\Omega}}} \psi a(x)u_t^2\, dx = \int_{\mathbb{R}^n} \sigma_0\psi a(x)u_t^2 dx \\
&\geq \frac{d}{dt}\int_{\mathbb{R}^n} \sigma_0 a(x)\psi u u_t\, dx + \sigma_0\varepsilon_0\int_{\hat{\Omega}\setminus\Omega} |\nabla_g u|_g^2\, dx \\
&\quad - \sigma_0\int_{\mathbb{R}^n} u^2 \operatorname{div} A\nabla\big(a(x)\psi\big)dx + \sigma_0\int_{\mathbb{R}^n} a(x)\psi u^2 dx \\
&\quad + \sigma_0\int_{\mathbb{R}^n} a(x)\psi\sigma(t)\rho(x, u_t)u\, dx.
\end{aligned} \tag{2.10}$$

since $\operatorname{supp}\psi \subset \hat{\hat{\Omega}}$ and $a(x) \geq \varepsilon_0 > 0$ for $x \in \Omega^c$.

Combining (2.8) with (2.9) and (2.10), we have

$$
\begin{aligned}
Y(t) \geq &\Big(\frac{k\sigma_0\varepsilon_0}{4} - (\sigma_1+\bar\sigma)\Big)\int_{\hat\Omega\setminus\Omega} |\nabla_g u|_g^2\,dx + \int_{\mathbb{R}^n} \bar\sigma u_t^2\,dx \\
&+ \frac{k}{2}\int_{\mathbb{R}^n} \sigma(t)\rho(x,u_t)u_t\,dx + \frac{\sigma_0 k}{4}\Big[\frac{d}{dt}\int_{\mathbb{R}^n} a(x)\psi u u_t\,dx \\
&- \int_{\mathbb{R}^n} u^2 \operatorname{div} A\nabla\big(a(x)\psi\big)dx + \int_{\mathbb{R}^n} a(x)\psi u^2 dx \\
&+ \int_{\mathbb{R}^n} a(x)\psi\sigma(t)\rho(x,u_t)u dx\Big] - \frac{k}{4}\int_{\Omega_1} \sigma(t)a(x)u_t^2\,dx \\
&+ \frac{k}{4}\int_{\Omega_1} \sigma(t)\rho(x,u_t)u_t\,dx
\end{aligned} \tag{2.11}
$$

Choose k large enough so that $\frac{k\sigma_0\varepsilon_0}{4} > \bar\sigma + \sigma_1$. Then it follows from (2.11) and (2.6) that

$$
\begin{aligned}
&\frac{d}{dt}\Big(\int_{\mathbb{R}^n} \big(u_t\varphi H(u) + q_0 u u_t\big)dx + kE(t) + \frac{\sigma_0 k}{4}\int_{\mathbb{R}^n} a(x)\psi u u_t\,dx\Big) \\
&+ \int_{\mathbb{R}^n} \bar\sigma u_t^2\,dx + \frac{k}{2}\int_{\Omega_1} \sigma(t)\rho(x,u_t)u_t\,dx - \frac{k}{4}\int_{\Omega_1} \sigma(t)a(x)\psi u_t^2\,dx \\
&+ \frac{k}{4}\int_{\Omega_1} \sigma(t)\rho(x,u_t)u_t\,dx + \frac{\sigma_0 k}{4}\Big[\int_{\mathbb{R}^n} \big(a\psi - \operatorname{div} A\nabla\big(a(x)\psi\big)\big)u^2\,dx \\
&+ \int_{\mathbb{R}^n} a(x)\psi\sigma(t)\rho(x,u_t)u dx\Big] + \int_{\mathbb{R}^n} \varphi H(u)u\,dx \\
&+ \int_{\mathbb{R}^n} \big(q_0 - \operatorname{div}\nabla_g q_0\big)u^2\,dx + \int_{\mathbb{R}^n} q_0 u\sigma(t)\rho(x,u_t)dx \\
&+ \int_{\mathbb{R}^n} \sigma(t)\rho(x,u_t)\varphi H(u)dx \leq 0.
\end{aligned} \tag{2.12}
$$

We may have equality by setting $q = \frac{\bar\sigma}{2}$ in (2.3). Then add that equality to (2.12) to yield

$$
\begin{aligned}
&\frac{d}{dt}\Big(\int_{\mathbb{R}^n} \big(u_t\varphi H(u) + q_0 u u_t\big)dx + kE(t) + \frac{\sigma_0 k}{4}\int_{\mathbb{R}^n} a(x)\psi u u_t\,dx \\
&+ \int_{\mathbb{R}^n} \frac{\bar\sigma}{2} u u_t\,dx\Big) + \int_{\mathbb{R}^n} \frac{\bar\sigma}{2}\big(|\nabla_g u|_g^2 + u_t^2 + u^2 + \sigma(t)\rho(x,u_t)u\big)dx \\
&+ \frac{k}{2}\int_{\Omega_1} \sigma(t)\rho(x,u_t)u_t\,dx - \frac{k}{4}\int_{\Omega_1} \sigma(t)a(x)\psi u_t^2\,dx \\
&+ \frac{k}{4}\int_{\Omega_1} \sigma(t)\rho(x,u_t)u_t\,dx + \frac{\sigma_0 k}{4}\Big[\int_{\mathbb{R}^n} \Big(a\psi - \operatorname{div} A\nabla\big(a(x)\psi\big)\Big)u^2 dx \\
&+ \int_{\mathbb{R}^n} a(x)\psi\sigma(t)\rho(x,u_t)u dx\Big] + \int_{\mathbb{R}^n} \varphi H(u)u\,dx \\
&+ \int_{\mathbb{R}^n} \big(q_0 - \operatorname{div}\nabla_g q_0\big)u^2\,dx + \int_{\mathbb{R}^n} q_0\sigma(t)\rho(x,u_t)u\,dx \\
&+ \int_{\mathbb{R}^n} \sigma(t)\rho(x,u_t)\varphi H(u)dx \leq 0.
\end{aligned} \tag{2.13}
$$

Set

$$X(t)=\int_{\mathbb{R}^n}\big(u_t\varphi H(u)+q_0uu_t\big)dx+kE(t)+\frac{\sigma_0 k}{4}\int_{\mathbb{R}^n}a(x)\psi uu_t\,dx+\int_{\mathbb{R}^n}\frac{\bar\sigma}{2}uu_t\,dx.$$

From (2.13) and the definition of $E(t)$, we have

$$\begin{aligned}
&\bar\sigma E(t)\\
&\le -\frac{d}{dt}X(t)-\int_{\mathbb{R}^n}\frac{\bar\sigma}{2}\sigma(t)\rho(x,u_t)u\,dx-\int_{\mathbb{R}^n}\sigma(t)\rho(x,u_t)\varphi H(u)\,dx\\
&\quad-\int_{\mathbb{R}^n}q_0\sigma(t)\rho(x,u_t)udx-\frac{k}{2}\int_{R^n}\sigma(t)\rho(x,u_t)u_t\,dx\\
&\quad-\frac{k}{4}\int_{\Omega_1}\sigma(t)\rho(x,u_t)u_t\,dx-\frac{\sigma_0 k}{4}\Big[\int_{\mathbb{R}^n}\Big(a\psi-\operatorname{div}A\nabla\big(a(x)\psi\big)\Big)u^2\,dx\\
&\quad+\int_{\mathbb{R}^n}a(x)\psi\sigma(t)\rho(x,u_t)u\,dx\Big]+\frac{k}{4}\int_{\Omega_1}\sigma(t)a(x)\psi u_t^2\,dx\\
&\quad-\int_{\mathbb{R}^n}\varphi H(u)u\,dx-\int_{\mathbb{R}^n}\big(q_0-\operatorname{div}\nabla_g q_0\big)u^2\,dx\\
&\le -\frac{d}{dt}X(t)+\sigma_4\Big|\int_{\mathbb{R}^n}\sigma(t)\rho(x,u_t)u\,dx\Big|-\frac{k}{2}E'(t)\\
&\quad+(\frac{\sigma_0k\sigma_2}{4}+\sigma_3)\int_{\hat{\hat\Omega}}|u|^2\,dx+\frac{k}{4}\int_{\Omega_1}\sigma(t)a(x)u_t^2\,dx+\sigma_5\int_{\hat\Omega}|\nabla_g u|_g u\,dx\\
&\quad+\sigma_5\int_{\mathbb{R}^n}\sigma(t)|\rho(x,u_t)||\nabla_g u|_g\,dx
\end{aligned}\tag{2.14}$$

where

$$\sigma_2=\sup_{x\in\bar{\hat{\hat\Omega}}}|a\psi-\operatorname{div}\nabla_g(a\psi)|,\quad \sigma_3=\sup_{x\in\bar{\hat\Omega}}|q_0-\operatorname{div}\nabla_g q_0|,$$
$$\sigma_4=\max\Big\{\frac{\sigma_0k|a|_\infty}{4},\frac{\bar\sigma}{2},\sup_{\bar{\hat\Omega}}|q_0|\Big\},\quad \sigma_5=\sup_{\bar{\hat\Omega}}|\varphi H|.$$

For the second term, the sixth term and the last term on the right of (2.14), we use Young's inequality to obtain

$$\begin{aligned}
\sigma_4\Big|\int_{\mathbb{R}^n}\sigma(t)\rho(x,u_t)udx\Big|&\le\sigma_4\int_{\mathbb{R}^n}\big(C_\varepsilon|\sigma(t)\rho(x,u_t)|^2+\varepsilon u^2\big)dx\\
&\le\sigma_4\int_{\mathbb{R}^n}C_\varepsilon|\sigma(t)\rho(x,u_t)|^2+2\varepsilon C_4E(t),
\end{aligned}\tag{2.15}$$

$$\begin{aligned}
\sigma_5\int_{\hat\Omega}|\nabla_g u|_g udx&\le\sigma_5\int_{\mathbb{R}^n}\varepsilon|\nabla_g u|_g^2\,dx+C_\varepsilon\sigma_5\int_{\hat\Omega}u^2\,dx,\\
&\le 2\varepsilon C_5E(t)+C_\varepsilon\sigma_5\int_{\hat\Omega}u^2\,dx,
\end{aligned}\tag{2.16}$$

and

$$\begin{aligned}
&\sigma_5\int_{\mathbb{R}^n}\sigma(t)|\rho(x,u_t)||\nabla_g u|_g\,dx\\
&\le\varepsilon\sigma_5\int_{\mathbb{R}^n}|\nabla_g u|_g^2\,dx+C_\varepsilon\sigma_5\int_{\mathbb{R}^n}|\sigma(t)\rho(x,u_t)|^2\,dx\\
&\le 2\varepsilon C_5E(t)+C_\varepsilon\sigma_5\int_{\mathbb{R}^n}|\sigma(t)\rho(x,u_t)|^2dx
\end{aligned}\tag{2.17}$$

where $\varepsilon > 0$ is small and C_ε is a constant dependent on ε.

Let ϕ be a function satisfying the conditions in Lemma 2.1. Let $r \geq 0$ be a constant determined later. Multiplying (2.14) by $E^r\phi'$ and integrating on $[S,T]$ with respect to t, from (2.15) -(2.17) we have

$$\begin{aligned}
&\int_S^T E^{r+1}(t)\phi'\bar{\sigma}dt \\
&\leq -\int_S^T E^r\phi' X'(t)dt + (\sigma_4 C_\varepsilon + \sigma_5 C_\varepsilon)\int_S^T E^r\phi'\int_{\mathbb{R}^n}|\sigma(t)\rho(x,u_t)|^2\,dx\,dt \\
&\quad + (2\varepsilon\sigma_4 + 4\varepsilon\sigma_5)\int_S^T E^{r+1}\phi' dt - \frac{k}{2}\int_S^T E^r\phi' E' dt \\
&\quad + (\frac{k\sigma_0\sigma_2}{4} + \sigma_3 + C_\varepsilon\sigma_5)\int_S^T E^r\phi'\int_{\hat{\Omega}} u^2\,dx\,dt \\
&\quad + \frac{k}{4}\int_S^T E^r\phi'\int_{\Omega_1}\sigma(t)a(x)u_t^2\,dx\,dt.
\end{aligned} \tag{2.18}$$

Using a similar argument as in [13] and [7], we can prove that if $T-S$ is large enough, then

$$\begin{aligned}
\int_S^T E^r\phi'\int_{\hat{\Omega}} u^2\,dx\,dt \leq C_\eta\int_S^T E^r\phi'\Big(\int_{\mathbb{R}^n}|\sigma(t)\rho(x,u_t)|^2\,dx \\
+ \int_{\Omega_1}\sigma(t)u_t^2 dx\Big)\,dt + \eta\int_S^T E^{r+1}(t)\phi'\,dt
\end{aligned} \tag{2.19}$$

holds for any $\eta > 0$.

For the rest of this article, let $\phi(t) = \int_0^t \sigma(s)ds$. It is clear that $\phi(t)$ satisfies the conditions of Lemma 2.1. Noticing that $|X(t)| \leq CE(t)$ and $E'(t) \leq 0$, we have by integrating by parts with respect to t,

$$\begin{aligned}
&-\int_S^T E^r\phi' X'(t)dt \\
&= -E^r(t)\phi'(t)X(t)\Big|_S^T + \int_S^T (E^r\phi')'X(t)dt \\
&\leq C\sigma(0)E^{r+1}(S) + \int_S^T |rE^{r-1}E'\phi' X(t)|dt + \int_S^T |E^r\phi'' X(t)|dt \\
&\leq C\sigma(0)E^{r+1}(S) + C\sigma(0)\int_S^T rE^r(-E')dt + \int_S^T E^{r+1}(-\phi'')dt \\
&\leq CE^{r+1}(S).
\end{aligned} \tag{2.20}$$

We also have

$$-\frac{k}{2}\int_S^T E^r\phi' E' dt \leq CE^{r+1}(S). \tag{2.21}$$

Now we estimate the second term on the right of (2.18). Set $\Omega_1^+ = \{x \in \Omega_1; |u_t| \geq 1\}$ and $\Omega_1^- = \{x \in \Omega_1; |u_t| < 1\}$. Then $\Omega_1 = \Omega_1^+ \cup \Omega_1^-$. Noticing (1.5), using Hölder inequality and Young's inequality, we have

$$
\begin{aligned}
&\int_S^T E^r\phi'\int_{\Omega_1^-}|\sigma(t)\rho(x,u_t)|^2\,dx\,dt\\
&=\int_S^T E^r\phi'\int_{\Omega_1^-}\sigma^2(t)|a(x)h(u_t)|^2\,dx\,dt\\
&\le c_2\int_S^T E^r\phi'\sigma^2(t)\int_{\Omega_1^-}a^2(x)\big(h(u_t)u_t\big)^{\frac{2}{m+1}}\,dx\,dt\\
&=c_2\int_S^T E^r\phi'\int_{\Omega_1^-}\big(\sigma(t)a(x)h(u_t)u_t\big)^{\frac{2}{m+1}}\big(a(x)\sigma(t)\big)^{2-\frac{2}{m+1}}\,dx\,dt\\
&\le C|a|_\infty^{2-\frac{2}{m+1}}\int_S^T E^r\phi'\sigma(t)^{2-\frac{2}{m+1}}\Big(\int_{\Omega_1^-}\sigma(t)a(x)h(u_t)u_t)dx\Big)^{\frac{2}{m+1}}dt\\
&\le C|a|_\infty^{2-\frac{2}{m+1}}\sigma^2(0)\Big[\varepsilon_1\int_S^T\big(E^r\phi'\sigma(t)^{-\frac{2}{m+1}}\big)^{\frac{m+1}{m-1}}dt\\
&\quad+C_{\varepsilon_1}\int_S^T\int_{\Omega_1^-}\sigma(t)a(x)h(u_t)u_t\,dx\,dt\Big].
\end{aligned}
\tag{2.22}
$$

Now, we set $r=\frac{m-1}{2}$. Thus, we have from (2.22) and (2.1)

$$
\begin{aligned}
&\int_S^T E^r\phi'\int_{\Omega_1^-}|\sigma(t)\rho(x,u_t)|^2\,dx\,dt\\
&\le C|a|_\infty^{2-\frac{2}{m+1}}\sigma^2(0)\Big[\varepsilon_1\int_S^T E^{\frac{m+1}{2}}\phi'dt+C_{\varepsilon_1}E(S)\Big].
\end{aligned}
\tag{2.23}
$$

Noticing (1.6) and (2.1), it is easy to obtain

$$
\begin{aligned}
\int_S^T E^r\phi'\int_{\Omega_1^+}|\sigma(t)\rho(x,u_t)|^2\,dx\,dt&\le c_4\sigma(0)|a|_\infty\int_S^T E^r\phi'(-E'(t))dt\\
&\le CE^{r+1}(S).
\end{aligned}
\tag{2.24}
$$

On the other hand, it is easy to obtain

$$
\int_S^T E^r\phi'\int_{\mathbb{R}^n\setminus\Omega_1}|\sigma(t)\rho(x,u_t)|^2\,dx\,dt\le CE^{r+1}(S)
\tag{2.25}
$$

since $\rho(x,u_t)=a(x)u_t$ when $x\in\mathbb{R}^n\setminus\Omega_1$.

From (2.23)-(2.25), we have

$$
\begin{aligned}
&\int_S^T E^r\phi'\int_{\mathbb{R}^n}|\sigma(t)\rho(x,u_t)|^2\,dx\,dt\\
&\le C\varepsilon_1|a|_\infty^{2-\frac{2}{m+1}}\sigma^2(0)\int_S^T E^{\frac{m+1}{2}}\phi'dt+C_{\varepsilon_1}E(S)+CE^{r+1}(S).
\end{aligned}
\tag{2.26}
$$

Similarly, for the last term on the right side of (2.18), we have

$$
\begin{aligned}
&\int_S^T E^r\phi'\int_{\Omega_1}\sigma(t)a(x)u_t^2\,dx\,dt\\
&\le C\varepsilon_1|a|_\infty^{\frac{m-1}{m+1}}\sigma(0)\int_S^T E^{\frac{m+1}{2}}\phi'dt+C_{\varepsilon_1}E(S)+CE^{r+1}(S).
\end{aligned}
\tag{2.27}
$$

At this point, we choose ε and η small enough so that

$$2\varepsilon\sigma_4 + 4\varepsilon\sigma_5 + \eta(\frac{k\sigma_0\sigma_2}{4} + \sigma_3 + C_\varepsilon\sigma_5) < \frac{\bar{\sigma}}{2}.$$

Then using (2.19)-(2.21), (2.18) becomes

$$\begin{aligned} &\frac{\bar{\sigma}}{2}\int_S^T E^{r+1}(t)\phi' dt \\ &\le CE^{r+1}(S) + \big(\sigma_4 C_\varepsilon + \sigma_5 C_\varepsilon + C_\eta(\frac{k\sigma_0\sigma_2}{4} + \sigma_3 + C_\varepsilon\sigma_5)\big) \\ &\times \int_S^T E^r\phi' \int_{\mathbb{R}^n} |\sigma(t)\rho(x,u_t)|^2\,dx\,dt - \frac{k}{2}\int_S^T E^r\phi' E' dt \\ &\quad + \big(\frac{k}{4} + C_\eta(\frac{k\sigma_0\sigma_2}{4} + \sigma_3 + C_\varepsilon\sigma_5)\big)\int_S^T E^r\phi' \int_{\Omega_1} \sigma(t)a(x)u_t^2\,dx\,dt. \end{aligned} \tag{2.28}$$

Once ε and η are fixed, we pick ε_1 small enough so that

$$C\varepsilon_1|a|_\infty^{\frac{m-1}{m+1}}\big(\sigma(0)+\sigma^2(0)\big)\big(\sigma_4 C_\varepsilon + \sigma_5 C_\varepsilon + \frac{k}{4} + C_\eta(\frac{k\sigma_0\sigma_2}{4} + \sigma_3 + C_\varepsilon\sigma_5)\big) \le \frac{\bar{\sigma}}{4}.$$

Then using (2.26)-(2.27) in (2.28), we have

$$\int_S^T E^{\frac{m+1}{2}}\phi' dt \le CE(S). \tag{2.29}$$

Let $T \to \infty$ in (2.29). Then Theorem 1.1 follows from Lemma 2.1 and (2.29) with $r = \frac{m-1}{2}$ and $\phi(t) = \int_0^t \sigma(s)ds$. $\square$

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