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COMBINED EFFECTS IN NONLINEAR SINGULAR SECOND-ORDER DIFFERENTIAL EQUATIONS ON THE HALF-LINE

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Abstract. We consider the existence, uniqueness and the asymptotic behavior of positive continuous solutions to the second-order boundary-value problem

$$\frac{1}{A}(Au')' + a_1(t)u^{\sigma_1} + a_2(t)u^{\sigma_2} = 0, \quad t \in (0,\infty),$$

$$\lim_{t\to 0^+} u(t) = 0, \quad \lim_{t\to\infty} \frac{u(t)}{\rho(t)} = 0,$$

where $\sigma_1, \sigma_2 \in (-1,1)$, A is a continuous function on $[0,\infty)$, positive and differentiable on $(0,\infty)$ such that $\int_0^1 \frac{1}{A(t)}dt < \infty$ and $\int_0^\infty \frac{1}{A(t)}dt = \infty$. Here $\rho(t) = \int_0^t \frac{1}{A(s)}ds$ and for $i \in \{1,2\}$, a_i is a nonnegative continuous function in $(0,\infty)$ such that there exists $c > 0$ satisfying for $t > 0$,

$$\frac{1}{c}\frac{h_i(m(t))}{A^2(t)(1+\rho(t))^{\mu_i}} \le a_i(t) \le c\frac{h_i(m(t))}{A^2(t)(1+\rho(t))^{\mu_i}},$$

where $m(t) = \frac{\rho(t)}{1+\rho(t)}$ and $h_i(t) = c_i t^{-\lambda_i}\exp(\int_t^\eta \frac{z_i(s)}{s}ds)$, $c_i > 0$, $\lambda_i \le 2$, $\mu_i > 2$ and z_i is continuous on $[0,\eta]$ for some $\eta > 1$ such that $z_i(0) = 0$. The comparable asymptotic rate of $a_i(t)$ determines the asymptotic behavior of the solution.

1. Introduction

Boundary-value problems on the half-line, have been studied widely in the literature (see, for example, [1, 13, 14, 16, 21, 23] and the references therein). The motivation for these studies stems from the fact that such problems arise naturally in the study of radially symmetric solutions of nonlinear elliptic equations (see, [4, 6, 13, 18, 19, 21, 22] and also many physical models, for example, the model of gas pressure in a semi-infinite porous medium, the Thomas-Fermi model for determining the electric potential in an isolated neutral atom (see, the Monographs, [1, 10] and the references therein). Therefore it is very important to investigate the boundary-value problems for differential equations on half-line.

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Zhao [23] considered the problem

$$\begin{gathered} u''+\varphi(.,u)=0, \quad \text{on } (0,\infty),\\ u>0, \quad \text{on } (0,\infty),\\ \lim_{t\to 0^+} u(t)=0, \end{gathered} \tag{1.1}$$

where φ is a measurable function on $(0,\infty)\times(0,\infty)$, dominated by a convex positive function. Then he showed that there exists $b>0$ such that for each $\mu\in(0,b]$, there exists a positive continuous solution u of (1.1) satisfying $\lim_{t\to\infty}\frac{u(t)}{t}=\mu$.

On the other hand, in [2], the author studied the singular problem

$$\begin{gathered} \frac{1}{A}(Au')'+\varphi(\cdot,u)=0, \quad t\in(0,\infty),\\ u>0, \quad \text{on } (0,\infty),\\ \lim_{t\to 0^+} u(t)=0, \quad \lim_{t\to\infty}\frac{u(t)}{\rho(t)}=0, \end{gathered} \tag{1.2}$$

where A is a continuous function on $[0,\infty)$, positive and differentiable on $(0,\infty)$ such that $\int_0^1\frac{1}{A(t)}dt<\infty$ and $\int_0^\infty\frac{1}{A(t)}dt=\infty$. Here $\rho(t)=\int_0^t\frac{1}{A(s)}ds$ and the function $\varphi:(0,\infty)\times(0,\infty)\to[0,\infty)$ is required to be continuous, non-increasing with respect to the second variable such that for each $c>0$, $\varphi(.,c)\neq 0$ and $\int_0^\infty A(s)\min(1,\rho(s))\varphi(s,c)ds<\infty$. The author proved the existence of a unique positive solution u in $C([0,\infty))\cap C^2((0,\infty))$ to problem (1.2).

Recently, in [3], the authors considered problem (1.2) with $\varphi(t,u)=a(t)u^\sigma$, $\sigma<1$, (which include the sublinear case) and a is a nonnegative continuous function on $(0,\infty)$ satisfying some appropriate assumptions related to Karamata regular variation theory. They have proved the existence, uniqueness and the global asymptotic behavior of positive solutions to problem (1.2).

In this article, we study the boundary-value problem

$$\begin{gathered} \frac{1}{A}(Au')'+a_1(t)u^{\sigma_1}+a_2(t)u^{\sigma_2}=0, \quad t\in(0,\infty),\\ u>0, \quad \text{on } (0,\infty),\\ \lim_{t\to 0^+} u(t)=0, \quad \lim_{t\to\infty}\frac{u(t)}{\rho(t)}=0, \end{gathered} \tag{1.3}$$

where $\sigma_1,\sigma_2\in(-1,1)$, A is a continuous function on $[0,\infty)$, positive and differentiable on $(0,\infty)$ such that $\int_0^1\frac{1}{A(t)}dt<\infty$, $\int_0^\infty\frac{1}{A(t)}dt=\infty$ and $\rho(t)=\int_0^t\frac{1}{A(s)}ds$, $t>0$.

Our goal is to study (1.3), especially, when the nonlinearity is the sum of a singular term and a sublinear term. Under appropriate assumptions on a_1 and a_2 related to the Karamata class $\mathcal{K}$ (see Definition 1.1), we prove the existence, uniqueness and the global asymptotic behavior of positive continuous solution to problem (1.3).

Throughout this paper and without loss of generality, we assume that $\int_0^1\frac{1}{A(t)}dt=1$. To state our result, we need some notation.

Definition 1.1. The class $\mathcal{K}$ is the set of all Karamata functions L defined on $(0,\eta]$ by

$$L(t):=c\exp\Big(\int_t^\eta\frac{z(s)}{s}ds\Big),$$

for some $\eta > 1$, where $c > 0$ and $z \in C([0,\eta])$ such that $z(0) = 0$.

The theory of such functions was initiated by Karamata in the fundamental paper [15]. On the other hand, we emphasize that the first use of the Karamata theory in the study of the growth rate of solutions near the boundary was done in the paper of Cirstea and Rădulescu [8].

Remark 1.2. A function L is in $\mathcal{K}$ if and only if L is a positive function in $C^1((0,\eta])$, for some $\eta > 1$, such that $\lim_{t\to 0^+} \frac{tL'(t)}{L(t)} = 0$.

As a typical example of function belonging to the class $\mathcal{K}$ (see [5, 17, 20]), we quote

$$L(t) = 2 + \sin(\log_2(\frac{\omega}{t})) \quad \text{and} \quad L(t) = \prod_{k=1}^{m} (\log_k(\frac{\omega}{t}))^{\xi_k},$$

where ξ_k are real numbers, $\log_k x = \log \circ \log \circ \dots \log x$ (k times) and ω is a sufficiently large positive real number such that L is defined and positive on $(0,\eta]$, for some $\eta > 1$.

In the sequel, we denote by $B^+((0,\infty))$ the set of nonnegative Borel measurable functions in $(0,\infty)$ and by $C_0([0,\infty))$ the set of continuous functions v on $[0,\infty)$ such that $\lim_{t\to\infty} v(t) = 0$. It is easy to see that $C_0([0,\infty))$ is a Banach space with the uniform norm $\|v\|_\infty = \sup_{t>0} |v(t)|$.

For two nonnegative functions f and g defined on a set S, the notation $f(t) \approx g(t)$, $t \in S$ means that there exists $c > 0$ such that $\frac{1}{c} f(t) \le g(t) \le c f(t)$, for all $t \in S$.

Furthermore, let $G(t,s) = A(s)\min(\rho(t), \rho(s)))$, be the Green's function of the operator $u \mapsto -\frac{1}{A}(Au')'$ on $(0,\infty)$ with the Dirichlet conditions $\lim_{t\to 0^+} u(t) = 0$ and $\lim_{t\to\infty} \frac{u(t)}{\rho(t)} = 0$.

For $f \in B^+((0,\infty))$, we put

$$Vf(t) = \int_0^\infty G(t,s) f(s) dt, \quad \text{for } t > 0.$$

We point out that if the map $s \to A(s)\min(1,\rho(s))f(s)$ is continuous and integrable on $(0,\infty)$, then Vf is the solution of the boundary-value problem

$$\begin{gathered} -\frac{1}{A}(Au')' = f, \quad \text{in } (0,\infty), \\ \lim_{t\to 0^+} u(t) = 0, \\ \lim_{t\to\infty} \frac{u(t)}{\rho(t)} = 0. \end{gathered} \tag{1.4}$$

For $\lambda \le 2$, $\sigma \in (-1,1)$ and $L \in \mathcal{K}$ defined on $(0,\eta]$ (for some $\eta > 1$), we put for $t \in (0,\eta)$

$$\Psi_{L,\lambda,\sigma}(t) = \begin{cases} \Big(\int_0^t \frac{L(s)}{s} ds\Big)^{\frac{1}{1-\sigma}}, & \text{if } \lambda = 2, \\ \big(L(t)\big)^{\frac{1}{1-\sigma}}, & \text{if } 1+\sigma < \lambda < 2, \\ \Big(\int_t^\eta \frac{L(s)}{s} ds\Big)^{\frac{1}{1-\sigma}}, & \text{if } \lambda = 1+\sigma, \\ 1, & \text{if } \lambda < 1+\sigma. \end{cases} \tag{1.5}$$

Throughout this article we assume the following condition:

(H1) For $i \in \{1,2\}$, a_i is a nonnegative continuous function on $(0,\infty)$ such that
$$a_i(t) \approx \frac{1}{(A(t))^2}(\rho(t))^{-\lambda_i}(1+\rho(t))^{\lambda_i-\mu_i}L_i(m(t)), \quad t>0, \tag{1.6}$$
where $\lambda_i \le 2$, $\mu_i > 2$, $m(t) := \frac{\rho(t)}{1+\rho(t)}$, for $t>0$ and $L_i \in \mathcal{K}$ defined on $(0,\eta]$ (for some $\eta > 1$) such that
$$\int_0^\eta s^{1-\lambda_i}L_i(s)ds < \infty. \tag{1.7}$$

As it will be seen, the numbers
$$\beta_1 = \min(1, \frac{2-\lambda_1}{1-\sigma_1}) \quad \text{and} \quad \beta_2 = \min(1, \frac{2-\lambda_2}{1-\sigma_2}) \tag{1.8}$$
will play an important role in the study of asymptotic behavior of solution. Without loss of generality, we may assume that
$$\frac{2-\lambda_1}{1-\sigma_1} \le \frac{2-\lambda_2}{1-\sigma_2}$$
and we define the function θ on $(0,\infty)$ by
$$\theta(t) = \begin{cases} (m(t))^{\beta_1}\Psi_{L_1,\lambda_1,\sigma_1}(m(t)) & \text{if } \beta_1 < \beta_2 \\ (m(t))^{\beta_1}(\Psi_{L_1,\lambda_1,\sigma_1}(m(t)) + \Psi_{L_2,\lambda_2,\sigma_2}(m(t))) & \text{if } \beta_1 = \beta_2, \end{cases} \tag{1.9}$$
where $m(t) := \frac{\rho(t)}{1+\rho(t)}$, for $t>0$.

For an explicit form of the function θ see (3.1). Now, we are ready to state our main results.

Theorem 1.3. *Let $\sigma_1, \sigma_2 \in (-1,1)$ and assume that* (H1) *is fulfilled. Then for $t \in (0,\infty)$,*
$$V(a_1\theta^{\sigma_1} + a_2\theta^{\sigma_2})(t) \approx \theta(t). \tag{1.10}$$

By applying the above theorem and using the Schauder fixed point theorem, we prove the following result.

Theorem 1.4. *Let $\sigma_1, \sigma_2 \in (-1,1)$ and assume that* (H1) *is fulfilled. Then* (1.3) *has a unique positive continuous solution u satisfying for $t \in (0,\infty)$*
$$u(t) \approx \theta(t). \tag{1.11}$$

The content of this paper is organized as follows. In Section 2, we present some fundamental properties of the Karamata class $\mathcal{K}$ including sharp estimates on some potential functions. In Section 3, exploiting the results of the previous section, we first prove Theorem 1.3 which allow us to prove Theorem 1.4 by means of the Schauder fixed point theorem.

2. Properties of Karamata regular variation theory

We collect in this section some properties of functions belonging to the Karamata class $\mathcal{K}$.

Proposition 2.1 ([17, 20])**.** *The following hold:*

(i) *Let $L_1, L_2 \in \mathcal{K}$ and $p \in \mathbb{R}$. Then $L_1 + L_2 \in \mathcal{K}$, $L_1L_2 \in \mathcal{K}$ and $L_1^p \in \mathcal{K}$.*
(ii) *Let $L \in \mathcal{K}$ and $\varepsilon > 0$. Then $\lim_{t\to 0^+} t^\varepsilon L(t) = 0$.*

Applying Karamata's theorem (see [17, 20]), we obtain the following result.

Lemma 2.2. *Let $\nu \in \mathbb{R}$ and L be a function in $\mathcal{K}$ defined on $(0,\eta]$. We have*

(i) *If $\nu < -1$, then $\int_0^\eta s^\nu L(s)ds$ diverges and $\int_t^\eta s^\nu L(s)ds \underset{t\to 0^+}{\sim} -\frac{t^{\nu+1}L(t)}{\nu+1}$.*

(ii) *If $\nu > -1$, then $\int_0^\eta s^\nu L(s)ds$ converges and $\int_0^t s^\nu L(s)ds \underset{t\to 0^+}{\sim} \frac{t^{\nu+1}L(t)}{\nu+1}$.*

The proof of the next lemmas can be found in [7].

Lemma 2.3. *Let L be a function in $\mathcal{K}$ defined on $(0,\eta]$ $(\eta > 1)$. Then*

$$\lim_{t\to 0^+} \frac{L(t)}{\int_t^\eta \frac{L(s)}{s} ds} = 0.$$

In particular $t \to \int_t^\eta \frac{L(s)}{s} ds \in \mathcal{K}$.

If further $\int_0^\eta \frac{L(s)}{s} ds$ converges, then

$$\lim_{t\to 0^+} \frac{L(t)}{\int_0^t \frac{L(s)}{s} ds} = 0.$$

In particular $t \to \int_0^t \frac{L(s)}{s} ds \in \mathcal{K}$.

Lemma 2.4. *For $i \in \{1,2\}$, let $L_i \in \mathcal{K}$ be defined on $(0,\eta]$ $(\eta > 1)$ and put for $t \in (0,\eta)$,*

$$M(t) = \Big(\int_t^\eta \frac{L_1(s)}{s} ds\Big)^{\frac{1}{1-\sigma_1}} + \Big(\int_t^\eta \frac{L_2(s)}{s} ds\Big)^{\frac{1}{1-\sigma_2}}.$$

Then for $t \in (0,\eta)$ we have

$$\int_t^\eta \frac{(M^{\sigma_1}L_1 + M^{\sigma_2}L_2)(s)}{s} ds \approx M(t).$$

Lemma 2.5. *For $i \in \{1,2\}$, let $L_i \in \mathcal{K}$ be defined on $(0,\eta]$ $(\eta > 1)$ such that $\int_0^\eta \frac{L_i(s)}{s} ds < \infty$. Put for $t \in (0,\eta)$,*

$$N(t) = \Big(\int_0^t \frac{L_1(s)}{s} ds\Big)^{\frac{1}{1-\sigma_1}} + \Big(\int_0^t \frac{L_2(s)}{s} ds\Big)^{\frac{1}{1-\sigma_2}}.$$

Then for $t \in (0,\eta)$ we have

$$\int_0^t \frac{(N^{\sigma_1}L_1 + N^{\sigma_2}L_2)(s)}{s} ds \approx N(t).$$

Next, we have the following fundamental sharp estimates on the potential function Vb, for

$$b(t) = \frac{1}{(A(t))^2}(\rho(t))^{-\beta}(1+\rho(t))^{\beta-\gamma}\widetilde{L}(m(t)),$$

where $\beta \le 2$, $\gamma > 2$, $\widetilde{L} \in \mathcal{K}$ and $m(t) = \frac{\rho(t)}{1+\rho(t)}$ for $t > 0$.

Proposition 2.6 ([3])**.** *Let $\beta \le 2$, $\gamma > 2$ and $\widetilde{L} \in \mathcal{K}$ be defined on $(0,\eta]$ $(\eta > 1)$ such that $\int_0^\eta s^{1-\beta}\widetilde{L}(s)ds < \infty$. Then for $t > 0$,*

$$Vb(t) \approx \phi_\beta(m(t)),$$

where for $r \in (0,1]$,

$$\phi_\beta(r) = \begin{cases} \int_0^r \frac{\widetilde{L}(s)}{s} ds & \text{if } \beta = 2, \\ r^{2-\beta}\widetilde{L}(r) & \text{if } 1 < \beta < 2, \\ r\int_r^\eta \frac{\widetilde{L}(s)}{s} ds & \text{if } \beta = 1, \\ r & \text{if } \beta < 1. \end{cases}$$

3. Proof of main results

Let $\sigma_1, \sigma_2 \in (-1,1)$, assume (H1) and for $i \in \{1,2\}$, let $L_i \in \mathcal{K}$ defined on $(0,\eta]$ (for some$\eta > 1$) satisfying (1.6) and (1.7). Let b, L, M and N be the nonnegative functions defined in $(0,\eta)$ by

$$b(t) := \Big(\int_t^\eta \frac{L_1(s)}{s} ds\Big)^{\frac{1}{1-\sigma_1}},$$
$$L(t) := (L_1(t))^{\frac{1}{1-\sigma_1}} + (L_2(t))^{\frac{1}{1-\sigma_2}},$$
$$M(t) := \Big(\int_t^\eta \frac{L_1(s)}{s} ds\Big)^{\frac{1}{1-\sigma_1}} + \Big(\int_t^\eta \frac{L_2(s)}{s} ds\Big)^{\frac{1}{1-\sigma_2}},$$
$$N(t) := \Big(\int_0^t \frac{L_1(s)}{s} ds\Big)^{\frac{1}{1-\sigma_1}} + \Big(\int_0^t \frac{L_2(s)}{s} ds\Big)^{\frac{1}{1-\sigma_2}}, \quad \text{if } \int_0^\eta \frac{L_i(s)}{s} ds < \infty.$$

First, we give an explicit form of the function θ defined by (1.9). We recall that for $i \in \{1,2\}$, $\lambda_i \le 2$ and $\beta_i = \min(1, \frac{2-\lambda_i}{1-\sigma_i})$. Since $\beta_1 < \beta_2$ is equivalent to $\frac{2-\lambda_1}{1-\sigma_1} < \frac{2-\lambda_2}{1-\sigma_2}$ and $1+\sigma_1 < \lambda_1$, we deduce that for $t \in (0,\infty)$,

$$\theta(t) = \begin{cases} \Big(\int_0^{m(t)} \frac{L_1(s)}{s} ds\Big)^{\frac{1}{1-\sigma_1}}, & \text{if } \lambda_1 = 2 \text{ and } \lambda_2 < 2, \\ (m(t))^{\frac{2-\lambda_1}{1-\sigma_1}} (L_1(m(t)))^{\frac{1}{1-\sigma_1}}, & \text{if } \frac{2-\lambda_1}{1-\sigma_1} < \frac{2-\lambda_2}{1-\sigma_2} \text{ and } 1+\sigma_1 < \lambda_1 < 2, \\ (m(t))^{\frac{2-\lambda_1}{1-\sigma_1}} L(m(t)), & \text{if } \frac{2-\lambda_1}{1-\sigma_1} = \frac{2-\lambda_2}{1-\sigma_2} \text{ and } 1+\sigma_1 < \lambda_1 < 2, \\ m(t)M(m(t)), & \text{if } \lambda_1 = 1+\sigma_1 \text{ and } \lambda_2 = 1+\sigma_2, \\ m(t)(1+b(m(t))), & \text{if } \lambda_1 = 1+\sigma_1 \text{ and } \lambda_2 < 1+\sigma_2, \\ 2m(t), & \text{if } \lambda_1 < 1+\sigma_1, \\ N(m(t)), & \text{if } \lambda_1 = \lambda_2 = 2, \end{cases} \tag{3.1}$$

where $m(t) = \frac{\rho(t)}{1+\rho(t)}$.

Proof of Theorem 1.3.

Lemma 3.1. *For* $r, s > 0$, *we have*

$$2^{-\max(1-\sigma_1, 1-\sigma_2)}(r+s) \le r^{1-\sigma_1}(r+s)^{\sigma_1} + s^{1-\sigma_2}(r+s)^{\sigma_2} \le 2(r+s). \tag{3.2}$$

Proof. Let $r, s > 0$ and put $t = \frac{r}{r+s}$. Since $0 \le t \le 1$, then we obtain

$$2^{-\max(1-\sigma_1, 1-\sigma_2)} \le t^{1-\sigma_1} + (1-t)^{1-\sigma_2} \le 2.$$

Which implies the result. □

Now we are ready to prove Theorem 1.3. We recall that for $i \in \{1,2\}$, a_i is a nonnegative continuous function on $(0,\infty)$ such that

$$a_i(t) \approx \frac{1}{(A(t))^2}(\rho(t))^{-\lambda_i}(1+\rho(t))^{\lambda_i-\mu_i} L_i(m(t)), \ t > 0,$$

where $\lambda_i \leq 2$ and $\mu_i > 2$.

Note that throughout the proof, we use Proposition 2.1 and Lemma 2.3 to verify that some functions are in $\mathcal{K}$. We distinguish the following cases.

Case 1: $\lambda_1 = 2$ and $\lambda_2 < 2$. We have

$$\theta(t) = \Big(\int_0^{m(t)} \frac{L_1(s)}{s} ds\Big)^{\frac{1}{1-\sigma_1}}.$$

Therefore

$$\begin{aligned}
&a_1(t)\theta^{\sigma_1}(t) + a_2(t)\theta^{\sigma_2}(t)\\
&\approx \frac{(\rho(t))^{-2}(1+\rho(t))^{2-\mu}}{(A(t))^2} L_1(m(t))\Big(\int_0^{m(t)} \frac{L_1(s)}{s} ds\Big)^{\frac{\sigma_1}{1-\sigma_1}}\\
&\quad + \frac{(\rho(t))^{-\lambda_2}(1+\rho(t))^{\lambda_2-\mu}}{(A(t))^2} L_2(m(t))\Big(\int_0^{m(t)} \frac{L_1(s)}{s} ds\Big)^{\frac{\sigma_2}{1-\sigma_1}}.
\end{aligned}$$

Since for $i \in \{1,2\}$, the function $t \to \widetilde{L}_i(t) := L_i(t)(\int_0^t \frac{L_1(s)}{s} ds)^{\frac{\sigma_i}{1-\sigma_1}} \in \mathcal{K}$ and $\lambda_2 < 2$, we deduce by Proposition 2.1 that

$$a_1(t)\theta^{\sigma_1}(t) + a_2(t)\theta^{\sigma_2}(t) \approx \frac{(\rho(t))^{-2}(1+\rho(t))^{2-\mu}}{(A(t))^2}\widetilde{L}_1(m(t)).$$

Moreover, since $\lambda_1 = 2$, we have

$$\int_0^{\eta} \frac{\widetilde{L}_1(s)}{s} ds \leq c\Big(\int_0^{\eta} \frac{L_1(r)}{r} dr\Big)^{\frac{1}{1-\sigma_1}} < \infty,$$

it follows by applying Proposition 2.6 with $\beta = \lambda_1 = 2$, $\gamma = \mu$, we obtain

$$V(a_1\theta^{\sigma_1} + a_2\theta^{\sigma_2})(t) \approx \int_0^{m(t)} \frac{L_1(s)}{s}\Big(\int_0^s \frac{L_1(r)}{r} dr\Big)^{\frac{\sigma_1}{1-\sigma_1}} ds \approx \theta(t).$$

Case 2: $\frac{2-\lambda_1}{1-\sigma_1} < \frac{2-\lambda_2}{1-\sigma_2}$ and $1 + \sigma_1 < \lambda_1 < 2$. Since

$$\theta(t) = (m(t))^{\frac{2-\lambda_1}{1-\sigma_1}}(L_1(m(t)))^{\frac{1}{1-\sigma_1}},$$

we obtain

$$\begin{aligned}
&a_1(t)\theta^{\sigma_1}(t) + a_2(t)\theta^{\sigma_2}(t)\\
&\approx \frac{(\rho(t))^{\frac{(2-\lambda_1)\sigma_1}{1-\sigma_1}-\lambda_1}(1+\rho(t))^{\lambda_1-\frac{(2-\lambda_1)\sigma_1}{1-\sigma_1}-\mu}}{(A(t))^2}(L_1L_1^{\frac{\sigma_1}{1-\sigma_1}})(m(t))\\
&\quad + \frac{(\rho(t))^{\frac{(2-\lambda_1)\sigma_2}{1-\sigma_1}-\lambda_2}(1+\rho(t))^{\lambda_2-\frac{(2-\lambda_1)\sigma_2}{1-\sigma_1}-\mu}}{(A(t))^2}(L_2L_1^{\frac{\sigma_2}{1-\sigma_1}})(m(t)).
\end{aligned}$$

Since in this case $\frac{(2-\lambda_1)\sigma_2}{1-\sigma_1} - \lambda_2 > \frac{(2-\lambda_1)\sigma_1}{1-\sigma_1} - \lambda_1$, we deduce by Proposition 2.1 that

$$\begin{aligned}
&a_1(t)\theta^{\sigma_1}(t) + a_2(t)\theta^{\sigma_2}(t)\\
&\approx \frac{(\rho(t))^{\frac{(2-\lambda_1)\sigma_1}{1-\sigma_1}-\lambda_1}(1+\rho(t))^{\lambda_1-\frac{(2-\lambda_1)\sigma_1}{1-\sigma_1}-\mu}}{(A(t))^2}(L_1L_1^{\frac{\sigma_1}{1-\sigma_1}})(m(t)).
\end{aligned}$$

Therefore applying Proposition 2.6 with $\beta = \lambda_1 - \frac{(2-\lambda_1)\sigma_1}{1-\sigma_1} \in (1,2)$, $\gamma = \mu$ and $\widetilde{L} = L_1 L_1^{\frac{\sigma_1}{1-\sigma_1}} = L_1^{\frac{1}{1-\sigma_1}} \in \mathcal{K}$, we obtain

$$V(a_1\theta^{\sigma_1} + a_2\theta^{\sigma_2})(t) \approx (m(t))^{2-\beta}(L_1(m(t)))^{\frac{1}{1-\sigma_1}} = \theta(t).$$

Case 3: $\frac{2-\lambda_1}{1-\sigma_1} = \frac{2-\lambda_2}{1-\sigma_2}$ and $1+\sigma_1 < \lambda_1 < 2$. We have

$$\theta(t) = (m(t))^{\frac{2-\lambda_1}{1-\sigma_1}} L(m(t)).$$

So

$$a_1(t)\theta^{\sigma_1}(t) \approx \frac{(\rho(t))^{\frac{(2-\lambda_1)\sigma_1}{1-\sigma_1}-\lambda_1}(1+\rho(t))^{\lambda_1-\frac{(2-\lambda_1)\sigma_1}{1-\sigma_1}-\mu}}{(A(t))^2}(L_1L^{\sigma_1})(m(t))$$

Hence using again Proposition 2.6 with $\beta = \lambda_1 - \frac{(2-\lambda_1)\sigma_1}{1-\sigma_1} \in (1,2)$, $\gamma = \mu$ and $\widetilde{L} = L_1L^{\sigma_1} \in \mathcal{K}$, we obtain

$$V(a_1\theta^{\sigma_1})(t) \approx (m(t))^{\frac{2-\lambda_1}{1-\sigma_1}}(L_1L^{\sigma_1})(m(t)).$$

On the other hand, since $1+\sigma_2 < \lambda_2 < 2$, we similarly obtain

$$V(a_2\theta^{\sigma_2})(t) \approx (m(t))^{\frac{2-\lambda_1}{1-\sigma_1}}(L_2L^{\sigma_2})(m(t)).$$

Using Lemma 3.1, we deduce that

$$V(a_1\theta^{\sigma_1} + a_2\theta^{\sigma_2})(t) \approx (m(t))^{\frac{2-\lambda_1}{1-\sigma_1}} L(m(t)) = \theta(t).$$

Case 4: $\lambda_1 = 1+\sigma_1$ and $\lambda_2 = 1+\sigma_2$. In this case we have $\theta(t) = m(t)M(m(t))$ By calculations, we obtain

$$a_1(t)\theta^{\sigma_1}(t) + a_2(t)\theta^{\sigma_2}(t) \approx \frac{(\rho(t))^{-1}(1+\rho(t))^{1-\mu}}{(A(t))^2}(M^{\sigma_1}L_1 + M^{\sigma_2}L_2)(m(t))$$

Using Proposition 2.6 with $\beta = 1$, $\gamma = \mu$ and $\widetilde{L} = M^{\sigma_1}L_1 + M^{\sigma_2}L_2$, we deduce that

$$V(a_1\theta^{\sigma_1} + a_2\theta^{\sigma_2})(t) \approx m(t)\int_{m(t)}^{\eta} \frac{\widetilde{L}(s)}{s} ds.$$

Hence the results follows from Lemma 2.4.

Case 5: $\lambda_1 = 1+\sigma_1$ and $\lambda_2 < 1+\sigma_2$. Since $\lim_{t\to 0^+} b(t) \in (0,\infty]$, it follows that $\theta(t) \approx m(t)b(m(t))$. So, we obtain that

$$\begin{aligned} a_1(t)\theta^{\sigma_1}(t) + a_2(t)\theta^{\sigma_2}(t) &\approx \frac{(\rho(t))^{-1}(1+\rho(t))^{1-\mu}}{(A(t))^2}(L_1b^{\sigma_1})(m(t)) \\ &\quad + \frac{(\rho(t))^{-\lambda_2+\sigma_2}(1+\rho(t))^{\lambda_2-\sigma_2-\mu}}{(A(t))^2}(L_2b^{\sigma_2})(m(t)). \end{aligned}$$

Using the fact that for $i \in \{1,2\}$, the function $t \to \widetilde{L}_i(t) := L_i(t)b^{\sigma_i}(t) \in \mathcal{K}$ and that $\lambda_2 - \sigma_2 < 1$, we deduce by Proposition 2.1 that

$$a_1(t)\theta^{\sigma_1}(t) + a_2(t)\theta^{\sigma_2}(t) \approx \frac{(\rho(t))^{-1}(1+\rho(t))^{1-\mu}}{(A(t))^2}(L_1b^{\sigma_1})(m(t)).$$

Hence applying Proposition 2.6 with $\beta = 1$, $\gamma = \mu$, we obtain that

$$V(a_1\theta^{\sigma_1} + a_2\theta^{\sigma_2})(t) \approx m(t)\int_{m(t)}^{\eta} \frac{\widetilde{L}_1(s)}{s}ds \approx \theta(t).$$

Case 6: $\lambda_1 < 1 + \sigma_1$. Since $\theta(t) = 2m(t)$, we obtain

$$a_1(t)\theta^{\sigma_1}(t) \approx \frac{(\rho(t))^{-\lambda_1+\sigma_1}(1+\rho(t))^{\lambda_1-\sigma_1-\mu}}{(A(t))^2} L_1(m(t)).$$

Applying Proposition 2.6 with $\beta = \lambda_1 - \sigma_1 < 1$, $\gamma = \mu$ and $\widetilde{L} = L_1$, we obtain

$$V(a_1\theta^{\sigma_1})(t) \approx m(t).$$

On the other hand, since also $\lambda_2 < 1 + \sigma_2$, we similarly obtain

$$V(a_2\theta^{\sigma_2})(t) \approx m(t).$$

Hence

$$V(a_1\theta^{\sigma_1} + a_2\theta^{\sigma_2})(t) \approx m(t) \approx \theta(t).$$

Case 7: $\lambda_1 = \lambda_2 = 2$. We have $\theta(t) = N(m(t))$. By calculus, we conclude that

$$a_1(t)\theta^{\sigma_1}(t) + a_2(t)\theta^{\sigma_2}(t) \approx \frac{(\rho(t))^{-2}(1+\rho(t))^{2-\mu}}{(A(t))^2}(N^{\sigma_1}L_1 + N^{\sigma_2}L_2)(m(t))$$

On the other hand, since $s \to (N^{\sigma_1}L_1 + N^{\sigma_2}L_2)(s) \in \mathcal{K}$ and from Lemma 2.5,

$$\int_0^1 \frac{(N^{\sigma_1}L_1 + N^{\sigma_2}L_2)(s)}{s}ds \approx N(1) < \infty,$$

then by Proposition 2.6 with $\beta = 2$, $\gamma = \mu$ and $\widetilde{L} = N^{\sigma_1}L_1 + N^{\sigma_2}L_2$, we deduce that

$$V(a_1\theta^{\sigma_1} + a_2\theta^{\sigma_2})(t) \approx \int_0^{m(t)} \frac{(N^{\sigma_1}L_1 + N^{\sigma_2}L_2)(s)}{s}ds.$$

Hence the results follows from Lemma 2.5.

Proof of Theorem 1.4. The next Lemma will be useful to prove the uniqueness.

Lemma 3.2 ([2]). *Let $a \geq 0$ and $u \in C^1((a,\infty))$ be a function satisfying*

$$\begin{gathered} -\frac{1}{A}(Au')' \geq 0, \quad \text{in } (a,\infty), \\ \lim_{t\to a^+} u(t) = 0, \quad \lim_{t\to\infty}\frac{u(t)}{\rho(t)} = 0. \end{gathered} \tag{3.3}$$

Then u is nondecreasing and nonnegative function on (a,∞).

Now we are ready to prove Theorem 1.4. Let $\sigma_1, \sigma_2 \in (-1, 1)$, assume (H1) and put $\widetilde{\omega} := a_1\theta^{\sigma_1} + a_2\theta^{\sigma_2}$ and $v := V(\widetilde{\omega})$. From Theorem 1.3, there exists $M > 1$ such that for each $t > 0$,

$$\frac{1}{M}\theta(t) \leq v(t) \leq M\theta(t). \tag{3.4}$$

On the other hand, using hypothesis (H1), (3.1), Lemma2.2 and Lemma 2.5, we verify that

$$\int_0^{\infty} A(s)\min(1,\rho(s))\widetilde{\omega}(s)ds < \infty. \tag{3.5}$$

Put $\sigma = \max(|\sigma_1|, |\sigma_2|)$, $c = M^{\frac{1+\sigma}{1-\sigma}}$, where the constant M is given in (3.4) and let

$$\Lambda = \{\omega \in C_0([0,\infty)) : \frac{\theta(t)}{c(1+\rho(t))} \leq \omega(t) \leq \frac{c\theta(t)}{1+\rho(t)}, \quad t > 0\}.$$

Using (3.1), Proposition 2.1 and Lemma 2.3, we verify that the function $t \mapsto \frac{\theta(t)}{1+\rho(t)} \in C_0([0,\infty))$ and so Λ is not empty.

We define the operator T on Λ by

$$\begin{aligned} T\omega(t) = \frac{1}{1+\rho(t)} \int_0^\infty G(t,s)\big[a_1(s)(1+\rho(s))^{\sigma_1}\omega^{\sigma_1}(s) \\ + a_2(s)(1+\rho(s))^{\sigma_2}\omega^{\sigma_2}(s)\big]\,ds. \end{aligned} \tag{3.6}$$

We shall prove that T has a fixed point in Λ.

First observe that for this choice of c, by using (3.4), we have for all $\omega \in \Lambda$ and $t > 0$

$$T\omega(t) \leq \frac{1}{1+\rho(t)} V(a_1 c^\sigma M^\sigma \theta^{\sigma_1} + a_2 c^\sigma M^\sigma \theta^{\sigma_2})(t) = \frac{c^\sigma M^\sigma}{1+\rho(t)} v(t) \leq \frac{c\theta(t)}{1+\rho(t)}$$

and

$$T\omega(t) \geq V(a_1 c^{-\sigma} M^{-\sigma} \theta^{\sigma_1} + a_2 c^{-\sigma} M^{-\sigma} \theta^{\sigma_2})(t) = \frac{c^{-\sigma} M^{-\sigma}}{1+\rho(t)} v(t) \geq \frac{\theta(t)}{c(1+\rho(t))}.$$

On the other hand, for all $\omega \in \Lambda$ we have

$$|a_1(t)(1+\rho(t))^{\sigma_1}\omega^{\sigma_1}(t) + a_2(t)(1+\rho(t))^{\sigma_2}\omega^{\sigma_2}(t)| \leq c^\sigma M^\sigma \widetilde{\omega}(t), \tag{3.7}$$

and for all $t, s > 0$, we have

$$\frac{G(t,s)}{1+\rho(t)} \leq A(s)\min(1, \rho(s)). \tag{3.8}$$

Since for each $s > 0$, the function $t \to \frac{G(t,s)}{1+\rho(t)}$ is in $C_0([0,\infty))$, we deduce by using (3.7), (3.8) and (3.5) that the family $\{t \to T\omega(t),\ \omega \in \Lambda\}$ is relatively compact in $C_0([0,\infty))$. Therefore, $T(\Lambda) \subset \Lambda$.

Now, we shall prove the continuity of the operator T in Λ in the supremum norm. Let $(\omega_k)_{k\in\mathbb{N}}$ be a sequence in Λ which converges uniformly to a function ω in Λ. Then, for each $t > 0$, we have

$$|T\omega_k(t) - T\omega(t)| \leq \frac{1}{1+\rho(t)} V[a_1(1+\rho(.))^{\sigma_1}|\omega_k^{\sigma_1} - \omega^{\sigma_1}| + a_2(1+\rho(.))^{\sigma_2}|\omega_k^{\sigma_2} - \omega^{\sigma_2}|](t).$$

On the other hand, by similar arguments as above, we have

$$a_1(1+\rho(.))^{\sigma_1}|\omega_k^{\sigma_1} - \omega^{\sigma_1}| + a_2(1+\rho(.))^{\sigma_2}|\omega_k^{\sigma_2} - \omega^{\sigma_2}| \leq \widetilde{c\omega}(s).$$

We conclude by (3.5) and the dominated convergence theorem that for all $t > 0$,

$$T\omega_k(t) \to T\omega(t) \quad \text{as } k \to +\infty.$$

Consequently, as $T(\Lambda)$ is relatively compact in $C_0([0,\infty))$, we deduce that the pointwise convergence implies the uniform convergence, namely,

$$\|T\omega_k - T\omega\|_\infty \to 0 \quad \text{as } k \to +\infty.$$

Therefore, T is a continuous mapping from Λ into itself. So the Schauder fixed point theorem implies the existence of $\omega \in \Lambda$ such that

$$\omega(t) = \frac{1}{1+\rho(t)} V(a_1(1+\rho(.))^{\sigma_1}\omega^{\sigma_1} + a_2(1+\rho(.))^{\sigma_2}\omega^{\sigma_2})(t).$$

Put $u(t)=1+\rho(t)\omega(t)$. Then u is continuous and satisfies

$$u(t)=V(a_1u^{\sigma_1}+a_2u^{\sigma_2})(t).$$

Since the function $s\to A(s)\min(1,\rho(s))[a_1(s)u^{\sigma_1}(s)+a_2(s)u^{\sigma_2}(s)]$ is continuous and integrable on $(0,\infty)$, then it follows that u is a solution of problem (1.3).

Finally, it remains to prove that u is the unique positive continuous solution satisfying (1.11). To this end, assume that problem (1.3) has two positive continuous solutions u,v satisfying (1.11). Then there exists a constant $m>1$ such that

$$\frac{1}{m}\le\frac{u}{v}\le m.$$

This implies that the set

$$J=\{m\ge 1:\ \frac{1}{m}\le\frac{u}{v}\le m\}$$

is not empty. Let $\sigma:=\max(|\sigma_1|,|\sigma_2|)$ and put $c_0:=\inf J$. Then $c_0\ge 1$ and we have $\frac{1}{c_0}v\le u\le c_0v$. It follows that for $i\in\{1,2\}$, $u^{\sigma_i}\le c_0^{\sigma}v^{\sigma_i}$ and that the function $w:=c_0^{\sigma}v-u$ satisfies

$$\begin{gathered}-\frac{1}{A}(Aw')'=a_1(c_0^{\sigma}v^{\sigma_1}-u^{\sigma_1})+a_2(c_0^{\sigma}v^{\sigma_2}-u^{\sigma_2})\ge 0,\\ \lim_{t\to 0^+}w(t)=0,\\ \lim_{t\to\infty}\frac{w(t)}{\rho(t)}=0.\end{gathered}$$

By Lemma 3.2, this implies that the function $w=c_0^{\sigma}v-u$ is nonnegative. By symmetry, we also have $v\le c_0^{\sigma}u$. Hence $c_0^{\sigma}\in J$ and $c_0\le c_0^{\sigma}$. Since $0\le\sigma<1$, then $c_0=1$ and therefore $u=v$. □

Example 3.3. Let $\sigma_1\in(-1,0)$, $\sigma_2\in(0,1)$ and $\lambda_1,\lambda_2<2$, such that $\frac{2-\lambda_1}{1-\sigma_1}\le\frac{2-\lambda_2}{1-\sigma_2}$. Let $\mu_1,\mu_2>2$ and a_1,a_2 be a positive continuous function on $(0,\infty)$ such that

$$a_i(t)\approx\frac{1}{(A(t))^2}(\rho(t))^{-\lambda_i}(1+\rho(t))^{\lambda_i-\mu_i},\quad\text{for } i\in\{1,2\}.$$

Then by Theorem 1.4, problem (1.3) has a unique positive continuous solution u satisfying for $t>0$,

$$u(t)\approx\begin{cases}(\frac{\rho(t)}{1+\rho(t)})^{\frac{2-\lambda_1}{1-\sigma_1}}, & \text{if } 1+\sigma_1<\lambda_1<2,\\ \frac{\rho(t)}{1+\rho(t)}(\log(\frac{2+2\rho(t)}{\rho(t)}))^{\frac{1}{1-\sigma_2}}, & \text{if } \lambda_1=1+\sigma_1 \text{ and } \lambda_2=1+\sigma_2,\\ \frac{\rho(t)}{1+\rho(t)}(\log(\frac{2+2\rho(t)}{\rho(t)}))^{\frac{1}{1-\sigma_1}}, & \text{if } \lambda_1=1+\sigma_1 \text{ and } \lambda_2<1+\sigma_2,\\ \frac{\rho(t)}{1+\rho(t)}, & \text{if } \lambda_1<1+\sigma_1.\end{cases}$$

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