Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 238, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

FOURTH-ORDER DISCRETE ANISOTROPIC BOUNDARY-VALUE PROBLEMS

MACIEJ LESZCZYŃSKI

ABSTRACT. In this article we consider the fourth-order discrete anisotropic boundary value problem with both advance and retardation. We apply the direct method of the calculus of variations and the mountain pass technique to prove the existence of at least one and at least two solutions. Non-existence of non-trivial solutions is also undertaken.

1. INTRODUCTION

Below $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of all natural numbers, integers and real numbers respectively. Let k denote a natural number. Let $a, b \in \mathbb{Z}$; we define $\mathbb{Z}(a) = \{a, a+1, \dots\}$, and when $a < b$, $\mathbb{Z}(a,b) = \{a, a+1, \dots b\}$.

We consider the difference equation with both advance and retardation,

$$\Delta^2(\gamma_{n-1}\phi_{p_n}(\Delta^2 u_{n-2})) = f(n, u_{n+1}, u_n, u_{n-1}), \quad n \in \mathbb{Z}(1,k), \tag{1.1}$$

with boundary values conditions

$$\Delta u_{-1} = \Delta u_0 = 0, \quad u_{k+1} = u_{k+2} = 0, \tag{1.2}$$

where γ_n is non-zero and real parameter for each $n \in \mathbb{Z}(0, k+1)$, p_n is real valued and $2 \le p_n < \infty$ for all $n \in \mathbb{Z}(0,k)$, Δ is the forward difference operator $\Delta u_n = u_{n+1} - u_n$, $\Delta^i u_n = \Delta(\Delta^{i-1} u_n)$ for $i \ge 2$, ϕ_{p_n} is the so called p_n-Laplacian operator defined as

$$\phi_{p_n}(s) = |s|^{p_n - 2} s,$$

and $f \in C(\mathbb{Z}(0,k) \times \mathbb{R}^3, \mathbb{R})$.

To determine whether boundary value problem (BVP) (1.1)-(1.2) has any solutions, we use the critical point theory. Let us denote

$$\overline{\gamma} = \max\{\gamma_n : n \in \mathbb{Z}(0, k+1)\}, \quad \underline{\gamma} = \min\{\gamma_n : n \in \mathbb{Z}(0, k+1)\},$$
$$\overline{p} = \max\{p_n : n \in \mathbb{Z}(0, k)\}, \quad \underline{p} = \min\{p_n : n \in \mathbb{Z}(0, k)\}.$$

Continuous versions of problem like(1.1)-(1.2) are known to be mathematical models of various phenomena arising in the study of elastic mechanics (see [23]),

---

2010 *Mathematics Subject Classification.* 39A10.

Key words and phrases. Discrete boundary value problem; critical point theory; discrete p-n-Laplacian; fourth-order.

Submitted May 15, 2015. Published September 17, 2015.

electrorheological fluids (see [17]), and image restoration (see [4]). Variational continuous anisotropic problems were started by Fan and Zhang in [5] and later considered by many methods and authors; see [10] for an extensive survey of such boundary value problems. The research concerning the discrete fourth-order anisotropic problems have only been started, see [11, 15], where known tools from the critical point theory are applied prove the existence of solutions. Concerning the investigation of discrete boundary value problems we mention, far from being exhaustive, the following recent papers that used critical point theory [1, 3, 12, 18, 19, 20, 21, 22]. These papers employ in the discrete setting the variational techniques already known for continuous problems of course with necessary modifications. The tools employed cover the Morse theory, mountain pass methodology, linking arguments. New critical point tool for fourth-order discrete BVPs are considered in [16]. In our setting, upon suitable changes, it seems possible to obtain similar results.

Concerning the fourth-order problems we mainly follow [13]. We use somehow simpler approach and consider a more complicated variable exponent case.

2. Variational framework

Let X be a k-dimensional Euclidean space consisting of functions $x : \mathbb{Z}(-1, k+1) \to \mathbb{R}$ satisfying (1.2) and equipped with a norm

$$\|u\| := \Big(\sum_{n=1}^{k} |\Delta^2 u_n|^2\Big)^{1/2}.$$

For $r \in [1, \infty)$, we define the norm

$$\|u\|_r = \Big(\sum_{j=1}^{k} |u_j|^r\Big)^{1/r}.$$

Since $\dim(X) = k$, all norms are equivalent, hence there exist constants $c_{1,r}, c_{2,r}$ for $r \in [1, \infty)$ such that $c_{2,r} \geq c_{1,r} \geq 0$, and

$$c_{1,r}\|u\| \leq \|u\|_r \leq c_{2,r}\|u\| \quad \forall u \in X.$$

Remark 2.1. Following some ideas from [7], the values of the above constants can be easily calculated as

$$c_{1,2} = \frac{\sqrt{6}}{6}, \quad c_{2,2} = \frac{k^2}{2}$$

For any $u \in X$ we have

$$|u_m| \leq \frac{k}{2} \sum_{n=1}^{k} |\Delta^2 u_n| \quad \forall m \in \mathbb{Z}(1, k).$$

Indeed, the following inequality is true

$$\max_{i \in \mathbb{Z}(1,k)} |u(i)| \leq \frac{1}{2} \sum_{n=1}^{k} |\Delta u_n|,$$

while the the boundary conditions $u_0 = u_{k+1} = 0$ are satisfied. So, analogously we obtain

$$\max_{i \in \mathbb{Z}(1,k)} |\Delta u(i)| \leq \frac{1}{2} \sum_{n=1}^{k} |\Delta^2 u_n|,$$

with conditions $\Delta u_0 = \Delta u_{k+1} = 0$, which are satisfied for every $u \in X$. Hence, for every $n \in \mathbb{Z}(1,k)$, we obtain

$$|u_n| \le \sum_{i=1}^{k} |\Delta u_i| \le k \max_{i\in\mathbb{Z}(1,k)} |\Delta u_i| \le \frac{k}{2}\sum_{i=1}^{k} |\Delta^2 u_n|.$$

Thus, for $m \in \mathbb{Z}(1,k)$ we have

$$\begin{aligned}
|u_m|^2 &\le \frac{k^2}{4}\Big(\sum_{n=1}^{k} |\Delta^2 u_n|\Big)^2 \\
&\le \frac{k^2}{4}\Big(\Big(\sum_{n=1}^{k} 1^2\Big)^{1/2}\Big(\sum_{n=1}^{k} |\Delta^2 u_n|^2\Big)^{1/2}\Big)^2 \\
&= \frac{k^3}{4}\sum_{n=1}^{k} |\Delta^2 u_n|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{n=1}^{k} |u_n|^2 = |u_1|^2 + |u_2|^2 + \ldots + |u_k|^2 \\
&\le \frac{k^3}{4}\sum_{n=1}^{k} |\Delta^2 u_n|^2 + k\sum_{n=1}^{k} |\Delta^2 u_n|^2 + \ldots + k\sum_{n=1}^{k} |\Delta^2 u_n|^2 \\
&= \frac{k^4}{4}\sum_{n=1}^{k} |\Delta^2 u_n|^2.
\end{aligned}$$

As a consequence,

$$\|u\|_2 = \Big(\sum_{n=1}^{k} |u_n|^2\Big)^{1/2} \le \Big(\frac{k^4}{4}\sum_{n=1}^{k} |\Delta^2 u_n|^2\Big)^{1/2} = \frac{k^2}{2}\Big(\sum_{n=1}^{k} |\Delta^2 u_n|^2\Big)^{1/2} = \frac{k^2}{2}\|u\|.$$

Additionally

$$\begin{aligned}
\|u\|^2 = \sum_{n=1}^{k} |\Delta^2 u_n|^2 &= \sum_{n=1}^{k} |u_{n+2} - 2u_{n+1} + u_n|^2 \\
&\le \sum_{n=1}^{k} |u_{n+2}|^2 + 4\sum_{n=1}^{k} |u_{n+1}|^2 + \sum_{n=1}^{k} |u_n|^2 \\
&\le \sum_{n=1}^{k} |u_n|^2 + 4\sum_{n=1}^{k} |u_n|^2 + \sum_{n=1}^{k} |u_n|^2 \\
&= 6\sum_{n=1}^{k} |u_n|^2 = 6\|u\|_2^2.
\end{aligned}$$

Hence, we have

$$\|u\|^2 \le 6\|u\|_2^2 \Leftrightarrow \frac{\sqrt{6}}{6}\|u\| \le \|u\|_2.$$

To sum up, we obtain

$$\frac{\sqrt{6}}{6}\|u\| \le \|u\|_2 \le \frac{k^2}{2}\|u\|$$

Let us consider a functional J defined on X as follows

$$J(u)=\sum_{n=1}^{k}\frac{\gamma_{n+1}}{p_n}|\Delta^2 u_n|^{p_n}-\sum_{n=1}^{k}F(n,u_{n+1},u_n), \tag{2.1}$$

where $F(n,x,y)$ is a function of three variables. We denote by $F_x(n,v_1,v_2)$ the derivative of F with respect to the second variable calculated at point (n,v_1,v_2) and by $F_y(n,v_1,v_2)$ the derivative of F with respect to the third one calculated at point (n,v_1,v_2).

$$F_x(n-1,v_2,v_3)+F_y(n,v_1,v_2)=f(n,v_1,v_2,v_3);$$

It is easy to see that $J\in C^1(\mathbb{R}^k,\mathbb{R})$ and for any $u\in\mathbb{R}^k$ exploiting boundary values (1.2) we can calculate the partial derivative as

$$\frac{\partial J}{\partial u_n}=\Delta^2(\gamma_{n-1}\phi_{p_n}(\Delta^2 u_{n-2}))-f(n,u_{n+1},u_n,u_{n-1}),\quad \forall_{n\in\mathbb{Z}(1,k)}.$$

As a consequence, u is a critical point of J on $\mathbb{R}^k$ if and only if

$$\Delta^2(\gamma_{n-1}\phi_{p_n}(\Delta^2 u_{n-2}))=f(n,u_{n+1},u_n,u_{n-1}),\forall n\in\mathbb{Z}(1,k)$$

We redefine problem of proving the existence of solutions of (1.1)-(1.2) to the existence of critical points of J on $\mathbb{R}^k$. Thus, the functional J is the variational framework of our problem.

Now we recall some necessary background from [6]. Let E be a real Banach space, $J\in C^1(E,\mathbb{R})$ i.e, J is a continuously Frechet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition (P.S. condition for short) if every sequence $\{u^{(l)}\}\subset E$ for which $\{J(u^{(l)})\}$ is bounded and $J'(u^{(l)})\to 0$ (as $l\to\infty$) possesses a convergent subsequence in E.

Lemma 2.2 (Mountain Pass Lemma). *Let E be a real Banach space and $J\in C^1(E,\mathbb{R})$ satisfying the P.S. condition. If $J(0)=0$ and*

- (J1) *there exist constants $\rho,a>0$ such that $J|_{\partial B_\rho}\geq a$, and*
- (J2) *there exists $e\in E\backslash B_\rho$ such that $J(e)\leq 0$.*

Then, J possesses a critical value $c\geq a$ given by

$$c=\inf_{g\in\Gamma}\max_{s\in[0,1]}J(g(s)),$$

where $\Gamma=\{g\in C([0,1],E);g(0)=0$ and $g(1)=e\}$.

3. Auxiliary results

In this article we use following inequalities

(A1) For every $u\in X$ and for every $m\geq 1$ we have

$$\sum_{n=1}^{k}|\Delta^2 u_n|^m\leq k\|u\|^m.$$

Proof. Of course we have

$$|\Delta^2 u_n|^2\leq\sum_{i=1}^{k}|\Delta^2 u_i|^2,\quad \text{for every } n\in\mathbb{Z}(1,k),$$

hence we have

$$|\Delta^2 u_n|^m \le ((\sum_{i=1}^{k} |\Delta^2 u_i|^2)^{1/2})^m, \quad \text{for every } n \in \mathbb{Z}(1,k),$$

Summing left hand side of the inequality from 1 to k we obtain

$$\sum_{n=1}^{k} |\Delta^2 u_n|^m \le k\Big(\Big(\sum_{n=1}^{k} |\Delta^2 u_i|^2\Big)^{1/2}\Big)^m.$$

which leads us to

$$\sum_{n=1}^{k} |\Delta^2 u_n|^m \le k\|u\|^m,$$

$\square$

(A2) For every $u \in X$ and every $m > 2$ we have

$$\sum_{n=1}^{k} |\Delta^2 u_n|^m \ge k^{\frac{2}{m-2}} \|u\|^m.$$

Proof. Using Hölder inequality for $m > 2$ we obtain

$$\sum_{n=1}^{k} |\Delta^2 u_n|^2 \le \Big(\sum_{n=1}^{k} 1^{\frac{m}{m-2}}\Big)^{\frac{m-2}{m}} \Big(\sum_{n=1}^{k} (|\Delta^2 u_n|^2)^{m/2}\Big)^{2/m} = k^{\frac{m-2}{m}} \sum_{n=1}^{k} (|\Delta^2 u_n|^m)^{2/m}.$$

Calculating further we obtain

$$\|u\| = \Big(\sum_{n=1}^{k} |\Delta^2 u_n|^2\Big)^{1/2} \le k^{\frac{m-2}{2m}} \sum_{n=1}^{k} (|\Delta^2 u_n|^m)^{1/m}.$$

Thus, we see the thesis

$$\|u\|^m \le k^{\frac{m-2}{2}} \sum_{n=1}^{k} |\Delta^2 u_n|^m \Leftrightarrow k^{\frac{2}{m-2}} \|u\|^m \le \sum_{n=1}^{k} |\Delta^2 u_n|^m.$$

$\square$

(A3) For every $u \in X$ such that $\|u\| \ge 1$ we have

$$\sum_{n=1}^{k} |\Delta^2 u_n|^{p_n} \ge k^{\frac{2}{\underline{p}-2}} \|u\|^{\underline{p}} - k.$$

Proof. Let $u \in X$ be such that $\|u\| \ge 1$. We obtain

$$\begin{aligned}
&\sum_{n=1}^{k} |\Delta^2 u_n|^{p_n} \\
&= \sum_{\{k \in \mathbb{Z}(1,k); |\Delta^2 u_k| \le 1\}} |\Delta^2 u_n|^{p_n} + \sum_{\{k \in \mathbb{Z}(1,k); |\Delta^2 u_k| > 1\}} |\Delta^2 u_n|^{p_n} \\
&\ge \sum_{\{k \in \mathbb{Z}(1,k); |\Delta^2 u_k| \le 1\}} |\Delta^2 u_n|^{\overline{p}} + \sum_{\{k \in \mathbb{Z}(1,k); |\Delta^2 u_k| > 1\}} |\Delta^2 u_n|^{\underline{p}} \\
&= \sum_{n=1}^{k} |\Delta^2 u_n|^{\underline{p}} - \sum_{\{k \in \mathbb{Z}(1,k); |\Delta^2 u_k| \le 1\}} |\Delta^2 u_n|^{\underline{p}} + \sum_{\{k \in \mathbb{Z}(1,k); |\Delta^2 u_k| \le 1\}} |\Delta^2 u_n|^{\overline{p}}
\end{aligned}$$

$$\begin{aligned}
&\geq \sum_{n=1}^{k} |\Delta^2 u_n|^{\underline{p}} - \sum_{n=1}^{k} 1 \\
&= \sum_{n=1}^{k} |\Delta^2 u_n|^{\underline{p}} - k.
\end{aligned}$$

Now we can use (A2) with $m := \underline{p}$ to get

$$\sum_{n=1}^{k} |\Delta^2 u_n|^{\underline{p}} - k \geq k^{\frac{2}{\underline{p}-2}} \|u\|^{\underline{p}} - k,$$

which is our assertion. □

(A4) For every $u \in X$ we have

$$\sum_{n=1}^{k} |\Delta^2 u_n|^{p_n} \leq k\|u\|^{\overline{p}} + k.$$

Proof. Let us decompose

$$\begin{aligned}
&\sum_{n=1}^{k} |\Delta^2 u_n|^{p_n} \\
&= \sum_{\{k\in\mathbb{Z}(1,k);|\Delta^2 u_k|\leq 1\}} |\Delta^2 u_n|^{p_n} + \sum_{\{k\in\mathbb{Z}(1,k);|\Delta^2 u_k|>1\}} |\Delta^2 u_n|^{p_n} \\
&\leq \sum_{\{k\in\mathbb{Z}(1,k);|\Delta^2 u_k|\leq 1\}} |\Delta^2 u_n|^{\underline{p}} + \sum_{\{k\in\mathbb{Z}(1,k);|\Delta^2 u_k|>1\}} |\Delta^2 u_n|^{\overline{p}} \\
&= \sum_{n=1}^{k} |\Delta^2 u_n|^{\overline{p}} + \sum_{\{k\in\mathbb{Z}(1,k);|\Delta^2 u_k|\leq 1\}} |\Delta^2 u_n|^{\underline{p}} - \sum_{\{k\in\mathbb{Z}(1,k);|\Delta^2 u_k|\leq 1\}} |\Delta^2 u_n|^{\overline{p}} \\
&\leq \sum_{n=1}^{k} |\Delta^2 u_n|^{\overline{p}} + \sum_{n=1}^{k} 1 \\
&= \sum_{n=1}^{k} |\Delta^2 u_n|^{\overline{p}} + k.
\end{aligned}$$

Now, using (A1) we have

$$\sum_{n=1}^{k} |\Delta^2 u_n|^{\overline{p}} + k \leq k\|u\|^{\overline{p}} + k.$$

□

(A5) For every $u \in X$ such that $\|u\| \leq 1$ we have

$$\sum_{n=1}^{k} |\Delta^2 u_n|^{p_n} \geq k^{-\frac{\overline{p}-2}{2}} \|u\|^{\overline{p}}.$$

In this section we have used some ideas from [8].

4. Existence of solutions

This section gives theorems with sufficient conditions for the existence of at least one solution to (1.1)-(1.2).

Theorem 4.1. *Assume that the following hypothesis are satisfied*

(F0) *for any* $n \in \mathbb{Z}(0, k+1)$, $\gamma_n < 0$;

(F1) *there exists a functional* $F \in C^1(\mathbb{Z} \times \mathbb{R}^2, \mathbb{R})$, *such that*
$$F_x(n-1, v_2, v_3) + F_y(n, v_1, v_2) = f(n, v_1, v_2, v_3);$$

(F2) *There exists* $M_0 > 0$ *such that, for all* $(n, v_1, v_2) \in \mathbb{Z}(1, k) \times \mathbb{R}^2$
$$F_x(n, v_1, v_2) \le M_0, F_y(n, v_1, v_2) \le M_0.$$

Then (1.1)-(1.2) *possesses at least one solution.*

Remark 4.2. Assumption (F2) implies that there exists a constant M_1 such that

(F2') $|F(n, v_1, v_2)| \le M_1 + M_0(|v_1| + |v_2|)$ for all $(n, v_1, v_2) \in \mathbb{Z}(1, k) \times \mathbb{R}^2$.

Let us define a function $H : [0, 1] \to \mathbb{R}$, $H(t) = F(n, tv_1, tv_2)$. Then H is differentiable, and
$$H'(t) = F_x(n, tv_1, tv_2)v_1 + F_y(n, tv_1, tv_2)v_2.$$
Using the mean value theorem on $[0, 1]$ we obtain
$$\begin{aligned} F(n, v_1, v_2) - F(n, 0, 0) &= H(1) - H(0) = H'(\theta)(1 - 0) \\ &= F_x(n, \theta v_1, \theta v_2)v_1 + F_y(n, \theta v_1, \theta v_2)v_2 \end{aligned}$$
for some $\theta \in [0, 1]$. Now, using assumption (F2) we obtain
$$\begin{aligned} |F(n, v_1, v_2) - F(n, 0, 0)| &= |F_x(n, \theta v_1, \theta v_2)v_1 + F_y(n, \theta v_1, \theta v_2)v_2| \\ &\le |F_x(n, \theta v_1, \theta v_2)||v_1| + |F_y(n, \theta v_1, \theta v_2)||v_2| \\ &\le M_0|v_1| + M_0|v_2|. \end{aligned}$$
On the other hand, using a well know inequality for absolute value we obtain
$$|F(n, v_1, v_2) - F(n, 0, 0)| \ge ||F(n, v_1, v_2)| - |F(n, 0, 0)||,$$
and combining both inequalities we produce the following statement
$$||F(n, v_1, v_2)| - |F(n, 0, 0)|| \le M_0|v_1| + M_0|v_2|.$$
By the definition of the absolute value, it is equivalent to
$$-M_0(|v_1| + |v_2|) \le |F(n, v_1, v_2)| - |F(n, 0, 0)| \le M_0|v_1| + M_0|v_2|,$$
which leads us to thesis substituting $M_1 := \max_{n \in \mathbb{Z}(1,k)}\{|F(n, 0, 0)|\}$.

Proof of Theorem 4.1. By (F2'), for any $u \in \mathbb{R}^k$, we have
$$\begin{aligned} J(u) &= \sum_{n=1}^{k} \frac{\gamma_{n+1}}{p_n}|\Delta^2 u_n|^{p_n} - \sum_{n=1}^{k} F(n, u_{n+1}, u_n) \\ &\le \frac{\overline{\gamma}}{\overline{p}} \sum_{n=1}^{k} |\Delta^2 u_n|^{p_n} - \sum_{n=1}^{k} F(n, u_{n+1}, u_n) \\ &\le \frac{\overline{\gamma}}{\overline{p}} \sum_{n=1}^{k} |\Delta^2 u_n|^{p_n} + M_0 \sum_{n=1}^{k} (|u_{n+1}| + |u_n|) + M_1 k \end{aligned}$$

$$\leq \frac{\overline{\gamma}}{\underline{p}} \sum_{n=1}^{k} |\Delta^2 u_n|^{p_n} + 2M_0 \sum_{n=1}^{k} |u_n| + M_1 k.$$

Now, using (A3) we obtain

$$\begin{aligned}
&\frac{\overline{\gamma}}{\underline{p}} \sum_{n=1}^{k} |\Delta^2 u_n|^{p_n} + 2M_0 \sum_{n=1}^{k} |u_n| + M_1 k \\
&\leq \frac{\overline{\gamma}}{\underline{p}} (k^{\frac{2}{\underline{p}-2}} \|u\|^{\underline{p}} - k) + 2M_0 \sum_{n=1}^{k} |u_n| + M_1 k \\
&\leq \frac{\overline{\gamma}}{\underline{p}} k^{\frac{2}{\underline{p}-2}} \|u\|^{\underline{p}} - \frac{\overline{\gamma}}{\underline{p}} k + 2M_0 c_{2,1} \|u\| + M_1 k \to -\infty \quad \text{as } \|u\| \to +\infty.
\end{aligned}$$

Above inequality means that J is anti coercive. With continuity of J, it attains its maximum at some point. From necessity condition of extremal point of differentiable functional, we acquire that $u_0 := \max\{J(u) : u \in X\}$ is a critical point of J. This finishes the proof. ☐

Theorem 4.3. *Suppose that* (F1) *and the following hypothesis are satisfied*

- (F0') *For every* $n \in \mathbb{Z}(1, k+1), \gamma_n > 0$*;*
- (F3) *There exist* $R > 0$, $1 < \alpha < 2$ *and constants* $a_1, a_2 > 0$ *such that for* $n \in \mathbb{Z}(1, k)$ *and* $\sqrt{(v_1^2 + v_2^2)} \geq R$*;*

$$F(n, v_1, v_2) \leq a_1 (\sqrt{v_1^2 + v_2^2})^{\frac{\alpha}{2} \underline{p}} - a_2 \,.$$

Then (1.1)-(1.2) *possesses at least one solution.*

Proof. By (F3) for any $u \in \mathbb{R}^k$, we have

$$\begin{aligned}
J(u) &= \sum_{n=1}^{k} \frac{\gamma_{n+1}}{p_n} |\Delta^2 u_n|^{p_n} - \sum_{n=1}^{k} F(n, u_{n+1}, u_n) \\
&\geq \frac{\underline{\gamma}}{\overline{p}} \sum_{n=1}^{k} |\Delta^2 u_n|^{p_n} - \sum_{n=1}^{k} F(n, u_{n+1}, u_n) \\
&\geq \frac{\underline{\gamma}}{\overline{p}} \sum_{n=1}^{k} |\Delta^2 u_n|^{p_n} - a_1 \sum_{n=1}^{k} (\sqrt{u_{n+1}^2 + u_n^2})^{\frac{\alpha}{2} \underline{p}} - a_2 k \\
&\geq \frac{\underline{\gamma}}{\overline{p}} \sum_{n=1}^{k} |\Delta^2 u_n|^{p_n} - a_1 \sum_{n=1}^{k} (\sqrt{\sum_{i=1}^{k} u_i^2})^{\frac{\alpha}{2} \underline{p}} - a_2 k \\
&\geq \frac{\underline{\gamma}}{\overline{p}} \sum_{n=1}^{k} |\Delta^2 u_n|^{p_n} - a_1 \sum_{n=1}^{k} \|u\|_2^{\frac{\alpha}{2} \underline{p}} - a_2 k \\
&\geq \frac{\underline{\gamma}}{\overline{p}} \sum_{n=1}^{k} |\Delta^2 u_n|^{p_n} - a_1 k \|u\|_2^{\frac{\alpha}{2} \underline{p}} - a_2 k.
\end{aligned}$$

Again we will use (A3). Indeed, we have

$$\frac{\underline{\gamma}}{\overline{p}} \sum_{n=1}^{k} |\Delta^2 u_n|^{p_n} - a_1 k \|u\|_2^{\alpha \underline{p}/2} - a_2 k$$

$$
\begin{aligned}
&\geq \frac{\underline{\gamma}}{\overline{p}}\big(k^{\frac{2}{\underline{p}-2}}\|u\|^{\underline{p}} - k\big) - a_1 k\|u\|_2^{\frac{\alpha}{2}\underline{p}} - a_2 k \\
&\geq \frac{\underline{\gamma}}{\overline{p}} k^{\frac{2}{\underline{p}-2}}\|u\|^{\underline{p}} - \frac{\underline{\gamma}}{\overline{p}}k - a_1 k\|u\|_2^{\frac{\alpha}{2}\underline{p}} - a_2 k \to +\infty \quad \text{as } \|u\| \to +\infty.
\end{aligned}
$$

This inequality implies that J is coercive, and using similar reasoning, we acquire that (1.1)-(1.2) possesses at least one solution. $\square$

5. Existence and multiplicity of solutions

This section will give sufficient conditions to existing at least two solutions to (1.1)-(1.2).

Theorem 5.1. *Suppose* (F0'), (F1) *and the following conditions are satisfied:*

(F4) *Functional F satisfies*

$$
\lim_{r\to 0}\frac{F(n,v_1,v_2)}{r^{\overline{p}}} = 0, \quad r = \sqrt{v_1^2+v_2^2};
$$

(F5) *There exist $\beta > \overline{p}$ and $a_3 > 0$ such that for $n \in \mathbb{Z}(1,k)$ and $\sqrt{(v_1^2+v_2^2)} \geq R$*

$$
F(n,v_1,v_2) > a_3\big(\sqrt{v_1^2+v_2^2}\big)^{\beta}.
$$

Then (1.1)-(1.2) *possesses at least two nontrivial solutions.*

Proof. To show that our functional satisfies the P.S. condition we use that any anti-coercive functional $T : X \to \mathbb{R}$, where $\dim X < \infty$, satisfies the P.S. condition. By (F5) we have

$$
\begin{aligned}
J(u) &= \sum_{n=1}^{k}\frac{\gamma_{n+1}}{p_n}|\Delta^2 u_n|^{p_n} - \sum_{n=1}^{k} F(n,u_{n+1},u_n) \\
&\leq \sum_{n=1}^{k}\frac{\gamma_{n+1}}{p_n}|\Delta^2 u_n|^{p_n} - \sum_{n=1}^{k} a_3\big(\sqrt{u_{n+1}^2+u_n^2}\big)^{\beta}.
\end{aligned}
$$

Now, using (A2) and (A3) we have

$$
\begin{aligned}
&\sum_{n=1}^{k}\frac{\gamma_{n+1}}{p_n}|\Delta^2 u_n|^{p_n} - \sum_{n=1}^{k} a_3\big(\sqrt{u_{n+1}^2+u_n^2}\big)^{\beta} \\
&\leq \frac{\overline{\gamma}}{\underline{p}}k\|u\|^{\overline{p}} + \frac{\overline{\gamma}}{\underline{p}}k - a_3\Big(\sum_{n=1}^{k}\sqrt{u_n^2}\Big)^{\beta} \\
&= \frac{\overline{\gamma}}{\underline{p}}k\|u\|^{\overline{p}} + \frac{\overline{\gamma}}{\underline{p}}k - a_3\|u\|_1^{\beta} \\
&\leq \frac{\overline{\gamma}}{\underline{p}}k\|u\|^{\overline{p}} + \frac{\overline{\gamma}}{\underline{p}}k - a_3 c_{2,1}^{\beta}\|u\|^{\beta} \to -\infty \quad \text{as } \|u\| \to \infty,
\end{aligned}
$$

We proved that J is anti coercive, thus, the P.S. condition is verified.

Now, we have to show that other conditions of Mountain Pass Lemma are satisfied. By (F4), for any

$$
\epsilon = k^{-\frac{\overline{p}-4}{2}}\frac{\underline{\gamma}}{2\overline{p}(c_{2,2})^{\overline{p}}},
$$

there exists $\rho > 0$ such that

$$
|F(n,v_1,v_2)| \leq \epsilon(v_1+v_2)^{\overline{p}/2} \quad \forall_{n\in\mathbb{Z}(1,k)}
$$

for $\sqrt{v_1^2+v_2^2} \le 2\sqrt{\rho}$. Then

$$
\begin{aligned}
J(u) &= \sum_{n=1}^{k} \frac{\gamma_{n+1}}{p_n} |\Delta^2 u_n|^{p_n} - \sum_{n=1}^{k} F(n, u_{n+1}, u_n) \\
&\ge \frac{\underline{\gamma}}{\overline{p}} \sum_{n=1}^{k} |\Delta^2 u_n|^{p_n} - \epsilon \sum_{n=1}^{k} (u_{n+1}^2 + u_n^2)^{\overline{p}/2}.
\end{aligned}
$$

Now, using (A5), we can estimate

$$
\begin{aligned}
&\frac{\underline{\gamma}}{\overline{p}} \sum_{n=1}^{k} |\Delta^2 u_n|^{p_n} - \epsilon \sum_{n=1}^{k} (u_{n+1}^2 + u_n^2)^{\overline{p}/2} \\
&\ge \frac{\underline{\gamma}}{\overline{p}} k^{-\frac{\overline{p}-2}{2}} \|u\|^{\overline{p}} - \epsilon \sum_{n=1}^{k} \Big(\sqrt{\sum_{i=1}^{k} u_i^2}\Big)^{\overline{p}} \\
&\ge \frac{\underline{\gamma}}{\overline{p}} k^{-\frac{\overline{p}-2}{2}} \|u\|^{\overline{p}} - \epsilon \sum_{n=1}^{k} \|u\|_2^{\overline{p}} \\
&\ge \frac{\underline{\gamma}}{\overline{p}} k^{-\frac{\overline{p}-2}{2}} \|u\|^{\overline{p}} - \epsilon k \|u\|_2^{\overline{p}} \\
&\ge \frac{\underline{\gamma}}{\overline{p}} k^{-\frac{\overline{p}-2}{2}} \|u\|^{\overline{p}} - \epsilon k c_{2,2}^{\overline{p}} \|u\|^{\overline{p}} \\
&= \frac{\underline{\gamma}}{\overline{p}} k^{-\frac{\overline{p}-2}{2}} \|u\|^{\overline{p}} - k^{-\frac{\overline{p}-2}{2}} \frac{\underline{\gamma}}{2\overline{p}(c_{2,2})^{\overline{p}}} (c_{2,2})^{\overline{p}} \|u\|^{\overline{p}} \\
&= \frac{\underline{\gamma}}{2\overline{p}} k^{-\frac{\overline{p}-2}{2}} \|u\|^{\overline{p}}
\end{aligned}
$$

Take $a = \frac{\underline{\gamma}}{2\overline{p}} k^{-\frac{\overline{p}-2}{2}} \rho^{\overline{p}} > 0$. Therefore

$$
J(u) \ge a > 0 \quad \forall_{u \in \partial B}.
$$

At the same time, we have also proved that there exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B} \ge a$. That is to say, J satisfies (J1) of the Mountain Pass Lemma.

For our setting, $J(0) = 0$. To exploit the Mountain Pass Lemma in critical point theory, we need to verify other conditions of the lemma. We have shown that J satisfies the P.S. condition. So, it suffices to verify the condition (J2). From the proof of the P.S. condition we know that

$$
J(u) \le \frac{\overline{\gamma}}{\underline{p}} k \|u\|^{\overline{p}} + \frac{\overline{\gamma}}{\underline{p}} k - a_3 c_{2,1}^{\beta} \|u\|^{\beta}.
$$

Since $\beta > \overline{p}$, we can choose u^* far enough to ensure that $J(u^*) < 0$. By the Mountain Pass Lemma, J possesses a critical value $c \ge a > 0$ where

$$
c = \inf_{h \in \Gamma} \sup_{s \in [0,1]} J(h(s)),
$$
$$
\Gamma = \{h \in C([0,1], \mathbb{R}^k) | h(0) = 0, h(1) = u^*\}.
$$

Let $\overline{u} \in \mathbb{R}^k$ be a critical point associated to the critical value c of J. Due to anti coercivity and continuity, we know that there exists $\hat{u}$ such that

$$
J(\hat{u}) = c_{\max} = \max_{s \in [0,1]} J(h(s)).
$$

Clearly, $\hat{u} \neq 0$. If $\overline{u} \neq \hat{u}$ we reach the assertion of the theorem.

Suppose that $\overline{u} = \hat{u}$. Itimplies that

$$J(\overline{u}) = \inf_{h \in \Gamma} \sup_{s \in [0,1]} J(h(s)).$$

Hence for any function $h \in \Gamma$, $\max_{t \in [0,1]} J(h(t)) = J(\overline{u})$. Indeed, for any $h \in \Gamma$ we have

$$J(\overline{u}) \geq \max_{t \in [0,1]} J(h(t))$$

since

$$J(\overline{p}) = \max_{x \in X} J(x) \text{ and } J(\overline{p}) \leq \max_{t \in [0,1]} J(h(t))$$

by the definition of the minimum. Since $k > 1$, the space $X \backslash \{\overline{u}\}$ is path connected. Then there exists a function $h_0 \in \Gamma$ such that $h_0(t) \neq \overline{u}$ for $t \in [0,1]$. Since $\max_{t \in [0,1]} J(h_0(t)) = J(\overline{u})$ it follows that there exists $t_0 \in (0,1)$ such that $J(h_0(t_0)) = \max_{x \in X} J(x)$ and by assertion $h_0(t_0) \neq \overline{u}$. Thus $h_0(t_0$ is a critical point different from $\overline{u}$.

The above argumentation implies that (1.1)-(1.2) possesses at least two nontrivial solutions. □

6. Nonexistence of solutions

This section give sufficient conditions for the nonexistence of nontrivial solutions to (1.1)-(1.2).

Theorem 6.1. *Let* (F0), (F1) *and the following conditions be satisfied.*

(F6) *For all* $n \in \mathbb{Z}(1,k)$, $v_2 \neq 0 \Rightarrow v_2 f(n, v_1, v_2, v_3) > 0$.

Then (1.1)-(1.2) *has no nontrivial solution.*

Proof. Assume in the sake of contradiction that (1.1)-(1.2) possesses a nontrivial solution. Then, functional J has a nonzero critical point u^*. Since

$$\frac{\partial J}{\partial u_n} = \Delta^2(\gamma_{n-1}\phi_{p_n}(\Delta^2 u_n)) - f(n, u_{n+1}, u_n, u_{n-1}),$$

it follows that

$$\begin{aligned}
&\sum_{n=1}^{k} f(n, u^*_{n+1}, u^*_n, u^*_{n-1}) \cdot u^*_n \\
&= \sum_{n=1}^{k} \Delta^2(\gamma_{n-1}\phi_{p_n}(\Delta^2 u^*_{n-2})) \cdot u^*_n \\
&= \sum_{n=1}^{k} \gamma_{n+1}|\Delta^2 u^*_n|^{p_{n+2}-2}(\Delta^2 u^*_n)u^*_n - 2\gamma_n|\Delta^2 u^*_{n-1}|^{p_{n+1}-2}(\Delta^2 u^*_{n-1})u^*_n \\
&\quad + \gamma_{n-1}|\Delta^2 u^*_{n-2}|^{p_n-2}(\Delta^2 u^*_{n-2})u^*_n \\
&= \sum_{n=1}^{k} \gamma_{n+1}|\Delta^2 u^*_n|^{p_{n+2}-2}(\Delta^2 u^*_n)u^*_n - 2\sum_{n=1}^{k} \gamma_n|\Delta^2 u^*_{n-1}|^{p_{n+1}-2}(\Delta^2 u^*_{n-1})u^*_n \\
&\quad + \sum_{n=1}^{k} \gamma_{n-1}|\Delta^2 u^*_{n-2}|^{p_n-2}(\Delta^2 u^*_{n-2})u^*_n
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{k} \gamma_{n+1}|\Delta^2 u_n^*|^{p_{n+2}-2}(\Delta^2 u_n^*)u_n^* - 2\sum_{n=0}^{k-1} \gamma_{n+1}|\Delta^2 u_n^*|^{p_{n+2}-2}(\Delta^2 u_n^*)u_{n+1}^* \\
&\quad + \sum_{n=-1}^{k-2} \gamma_{n+1}|\Delta^2 u_n^*|^{p_{n+2}-2}(\Delta^2 u_n^*)u_{n+2}^* \\
&= \sum_{n=1}^{k} \gamma_{n+1}|\Delta^2 u_n^*|^{p_{n+2}-2}(\Delta^2 u_n^*)u_n^* - 2\sum_{n=1}^{k} \gamma_{n+1}|\Delta^2 u_n^*|^{p_{n+2}-2}(\Delta^2 u_n^*)u_{n+1}^* \\
&\quad + \sum_{n=1}^{k} \gamma_{n+1}|\Delta^2 u_n^*|^{p_{n+2}-2}(\Delta^2 u_n^*)u_{n+2}^* + [-2\gamma_1|\Delta^2 u_0^*|^{p_2-2}(\Delta^2 u_0^*)u_1^* \\
&\quad + 2\gamma_{k+1}|\Delta^2 u_k^*|^{p_k+2-2}(\Delta^2 u_k^*)u_{k+1}^* + \gamma_0|\Delta^2 u_{-1}^*|^{p_1-2}(\Delta^2 u_{-1}^*)u_1^* \\
&\quad + \gamma_1|\Delta^2 u_0^*|^{p_2-2}(\Delta^2 u_0^*)u_2^* - \gamma_k|\Delta^2 u_{k-1}^*|^{p_k+1-2}(\Delta^2 u_{k-1}^*)u_{k+1}^* \\
&\quad - \gamma_{k+1}|\Delta^2 u_k^*|^{p_{k+2}-2}(\Delta^2 u_k^*)u_{k+2}^*].
\end{aligned}
$$

Using boundary values, it is easy to see that the expression in square bracket is equal to zero. This implies

$$
\begin{aligned}
&\sum_{n=1}^{k} \gamma_{n+1}|\Delta^2 u_n^*|^{p_{n+2}-2}(\Delta^2 u_n^*)(u_n^* - 2u_{n+1}^* + u_{n+2}^*) \\
&= \sum_{n=1}^{k} \gamma_{n+1}|\Delta^2 u_n^*|^{p_{n+2}-2}(\Delta^2 u_n^*)^2 < 0,
\end{aligned}
$$

which is a contradiction with assumption. Hence, the only critical point of J is 0. □

7. Final comments and examples

Firstly note that the classical approach to the positive solutions do not apply to the fourth-order problems. It is so because of the inequality

$$\Delta^2 u_n \cdot \Delta^2 u_{-n} \le 0,$$

where $u_- = \max\{-u, 0\}$, is not satisfied for all $u \in X$. Indeed, take

$$u_n = 5, \quad u_{n+1} = 1, \quad u_{n+2} = -2.$$

Substituting symbols by numbers we obtain

$$
\begin{aligned}
\Delta^2 u_n \cdot \Delta^2 u_{-n} &= (u_{n+2} - 2u_{n+1} + u_n)(u_{-n+2} - 2u_{-n+1} + u_{-n}) \\
&= (-2 - 2 + 5) \cdot (2 + 0 + 0) = 2 > 0.
\end{aligned}
$$

Now, we shown four examples to illustrate the main results.

Example 7.1. For $n \in \mathbb{Z}(1,k)$, assume that

$$\Delta^2(-2n\phi_{p_n}(\Delta^2 u_{n-2})) = \Phi(n-1)\cos u_n \cos u_{n-1} - \Phi(n)\sin u_{n+1}\sin u_n,$$

with boundary value conditions (1.2), where $p_n : \mathbb{Z}(1,k) \to \mathbb{R}$, $\Phi(n) > 0$, $n \in \mathbb{Z}(1,k)$. We have

$$
\begin{gathered}
\gamma_n = -2(n+1), \quad f(n, v_1, v_2, v_3) = \Phi(n-1)\cos v_2 \cos v_3 - \Phi(n)\sin v_1 \sin v_2, \\
F(n, v_1, v_2) = \Phi(n)\sin v_1 \cos v_2.
\end{gathered}
$$

It is easy to verify that all the assumptions of Theorem 4.1 are satisfied, thus our problem possesses at least one solution.

Example 7.2. For $n \in \mathbb{Z}(1,k)$, assume that

$$\Delta^2(6^{n-1}\phi_{p_n}(\Delta^2 u_{n-2})) = \alpha u_n[\psi(n)(u_{n+1}^2+u_n^2)^{\frac{\alpha}{4}\underline{p}-1}+\psi(n-1)(u_n^2+u_{n-1}^2)^{\frac{\alpha}{4}\underline{p}-1}],$$

with boundary value conditions (1.2), where $p_n : \mathbb{Z}(1,k) \to \mathbb{R}$, $\psi(n) > 0$, $n \in \mathbb{Z}(1,k)$, $1 < \alpha < 2$. We have

$$\gamma_n = 6^n, \quad f(n,v_1,v_2,v_3) = \alpha v_2[\psi(n)(v_1^2+v_2^2)^{\frac{\alpha}{4}\underline{p}-1}+\psi(n-1)(v_2^2+v_3^2)^{\frac{\alpha}{4}\underline{p}-1}],$$
$$F(n,v_1,v_2) = \psi(n)(v_1^2+v_2^2)^{\frac{\alpha}{4}\underline{p}}.$$

We can easily check that all the assumptions of Theorem 4.1 are satisfied, hence our problem possesses at least one solution.

Example 7.3. For $n \in \mathbb{Z}(1,k)$, assume that

$$\Delta^2(\phi_{p_n}(\Delta^2 u_{n-2})) = \beta u_n[\psi(n)(u_{n+1}^2+u_n^2)^{\frac{\beta}{2}-1}+\psi(n-1)(u_n^2+u_{n-1}^2)^{\frac{\beta}{2}-1}]$$

with boundary value conditions (1.2), where $p_n : \mathbb{Z}(1,k) \to \mathbb{R}$, $\psi(n) > 0$, $n \in \mathbb{Z}(1,k)$, $\beta > \overline{p}$. We have

$$\gamma_n \equiv 1, \quad f(n,v_1,v_2,v_3) = \beta v_2[\psi(n)(v_1^2+v_2^2)^{\frac{\beta}{2}-1}+\psi(n-1)(v_2^2+v_3^2)^{\frac{\beta}{2}-1}],$$
$$F(n,v_1,v_2) = \psi(n)(v_1^2+v_2^2)^{\frac{\beta}{2}}.$$

Then, the assumptions of Theorem 5.1 are satisfied, hence our problem possesses at least two solutions.

Example 7.4. For $n \in \mathbb{Z}(1,k)$, assume that

$$\Delta^2(-\phi_{p_n}(\Delta^2 u_{n-2})) = 4u_n[(u_{n+1}^2+u_n^2)+(u_n^2+u_{n-1}^2)],$$

with boundary value conditions (1.2), where $p_n : \mathbb{Z}(1,k) \to \mathbb{R}$. We have

$$\gamma_n \equiv -1,\ f(n,v_1,v_2,v_3) = 4v_2[(v_1^2+v_2^2)+(v_2^2+v_3^2)],$$
$$F(n,v_1,v_2) = (v_1^2+v_2^2)^2.$$

Then, the assumptions of Theorem 6.1 are satisfied, hence our problem has no nontrivial solutions.

Acknowledgements. The Author thanks Professor Marek Galewski from Lodz University of Technology for some suggestions on the topic of this research.

References

[1] R. P. Agarwal, K. Perera, D. O'Regan; *Multiple positive solutions of singular discrete p-Laplacian problems via variational methods*, Adv. Difference Equ. 2005 (2) (2005) 93–99.

[2] A. Cabada, A. Iannizzotto, S. Tersian; *Multiple solutions for discrete boundary value problems.* J. Math. Anal. Appl. 356 (2009), no. 2, 418–428.

[3] X. Cai, J. Yu; *Existence theorems of periodic solutions for second-order nonlinear difference equations*, Adv. Difference Equ. 2008 (2008) Article ID 247071.

[4] Y. Chen, S. Levine, M. Rao; *Variable exponent, linear growth functionals in image processing*, SIAM J. Appl. Math. 66 (2006), No. 4, 1383-1406.

[5] X. L. Fan, H. Zhang; *Existence of Solutions for $p(x)$-Lapacian Dirichlet Problem*, Nonlinear Anal., Theory Methods Appl. 52, No. 8, A, 1843-1852 (2003).

[6] D. G. Figueredo; *Lectures on the Ekeland Variational Principle with Applications and Detours*, Preliminary Lecture Notes, SISSA, 1988.

[7] M. Galewski, J. Smejda; *On the dependence on parameters for mountain pass solutions of second-order discrete BVP's*, Appl. Math. Comput. 219, No. 11, 5963-5971 (2013).

[8] M. Galewski, R. Wieteska; *Existence and multiplicity of positive solutions for discrete anisotropic equations*, Turk. J. Math. 38, No. 2, 297-310 (2014).
[9] A. Guiro, B. Kone, S. Ouaro; *Weak homoclinic solutions of anisotropic difference equation with variable exponents*, Advances in Difference Equations 2012, 2012:154
[10] P. Harjulehto, P. Hästö, U. V. Le, M. Nuortio; *Overview of differential equations with non-standard growth*, Nonlinear Anal. 72 (2010), 4551-4574.
[11] B. Kone. S. Ouaro; *Weak solutions for anisotropic discrete boundary value problems*, J. Difference Equ. Appl. 17 (2011), no. 10, 1537–1547.
[12] J. Q. Liu, J. B. Su; *Remarks on multiple nontrivial solutions for quasi-linear resonant problemes*, J. Math. Anal. Appl. 258 (2001) 209–222.
[13] X. Liy, Y. Zhang, H. Shi; *Nonexistence and existence result for a fourth-order p-Laplacian discrete mixed boundary value problem*, Mediterr. J. Math. (2014)
[14] J. Mawhin; *Problèmes de Dirichlet variationnels non liné aires*, Les Presses de l'Université de Montréal, 1987.
[15] M. Mihăilescu, V. Rădulescu, S. Tersian; *Eigenvalue problems for anisotropic discrete boundary value problems.* J. Difference Equ. Appl. 15 (2009), no. 6, 557–567.
[16] G. Molica Bisci, D. Repovš; *On sequences of solutions for discrete anisotropic equations*, Expo. Math. 32 (2014), 284-295.
[17] M. Růžička; *Electrorheological fluids: Modelling and Mathematical Theory*, in: Lecture Notes in Mathematics, vol. 1748, Springer-Verlag, Berlin, 2000.
[18] P. Stehlík; *On variational methods for periodic discrete problems*, J. Difference Equ. Appl. 14 (3) (2008), 259–273.
[19] Y. Tian, Z. Du, W. Ge; *Existence results for discrete Sturm-Liouville problem via variational methods*, J. Difference Equ. Appl. 13 (6) (2007), 467–478.
[20] Y. Yang, J. Zhang; *Existence of solution for some discrete value problems with a parameter*, Appl. Math. Comput. 211 (2009), 293–302.
[21] G. Zhang, S. S. Cheng; *Existence of solutions for a nonlinear system with a parameter*, J. Math. Anal. Appl. 314 (1) (2006), 311–319.
[22] G. Zhang; *Existence of non-zero solutions for a nonlinear system with a parameter*, Nonlinear Anal. 66 (6) (2007), 1400–1416.
[23] V. V. Zhikov; *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR Izv. 29 (1987), 33-66.

Maciej Leszczyński
Institute of Mathematics, Technical University of Lodz, Wolczanska 215, 90-924 Lodz, Poland
E-mail address: 165708@edu.p.lodz.pl