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EXISTENCE OF GLOBAL SOLUTIONS TO A MUTUALISTIC MODEL WITH DOUBLE FRONTS

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Abstract. We study a system of semilinear parabolic equations with two free boundaries describing the spreading fronts of the invasive species in a mutualistic ecological model. We establish the existence and uniqueness of a local classical solution and then study the asymptotic behavior of the free boundary problem. The results indicate that two free boundaries tend monotonically to finite values at the same time, or to infinite simultaneously. Also the free boundary problem admits a global slow solution with unbounded free boundaries if the geometric average of the interaction coefficients is less than 1, while if it is bigger than 1 there exist the grow-up solution and global fast solution with bounded free boundaries.

1. Introduction

Free boundary problems associated with the ecological models have attracted considerable research attention in the past because of their relevance in applications. For example, Lin [12] introduced the free boundary in a predator-prey model. Du and Lou [6] considered a two free boundaries problem with a general nonlinear term. Wang and Zhao [18, 21] studied the Lotka-Volterra type prey-predator model. While Lotka-Volterra type competition models had been discussed by Du and Lin [5], and Guo and Wu [9]. Some free boundary problems describing tumor growth had been considered by Tao and Xu [17, 19].

For the mutualistic model, Kim and Lin [10] studied the free boundary problem

$$
\begin{gathered}
u_t - d_1 u_{xx} = u(a_1 - b_1 u + c_1 v), \quad t > 0,\ 0 < x < h(t),\\
v_t - d_2 v_{xx} = v(a_2 + b_2 u - c_2 v), \quad t > 0,\ 0 < x < \infty,\\
u(t,x) = 0, \quad t \ge 0,\ h(t) < x < \infty,\\
u = 0, \quad h'(t) = -\mu \frac{\partial u}{\partial x}, \quad t > 0,\ x = h(t),\\
\frac{\partial u}{\partial x}(t,0) = \frac{\partial v}{\partial x}(t,0) = 0, \quad t > 0,\\
h(0) = b, \quad (0 < b < \infty),\\
u(0,x) = u_0(x) \ge 0, \quad 0 \le x \le b,
\end{gathered}
$$

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$$v(0,x) = v_0(x) \geq 0, \quad 0 \leq x \leq \infty, \tag{1.1}$$

and found blowup and global solutions.

The condition on the free boundary is $h'(t) = -\mu u_x(t, h(t))$ called the one-phase Stefan condition, and it was given by Josef Stefan in his papers published in 1989. Ecologically, it means that the amount of the species flowing across the free boundary is increasing with respect to the moving length [12].

As for the one-phase Stefan problem for the heat equation with a superlinear reaction term

$$\begin{gathered}
u_t - u_{xx} = u^{1+p}, \quad t > 0,\ 0 < x < h(t), \\
h'(t) = -\frac{\partial u}{\partial x}, \quad t > 0,\ x = h(t), \\
\frac{\partial u}{\partial x}(t,0) = u(0,h(t)) = 0, \quad t > 0, \\
h(0) = b, \quad (0 < b < \infty), \\
u(0,x) = u_0(x) \geq 0, \quad 0 \leq x \leq b,
\end{gathered} \tag{1.2}$$

it was shown in [7, 8] that all global solutions are bounded and decay uniformly to 0 as $t \to \infty$ if the initial data is small, while if it is big, the solution will blow up in a finite time. Moreover they showed that there exist global solutions with slow decay and unbounded free boundary.

Considering two species mutualistic model proposed by May [13] in 1976, and the model is described by the following coupled ODE system:

$$\begin{gathered}
\dot{u}(t) = r_1 u\big(1 - \frac{u}{K_1 + \alpha_1 v}\big), \\
\dot{v}(t) = r_2 v\big(1 - \frac{v}{K_2 + \alpha_2 u}\big),
\end{gathered} \tag{1.3}$$

where $r_i, K_i, \alpha_i, (i = 1, 2)$ are positive constants. We deduce that, if $\alpha_1\alpha_2 > 1$, the solution would grow up, which means that it becomes infinite as the time goes to infinity, while if $\alpha_1\alpha_2 < 1$ there is an unique positive equilibrium $(\frac{K_1+\alpha_1 K_2}{1-\alpha_1\alpha_2}, \frac{K_2+\alpha_2 K_1}{1-\alpha_1\alpha_2})$. Linearization and spectrum analysis shows that the unique positive equilibrium is locally asymptotically stable, and it is globally asymptotically stable in the positive quadrant by constructing the Lyapunov function.

Motivated by the former work, we study the following mutualistic model with double fronts,

$$\begin{gathered}
\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + r_1 u\big(1 - \frac{u}{K_1 + \alpha_1 v}\big), \quad t > 0,\ g(t) < x < h(t), \\
\frac{\partial v}{\partial t} = b\frac{\partial^2 v}{\partial x^2} + r_2 v\big(1 - \frac{v}{K_2 + \alpha_2 u}\big), \quad t > 0,\ -\infty < x < \infty, \\
u(t,x) = 0, \quad t > 0,\ x \leq g(t) \text{ or } x \geq h(t), \\
g(0) = -h_0, \quad g'(t) = -\mu\frac{\partial u}{\partial x}(t, g(t)), \quad t > 0, \\
h(0) = h_0, \quad h'(t) = -\mu\frac{\partial u}{\partial x}(t, h(t)), \quad t > 0, \\
u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad -\infty < x < \infty,
\end{gathered} \tag{1.4}$$

where $x = g(t)$ and $x = h(t)$ are the moving left and right boundaries to be determined, and h_0 and μ are positive constants. Throughout this paper the initial functions u_0 and v_0 are nonnegative and satisfy

$$\begin{gathered} u_0 \in C^2([-h_0,h_0]),\ u_0(\pm h_0)=0, \quad u_0(x)>0, \quad x\in(-h_0,h_0),\\ v_0\in C^2(-\infty,\infty)\cap L^\infty(-\infty,\infty), \quad v_0(x)=0,\ x\in(-\infty,-h_0]\cup[h_0,\infty). \end{gathered} \tag{1.5}$$

The paper is organized as follows. In the next section, existence and uniqueness of local solutions for two free boundaries problem (1.4) is established by using contraction mapping theorem. Results relating to global slow solution for $\alpha_1\alpha_2<1$ are presented in Section 3. In Section 4, the grow-up solution and global fast solution for $\alpha_1\alpha_2>1$ are established.

We end this section by recalling two definitions which will be used in next sections.

Definition 1.1 ([7, 8])**.** A solution $(u,v;g,h)$ of (1.4) is said to be classical if $u\in C([0,T_{\max})\times[g(t),h(t)])\cap C^{1,2}((0,T_{\max})\times(g(t),h(t))$, $v\in C([0,T_{\max})\times(-\infty,\infty))\cap C^{1,2}((0,T_{\max})\times(-\infty,\infty))\cap C([0,T_{\max})\times L^\infty(-\infty,\infty))$ and $h,g\in C^1[0,T_{\max})$ with $T_{\max}\le+\infty$ and satisfy (1.4), where $T_{\max}$ denotes the maximal existing time of solution.

Definition 1.2 ([1, 7, 8])**.** A solution $(u,v;g,h)$ of (1.4) is said to be global if $T_{\max}=+\infty$. If $T_{\max}=\infty$ and $\lim_{t\to T_{max}}(\|u(t,x)\|_{L^\infty[g(t),h(t)]}+\|v(t,x)\|_{L^\infty(-\infty,+\infty)})\to+\infty$, we say that the solution grows up. If $T_{\max}=\infty$ and $h_\infty:=\lim_{t\to\infty}h(t)<\infty$, $g_\infty:=\lim_{t\to\infty}g(t)>-\infty$, the solution is called global fast solution since that the solution decays uniformly to 0 at an exponential rate, while If $T_{\max}=\infty$ and $h_\infty=\infty$, $g_\infty=-\infty$, it is called global slow solution, whose decay rate is at most polynomial.

2. Existence and uniqueness

In this section, we first present the following local existence and uniqueness result by the contraction mapping theorem and then give the property of the double fronts.

Theorem 2.1. *For any given (u_0,v_0) satisfying (1.5), and any $\alpha\in(0,1)$, there exists a $T>0$ such that problem (1.4) admits a unique solution*

$$(u,v;g,h)\in C^{1+\alpha,(1+\alpha)/2}(D_T)\times C^{1+\alpha,(1+\alpha)/2}(D_T^\infty)\times[C^{1+\alpha/2}([0,T])]^2,$$

moreover,

$$\begin{aligned} &\|u\|_{C^{1+\alpha,(1+\alpha)/2}(D_T)}+\|v\|_{C^{1+\alpha,(1+\alpha)/2}(D_T)}\\ &+\|g\|_{C^{1+\alpha/2}([0,T])}+\|h\|_{C^{1+\alpha/2}([0,T])}\le K, \end{aligned}\tag{2.1}$$

where $D_T=\{(t,x)\in\mathbb{R}^2: t\in[0,T], x\in[g(t),h(t)]\}$, $D_T^\infty=\{(t,x): t\in[0,T], x\in\mathbb{R}\}$, K and T only depend on $h_0,\alpha,\|u_0\|_{C^2([-h_0,h_0])}$, $\|v_0\|_{C^2([-h_0,h_0])}$ and $\|v_0\|_{L^\infty(-\infty,\infty)}$.

Proof. As in [20], we first straighten the double free boundary fronts by making the following change of variable:

$$x=\frac{h(t)-g(t)}{2h_0}y+\frac{h(t)+g(t)}{2}.$$

Now, a straightforward computation yields

$$\frac{\partial y}{\partial x}=\frac{2h_0}{h(t)-g(t)},$$

$$\frac{\partial y}{\partial t} = -2h_0 \frac{x(h'(t) - g'(t)) + h(t)g'(t) - h'(t)g(t)}{(h(t) - g(t))^2}.$$

If we set

$$u(t, x) = u(t, \frac{h(t) - g(t)}{2h_0} y + \frac{h(t) + g(t)}{2}) := w(t, y),$$
$$v(t, x) = v(t, \frac{h(t) - g(t)}{2h_0} y + \frac{h(t) + g(t)}{2}) := z(t, y),$$

then

$$u_t = w_t - 2h_0 \frac{x[h'(t) - g'(t)] + h(t)g'(t) - h'(t)g(t)}{[h(t) - g(t)]^2} w_y = w_t - Aw_y,$$
$$v_t = z_t - 2h_0 \frac{x[h'(t) - g'(t)] + h(t)g'(t) - h'(t)g(t)}{[h(t) - g(t)]^2} z_y = z_t - Az_y,$$
$$u_{xx} = Bw_{yy}, \quad v_{xx} = Bz_{yy},$$

where

$$A = A(h, g, y) = \frac{y[h'(t) - g'(t)] + h_0[h'(t) + g'(t)]}{h(t) - g(t)},$$
$$B = B(h, g) = \frac{4h_0^2}{[h(t) - g(t)]^2}.$$

Problem (1.4) can be reduced to

$$\begin{gathered}
w_t = Aw_y + aBw_{yy} + r_1 w(1 - \frac{w}{K_1 + \alpha_1 z}), \quad t > 0,\ -h_0 < y < h_0, \\
z_t = Az_y + bBw_{yy} + r_2 z(1 - \frac{z}{K_2 + \alpha_2 w}), \quad t > 0,\ -\infty < y < \infty, \\
w = 0, \quad h'(t) = -\frac{2h_0\mu}{h(t) - g(t)} \frac{\partial w}{\partial y}, \quad t > 0,\ y \geq h_0, \\
w = 0, \quad g'(t) = -\frac{2h_0\mu}{h(t) - g(t)} \frac{\partial w}{\partial y}, \quad t > 0,\ y \leq -h_0, \\
h(0) = h_0, \quad g(0) = -h_0, \\
w(0, y) = w_0(y) := u_0(y), \quad z(0, y) = z_0(y) := v_0(y), \quad -\infty \leq y \leq \infty.
\end{gathered} \tag{2.2}$$

Now the free boundaries $x = h(t)$ and $x = g(t)$ become the fixed lines $y = h_0$ and $y = -h_0$ respectively, and the equations become more complex, since the coefficients in the first and second equations of (2.2) contain unknown functions $h(t)$, $g(t)$ and their derivatives.

The rest of the proof is by the contraction mapping argument as in [4, 20] with suitable modifications, and we omit the details here. □

To discuss further on (1.4), we need some preliminary theorems which will be used in the sequel. Next we present the monotonicity of the double fronts.

Theorem 2.2. *The two free boundaries for problem* (1.4) *are strictly monotone, namely, for any solution on* $[0, T]$, *we have*

$$h'(t) > 0 \quad \textit{and} \quad g'(t) < 0 \quad \textit{for } 0 \leq t \leq T.$$

Proof. Using the Hopf Lemma to the system of (1.4), we immediately deduce that

$$u_x(t, h(t)) < 0,\ u_x(t, g(t)) > 0 \quad \text{for } 0 \leq t \leq T.$$

Then, combining the above two inequalities with the Stefan conditions in (1.4), the result can be obtained. □

The above theorem indicates that $h(t)$ and $g(t)$ are strictly monotone, and therefore there exists $h_\infty, -g_\infty \in (0, +\infty]$ such that $\lim_{t\to+\infty} h(t) = h_\infty$ and $\lim_{t\to+\infty} g(t) = g_\infty$. Thus, we have four possible cases: (I) $h_\infty = \infty = -g_\infty$, (II) $h_\infty < \infty, g_\infty > -\infty$, (III) $h_\infty < \infty, g_\infty = -\infty$ and (IV) $h_\infty = \infty, g_\infty > -\infty$. The following theorem shows that the last two cases are unlikely to occur. It indicates that both h_∞ and g_∞ are finite or infinite simultaneously.

Theorem 2.3. *Let $(u, v; g, h)$ be a solution of (1.4) in $[0, T_{\max}) \times [g(t), h(t)]$. Then $g(t)$ and $h(t)$ satisfy*

$$-2h_0 < g(t) + h(t) < 2h_0, \quad t \in [0, T_{\max}).$$

Proof. It follows from continuity that $g(t) + h(t) < 2h_0$ for small $t > 0$. Define

$$T := \sup\{s : g(t) + h(t) < 2h_0,\ t \in [0, s)\}.$$

We can deduce that $T = T_{\max}$ in the following proof by contradiction. Suppose that $T < T_{\max}$, Then we have

$$g(t) + h(t) < 2h_0, \quad t \in [0, T), \quad g(T) + h(T) = 2h_0.$$

Hence

$$g'(T) + h'(T) \geq 0. \tag{2.3}$$

To obtain a contradiction, we define the function $\mathcal{F}(t, x) := u(t, x) - u(t, -x + 2h_0)$ on the region

$$\Omega' = \{(t, x) : 0 \leq t \leq T,\ h_0 \leq x \leq h(t)\}.$$

A straightforward computation yields

$$\mathcal{F}_t = \mathcal{F}_{xx} + c(t, x)\mathcal{F}, \quad 0 < t \leq T,\ h_0 < x < h(t),$$

with some $c(t, x) \in L^\infty(\Omega')$ and

$$\mathcal{F}(t, h_0) = 0,\ \mathcal{F}(t, h(t)) < 0,\ 0 < t < T.$$

Moreover,

$$\mathcal{F}(T, h(T)) = u(T, h(T)) - u(T, -h(T) + 2h_0) = u(T, h(T)) - u(T, g(T)) = 0.$$

Then

$$\begin{gathered} \mathcal{F}(t, x) < 0, \quad (t, x) \in (0, T] \times (h_0, h(t)), \\ \mathcal{F}_x(T, h(T)) < 0, \end{gathered}$$

by applying the strong maximum principle and the Hopf Lemma. However

$$\mathcal{F}_x(T, h(T)) = u_x(T, h(T)) + u_x(T, g(T)) = -[g'(T) + h'(T)]/\mu,$$

namely

$$g'(T) + h'(T) > 0,$$

which contradicts (2.1). Therefore $g(t) + h(t) < 2h_0$ for all $0 < t < T_{\max}$. Similarly we can prove $g(t) + h(t) > -2h_0$ for all $0 < t < T_{\max}$. □

Theorem 2.1 implies that there exists a T such that the solution exists in time interval $[0,T]$, and the solution can be further extended to $[0,T_{\max})$ with $T_{\max}\le +\infty$ by Zorn's lemma. The maximal exist time of the solution $T_{\max}$ depends on a prior estimate with respect to $\|u\|_{L^\infty}$, $\|v\|_{L^\infty}$ and $g'(t),h'(t)$. Next we show that if $\|u\|_{L^\infty}<\infty$, then the solution is global. For this purpose we first provide the following lemma.

Lemma 2.4. *Suppose that $\overline{M}:=\|u\|_{L^\infty([0,T]\times[g(t),h(t)])}<\infty$. Then the solution of the free boundary problem (1.4) satisfies*

$$0\le v\le M_2(\overline{M})\quad \text{for } 0\le t\le T,\ -\infty\le x<\infty,$$
$$0<-g'(t),h'(t)\le M_3(\overline{M})\quad \text{for } 0\le t\le T,$$

where M_2,M_3 are independent of T.

Proof. Because of $\overline{M}:=\|u\|_{L^\infty([0,T]\times[g(t),h(t)])}<\infty$, we obtain

$$v_t-bv_{xx}\le r_2v(1-\frac{v}{K_2+\alpha_2\bar{M}})$$

for $0<t\le T$, $-\infty<x<\infty$, then we deduce the estimate for v by the Phragman-Lindelof principle. Set

$$\Omega=\{(t,x):0<t\le T, g(t)<x<g(t)+\frac{1}{M}\}$$

and define an auxiliary function

$$w(t,x)=\overline{M}[2M(x-g(t))-M^2(x-g(t))^2].$$

Next, we choose M such that $w(t,x)$ is the supersolution of $u(t,x)$ in Ω. Directly computations show that

$$w_t=-2\overline{M}Mg'(t)\big[1-M(x-g(t))\big]\ge 0,$$
$$-w_{xx}=2\overline{M}M^2,$$
$$r_1u(1-\frac{u}{K_1+\alpha_1v})\le r_1\overline{M}.$$

If $M^2\ge r_1/(2a)$, we have

$$w_t-aw_{xx}\ge 2a\overline{M}M^2\ge r_1\overline{M}\ge r_1u(1-\frac{u}{K_1+\alpha_1v}).$$

On the other hand,

$$w(t,g(t)+\frac{1}{M})=\overline{M}\ge u(t,g(t)+\frac{1}{M}),$$
$$w(t,g(t))=0=u(t,g(t)).$$

Recalling that $u_0(-h_0)=0$ and $u_0'(-h_0)=-g_1/\mu$ gives that there exists $0<\delta<h_0$ such that $u_0(x)\le\frac{3}{4}\overline{M}$ and $|u_0'(x)|\le|g_1/\mu|+1$ for $x\in[-h_0,-h_0+\delta]$, we then have $w(0,x)\ge u_0(x)$ in $[-h_0,-h_0+\frac{1}{M}]$ if $M\ge\max\{\frac{1}{\delta},\frac{|g_1|/\mu+1}{M_1}\}$. Using the comparison principle yields $u(t,x)\le w(t,x)$ in Ω. Noticing that $u(t,g(t))=w(t,g(t))=0$, we have

$$u_x(t,g(t))\le w_x(t,g(t))=2M\overline{M}.$$

Note that the free boundary condition in (1.4) deduces to

$$0<-g'(t)\le 2\mu M\overline{M}:=M_3,\quad 0<t\le T,$$

where M_3 is independent of T. Analogously, we can define

$$w(t,x) = \overline{M}[2M(h(t)-x) - M^2(h(t)-x)^2]$$

over the region

$$\Omega' = \{(t,x) : 0 < t \le T, h(t) - \frac{1}{M} < x < h(t)\},$$

and derive that $0 < h'(t) \le M_3, \quad 0 < t \le T$. □

Theorem 2.5. *Problem* (1.4) *admits a unique global solution.*

Proof. It follows from the uniqueness that there is a number $T_{\max}$ such that $[0, T_{\max})$ is the maximal time interval in which the solution exists. Next we show that $T_{\max} = \infty$. Arguing indirectly, we assume that $T_{\max} < \infty$. It is easy to see that $(le^{r_1 t}, le^{r_2 t})$ is the upper solution of the (1.4), where

$$l = \max\{\max_{[-h_0,h_0]} u(x,0), \|u(x,0)\|_{L^\infty(-\infty,\infty)}\}.$$

We now fix $M > T_{\max}$. Then $u(x,t) \le le^{r_1 M}$ in $[0, T_{\max}) \times [g(t), h(t)]$. By Lemma 2.4, we can find M_2, M_3 independent of T such that

$$\begin{gathered} 0 \le v \le M_2 \quad \text{for } 0 \le t < T_{\max},\ -\infty \le x < \infty, \\ 0 < -g'(t), h'(t) \le M_3 \quad \text{for } 0 \le t < T_{\max}. \end{gathered}$$

It then follows from the proof of Theorem 2.1 that there exists a $\tau > 0$ depending only on M, M_2 and M_3 such that the solution (1.4) with initial time $T_{\max} - \tau/2$ can be extended uniquely to the time $T_{\max} - \tau/2 + \tau$. But this contradicts the assumption. The proof is complete. □

3. Global bounded solution

To obtain the existence of a global solution, we first derive a priori estimate for the solution of (1.4).

Lemma 3.1. *If $\alpha_1\alpha_2 < 1$, then the solution of the free boundary problem* (1.4) *satisfies*

$$\begin{gathered} 0 < u(t,x) \le C_1 \quad \textit{for } 0 \le t \le T,\ g(t) < x < h(t), \\ 0 \le v(t,x) \le C_2 \quad \textit{for } 0 \le t \le T,\ -\infty < x < \infty, \end{gathered}$$

where C_i is independent of T for $i = 1, 2$.

Proof. Firstly we have that $u > 0$ in $[g(t), h(t)] \times [0, T]$ and $v \ge 0$ in $(-\infty, \infty) \times [0, T]$ provided that solution exists.

Since the solution is classical in $[0, T]$, there exists a $\tilde{K}(T)$ such that $u(t,x) \le \alpha_1 \tilde{K}$ and $v(t,x) \le \tilde{K}$. Next we give the proof for $u(t,x) \le C_1$ and $v(t,x) \le C_2$, where

$$\begin{gathered} C_1 := m\frac{K_1 + K_2\alpha_1}{1 - \alpha_1\alpha_2} > \max_{[-h_0,h_0]} u_0(x), \\ C_2 := m\frac{K_1\alpha_2 + K_2}{1 - \alpha_1\alpha_2} > \|v_0\|_{L^\infty(-\infty,\infty)} \end{gathered}$$

for some $m > 1$.

Because the interval $(-\infty,\infty)$ is unbounded, maximum principle does not apply. Next we prove that for any $l > h_0$,

$$
\begin{gathered}
u(t,x) \le C_1 + \alpha_1 \frac{\tilde{K}[x^2 + 2\max(a,b)t]}{l^2}, \\
v(t,x) \le C_2 + \frac{\tilde{K}[x^2 + 2\max(a,b)t]}{l^2}
\end{gathered}
$$

for $0 \le t \le T$, $-l \le x \le l$. Setting

$$
\begin{gathered}
\overline{u}(t,x) = C_1 + \alpha_1 \frac{\tilde{K}[x^2 + 2\max(a,b)t]}{l^2}, \\
\overline{v}(t,x) = C_2 + \frac{\tilde{K}[x^2 + 2\max(a,b)t]}{l^2},
\end{gathered}
$$

then $(\overline{u},\overline{v})$ satisfies

$$
\begin{gathered}
\overline{u}_t - a\overline{u}_{xx} \ge r_1\overline{u}(1 - \frac{\overline{u}}{K_1 + \alpha_1\overline{v}}), \quad 0 < t \le T,\ -l < x < l, \\
\overline{v}_t - b\overline{v}_{xx} \ge r_2\overline{v}(1 - \frac{\overline{v}}{K_2 + \alpha_2\overline{u}}), \quad 0 < t \le T,\ ; -l < x < l, \\
\overline{u} \ge C_1 + \alpha_1\tilde{K} > u, \quad \overline{v} \ge C_2 + \tilde{K} > v, \quad 0 < t \le T,\ x = \pm l, \\
\overline{u}(0,x) \ge C_1 > u_0(x), \quad -l \le x \le l \\
\overline{v}(0,x) \ge C_2 > v_0(x), \quad -l \le x \le l.
\end{gathered}
$$

It follows that $u \le \overline{u}$ and $v \le \overline{v}$ by using the maximum principle on $[0,T] \times [-l,l]$. Now for any fixed $(t_0,x_0) \in [0,T] \times (-\infty,\infty)$, letting l sufficiently large so that $(t_0,x_0) \in [0,T] \times [-l,l]$, we deduce from the above proof that

$$
\begin{gathered}
u(t_0,x_0) \le \overline{u}(t_0,x_0) = C_1 + \alpha_1 \frac{\tilde{K}[x_0^2 + 2\max(a,b)t_0]}{l^2}, \\
v(t_0,x_0) \le \overline{v}(t_0,x_0) = C_2 + \frac{\tilde{K}[x_0^2 + 2\max(a,b)t_0]}{l^2}.
\end{gathered}
$$

Taking $l \to \infty$ gives the desired estimates. □

Combing Theorem 2.5 with Lemma 3.1 yields the following result.

Theorem 3.2. *If parameters α_1, α_2 in double free boundaries problem* (1.4) *satisfy $\alpha_1\alpha_2 < 1$, then* (1.4) *admits a unique global bounded solution.*

Next we discuss the long-time behavior of the free boundary problem (1.4). We first present the slow solution.

Theorem 3.3. *If $\alpha_1\alpha_2 < 1$ and $h_0 > \frac{\pi}{2}\sqrt{a/r_1}$, the free boundaries of the problem* (1.4) *satisfy $h_\infty = \infty$ and $g_\infty = -\infty$.*

Proof. Combing Lemma 2.4 with Theorem 3.2, we know that the solution is global, $x = g(t)$ is monotonic decreasing and $x = h(t)$ is monotonic increasing. Assuming that $g_\infty > -\infty$ by contradiction, then we have $\lim_{t\to+\infty} g'(t) = 0$.

On the other hand, the condition $1 > a/r_1(\frac{\pi}{2h_0})^2$ implies that $1 > \lambda_1$, where λ_1 denotes the first eigenvalue of the problem

$$
-(a/r_1)\phi'' = \lambda\phi \ \text{ in } \ (-h_0,h_0), \quad \phi(\pm h_0) = 0.
$$

Therefore, for all small $\delta > 0$, the first eigenvalue λ_1^δ of the problem

$$-a\phi'' + \delta\phi' = \lambda r_1 \phi \quad \text{in } (-h_0, h_0), \quad \phi(\pm h_0) = 0$$

satisfies $\lambda_1^\delta < 1$. Fix such a $\delta > 0$ and consider the problem

$$L_\delta \psi = \psi - \frac{\psi^2}{K_1} \quad \text{in } (-h_0, h_0), \quad \psi(\pm h_0) = 0, \tag{3.1}$$

where $L_\delta \psi = -(a\psi'' - \delta\psi')/r_1$. It is well known [2, Proposition 3.3] that (3.1) admits a unique positive solution $\psi = \psi_\delta$. By the moving plane method one easily sees that $\psi(x)$ is symmetric about $x = 0$ with $\psi'(x) > 0$ for $x \in [-h_0, 0)$. Moreover using the comparison principle, we have $\psi < K_1$ in $[-h_0, h_0]$. We now set

$$\mathcal{F}(t,x) = \psi\Big(\frac{-h_0}{g(t)} x\Big),$$

and directly compute

$$\mathcal{F}_t - a\mathcal{F}_{xx} = \frac{h_0 x}{g^2(t)} g'(t)\psi' - a\frac{h_0^2}{g^2(t)}\psi'' = \frac{h_0^2}{g^2(t)}[-a\psi'' + \frac{xg'(t)}{h_0}\psi'].$$

Note that $g'(t) \to 0$ as $t \to +\infty$, we can choose $T_0 > 0$ such that $g'(t) > \delta \frac{h_0}{g_\infty}$ for $t \geq T_0$, then, we obtain $\frac{xg'(t)}{-h_0} \geq -\delta$ for $t \geq T_0$ and $x \in [g(t), 0]$, which leads to

$$\mathcal{F}_t - a\mathcal{F}_{xx} \leq \frac{h_0^2}{g^2(t)}(-a\psi'' + \delta\psi') = \frac{h_0^2}{g^2(t)} r_1 (\psi - \frac{\psi^2}{K_1}).$$

Because of $0 \leq \psi < K_1$ and $\frac{-h_0}{g(t)} \leq 1$, we obtain

$$\mathcal{F}_t - a\mathcal{F}_{xx} \leq r_1(\psi - \frac{\psi^2}{K_1}) = r_1(\mathcal{F} - \frac{\mathcal{F}^2}{K_1}) \quad \text{for } t \geq T_0,\ x \in [g(t), 0].$$

Now we choose $\delta \in (0,1)$ sufficiently small so that $\delta\mathcal{F}(T_0, x) \leq u(T_0, x)$. Then $\underline{u}(t,x) := \delta\mathcal{F}(t,x)$ satisfies

$$\begin{gathered}
\underline{u}_t - a\underline{u}_{xx} \leq r_1(\underline{u} - \frac{\underline{u}^2}{K_1}), \quad t \geq T_0,\ x \in [g(t), 0], \\
\underline{u}(t, g(t)) = 0, \quad \underline{u}_x(t, 0) = 0, \quad t \geq T_0, \\
\underline{u}(T_0, x) \leq u(T_0, x), \quad g(T_0) \leq x \leq 0.
\end{gathered}$$

So we can use the comparison principle to conclude that

$$\underline{u}(t,x) \leq u(t,x) \quad \text{for } t \geq T_0,\ x \in [g(t), 0].$$

It follows that

$$u_x(t, g(t)) \geq \underline{u}_x(t, g(t)) = \delta\frac{h_0}{g(t)}\psi'(h_0) \to \delta\frac{h_0}{g_\infty}\psi'(h_0) > 0,$$

which means that $g'(t) \leq -\mu\delta\frac{h_0}{g_\infty}\psi'(h_0) < 0$. This is a contradiction to the fact that $g'(t) \to 0$ as $t \to \infty$. This contradiction implies that $g_\infty = -\infty$. Likewise, we can set

$$\mathcal{F}(t,x) = \psi\Big(\frac{h_0}{h(t)} x\Big), \quad x \in [0, h(t)]$$

to prove that $h_\infty = +\infty$. □

4. Global fast solution and grow up solution

In this section, we discuss the asymptotic behavior of the solution for the case $\alpha_1\alpha_2 > 1$, which is more complicated than that for the case $\alpha_1\alpha_2 < 1$. At first, we give the grow-up result.

Theorem 4.1. *Assume that $\alpha_1\alpha_2 > 1$, then the solution of (1.4) with any nontrivial nonnegative initial data grows up when h_0 is sufficiently large.*

Proof. We first show that the solution cannot blow up in any finite time. In fact, it has a upper solution $(\overline{u}(t), \overline{v}(t))$ satisfies

$$\begin{gathered}
\overline{u}_t = r_1\overline{u}, \quad \overline{v}_t = r_2\overline{v}, \quad t > 0, \\
\overline{u}(0) = \max_{[-h_0,h_0]} u(x,0) \geq 0, \\
\overline{v}(0) = \max_{[-h_0,h_0]} v(x,0) \geq 0
\end{gathered}$$

and the upper solution cannot blow up in finite time.

To prove the solution of (1.4) grows up, it suffices to compare the free boundary problem with the corresponding problem in the fixed domain:

$$\begin{gathered}
u_t - au_{xx} = r_1u(1 - \frac{u}{K_1 + \alpha_1 v}), \quad t > 0,\ -h_0 < x < h_0, \\
v_t - bv_{xx} = r_2v(1 - \frac{v}{K_2 + \alpha_2 u}), \quad t > 0,\ -h_0 < x < h_0, \\
u(t, -h_0) = v(t, -h_0) = 0, \quad t > 0, \\
u(t, h_0) = v(t, h_0) = 0, \quad t > 0, \\
u(0, x) = u_0(x) \geq 0, \quad -h_0 \leq x \leq h_0, \\
v(0, x) = v_0(x) \geq 0, \quad -h_0 \leq x \leq h_0.
\end{gathered} \tag{4.1}$$

On the other hand, we want to find a lower solution of (4.1) that increases exponentially. Let $(\hat{u}, \hat{v}) = (\delta_1 w, \delta_2 w)$, where $\delta_i (i = 1, 2)$ is some positive constant. Then $(\hat{u}, \hat{v})$ is a lower solution of (4.1) if $(\delta_1 w, \delta_2 w)$ satisfies the relations

$$\begin{gathered}
w_t - aw_{xx} \leq r_1w(1 - \frac{\delta_1 w}{K_1 + \alpha_1\delta_2 w}), \quad t > 0,\ -h_0 < x < h_0, \\
w_t - bw_{xx} \leq r_2w(1 - \frac{\delta_2 w}{K_2 + \alpha_2\delta_1 w}), \quad t > 0,\ -h_0 < x < h_0, \\
w(t, -h_0) = w(t, -h_0) = 0, \quad t > 0, \\
\delta_1 w(0, x) \leq u_0(x), \quad -h_0 \leq x \leq h_0, \\
\delta_2 w(0, x) \leq v_0(x), \quad -h_0 \leq x \leq h_0.
\end{gathered} \tag{4.2}$$

Then (4.2) holds if

$$\begin{gathered}
w_t - aw_{xx} \leq \frac{r_1(\alpha_1\delta_2 - \delta_1)w}{\alpha_1\delta_2}, \quad t > 0,\ -h_0 < x < h_0, \\
w_t - bw_{xx} \leq \frac{r_2(\alpha_2\delta_1 - \delta_2)w}{\alpha_2\delta_1}, \quad t > 0,\ -h_0 < x < h_0, \\
w(t, -h_0) = w(t, -h_0) = 0, \quad t > 0, \\
\delta_1 w(0, x) \leq u_0(x), \quad -h_0 \leq x \leq h_0, \\
\delta_1 w(0, x) \leq v_0(x), \quad -h_0 \leq x \leq h_0.
\end{gathered} \tag{4.3}$$

Recall the assumption in the theorem, let $\delta_i > 0$ such that

$$\frac{1}{\alpha_2} < \frac{\delta_1}{\delta_2} < \alpha_1$$

and set

$$D = \max\{\frac{1}{a}, \frac{1}{b}\}, \quad d^* = \min\big\{\frac{r_1(\alpha_1\delta_2 - \delta_1)}{\alpha_1\delta_2 a}, \frac{r_2(\alpha_2\delta_1 - \delta_2)}{\alpha_2\delta_1 b}\big\}.$$

Then $d^* > 0$ and thus (4.3) holds if

$$\begin{gathered} Dw_t - w_{xx} \le d^* w, \quad t > 0,\ -h_0 < x < h_0, \\ w(t, -h_0) = w(t, h_0) = 0, \quad t > 0, \\ w_0(x) \le \min\big\{\frac{u_0(x)}{\delta_1}, \frac{v_0(x)}{\delta_2}\big\}, \quad -h_0 \le x \le h_0. \end{gathered} \tag{4.4}$$

Let $w(x,t) = \delta e^{\varepsilon t}\cos(\frac{\pi}{2h_0}x)$. Direct calculations show that if $h_0 > \frac{\pi}{2}\frac{1}{\sqrt{d^*}}$, then we can choose small δ and ε such that $w_t \ge 0$ and (4.4) holds. Therefore the lower solution $(\hat{u}, \hat{v})$ increases exponentially, so does the solution of (1.4). $\square$

Next we introduce a comparison principle for double free boundaries $x = h(t)$ and $x = g(t)$, which can be proved similarly as [4, Lemma 3.5].

Lemma 4.2. *Suppose that* $T \in (0,\infty)$, $\overline{h}, \overline{g} \in C^1([0,T])$, $\overline{u} \in C(\overline{D}^*_{1,T}) \cap C^{1,2}(D^*_{1,T})$ *and* $\overline{v} \in C(\overline{D}^*_{2,T}) \cap C^{1,2}(D^*_{2,T})$ *with* $D^*_{1,T} = (0,T] \times (\overline{g}(t), \overline{h}(t))$, $D^*_{2,T} = (0,T] \times (-\infty, +\infty)$, *and*

$$\begin{gathered} \overline{u}_t - a\overline{u}_{xx} \ge \overline{r}_1 u(1 - \frac{\overline{u}}{K_1 + \alpha_1\overline{v}}), \quad t > 0,\ \overline{g}(t) < x < \overline{h}(t), \\ \overline{v}_t - b\overline{v}_{xx} \ge \overline{r}_2 v(1 - \frac{\overline{v}}{K_2 + \alpha_2\overline{u}}), \quad t > 0,\ -\infty < x < \infty, \\ \overline{u}(t,x) = 0, \quad t > 0,\ -\infty < x < g(t), \\ \overline{u}(t,x) = 0, \quad t > 0,\ h(t) < x < \infty, \\ \overline{u} = 0, \quad \overline{h}'(t) \ge -\mu\frac{\partial \overline{u}}{\partial x}, \quad t > 0,\ x = \overline{h}(t), \\ \overline{u} = 0, \quad \overline{g}'(t) \le -\mu\frac{\partial \overline{u}}{\partial x}, \quad t > 0,\ x = \overline{g}(t). \end{gathered} \tag{4.5}$$

If $-h_0 \ge \overline{g}(0), h_0 \le \overline{h}(0)$, $u_0(x) \le \overline{u}(0,x)$ *in* $[-h_0, h_0]$ *and* $v_0(x) \le \overline{v}(0,x)$ *in* $(-\infty, +\infty)$, *then the solution* $(u, v; g, h)$ *of the free boundary problem* (1.4) *satisfies*

$$\begin{gathered} g(t) \ge \overline{g}(t), \quad h(t) \le \overline{h}(t) \quad \text{in } (0,T], \\ u(t,x) \le \overline{u}(t,x) \quad \text{in } [0,T] \times (g(t), h(t)), \\ v(t,x) \le \overline{v}(t,x) \quad \text{in } [0,T] \times (-\infty, +\infty). \end{gathered}$$

Remark 4.3. The $(\overline{u}, \overline{v}; \overline{h}, \overline{g})$ in Lemma 4.2 is usually called an upper solution of the problem (1.4). We can define a lower solution by reversing all the inequalities in the obvious places. Moreover, one can easily prove an analogue of Lemma 4.2 for lower solutions.

In the following theorem, we show existence of a global fast solution.

Theorem 4.4. *If $\alpha_1\alpha_2 > 1$, then the free boundary problem (1.4) admits a global fast solution provided that the initial data u_0 and h_0 are suitably small. Moreover, there exist constant $\beta = r_1/2$ and $\eta = \eta(h_0, K_2, a, \mu, \alpha_2)$ such that*

$$\|u\|_\infty \le \eta e^{-\beta t}, \quad t \ge 0.$$

Proof. As in [16], we have only to find a suitable supersolution. For $t \ge 0$, define

$$\sigma(t) = 2h_0(2 - e^{-\gamma t}), \quad \lambda(t) = -\sigma(t), \quad \mathcal{F}(y) = \cos(\frac{\pi}{2}y), \quad -1 \le y \le 1,$$
$$\overline{u}(t,x) = \eta e^{-\beta t}\mathcal{F}(x/\sigma(t)), \quad t \ge 0, \quad \lambda(t) \le x \le \sigma(t),$$
$$\overline{v}(t,x) = \max\{2K_2, \|v_0(x)\|_{L^\infty(-\infty,+\infty)}\}, \quad t \ge 0, \ -\infty \le x \le \infty,$$

where γ, β and $\eta > 0$ to be determined later.

Straightforward calculations yield

$$\begin{aligned}
&\overline{u}_t - a\overline{u}_{xx} - r_1\overline{u}(1 - \frac{\overline{u}}{K_1 + \alpha_1\overline{v}}) \\
&= \eta e^{-\beta t}[-\beta\mathcal{F} - x\sigma'\sigma^{-2}\mathcal{F}' - a\sigma^{-2}\mathcal{F}'' - r_1\mathcal{F}(1 - \frac{\eta e^{-\beta t}\mathcal{F}}{K_1 + \alpha_1\overline{v}})] \\
&\ge \eta e^{-\beta t}\mathcal{F}[-\beta + (\frac{\pi}{2})^2\frac{a}{16h_0^2} - r_1]
\end{aligned}$$

for all $t > 0$ and $\lambda(t) < x < \sigma(t)$ and

$$\begin{aligned}
\overline{v}_t - b\overline{v}_{xx} - r_2\overline{v}(1 - \frac{\overline{v}}{K_2 + \alpha_2\overline{u}}) &= \overline{v}(-r_2 + \frac{r_2\overline{v}}{K_2 + \alpha_2\eta e^{-\beta t}\mathcal{F}}) \\
&\ge 2r_2K_2(-1 + \frac{2K_2}{K_2 + \alpha_2\eta})
\end{aligned}$$

for all $t > 0$ and $-\infty < x < \infty$. On the other hand, we can easily deduce $\sigma'(t) = 2\gamma h_0 e^{-\gamma t} > 0$, $-\overline{u}_x(t,\sigma(t)) = \frac{\pi}{2}\eta\sigma^{-1}(t)e^{-\beta t}$ and $-\overline{u}_x(t,\lambda(t)) = \frac{\pi}{2}\eta\lambda^{-1}(t)e^{-\beta t}$. Now we set

$$h = \frac{\pi}{16}\sqrt{\frac{2a}{r_1}},$$

choosing

$$0 < h_0 \le h, \quad \eta = \min\{\frac{K_2}{\alpha_2}, \ \frac{a\pi}{8\mu}(\frac{h_0}{2h})^2\}, \quad \beta = \gamma = (\frac{\pi}{2})^2\frac{a}{64h^2} = \frac{r_1}{2}.$$

It follows that

$$\begin{gathered}
\overline{u}_t - a\overline{u}_{xx} \ge r_1\overline{u}(1 - \frac{\overline{u}}{K_1 + \alpha_1\overline{v}}), \quad t > 0, \ \lambda(t) < x < \sigma(t), \\
\overline{v}_t - b\overline{v}_{xx} \ge r_2\overline{v}(1 - \frac{\overline{v}}{K_2 + \alpha_2\overline{u}}), \quad t > 0, \ -\infty < x < \infty, \\
\overline{u} = 0, \quad \sigma'(t) > -\mu\frac{\partial\overline{u}}{\partial x}, \quad t > 0, \ x = \sigma(t), \\
\overline{u} = 0, \quad \lambda'(t) < -\mu\frac{\partial\overline{u}}{\partial x}, \quad t > 0, \ x = \lambda(t), \\
\sigma(0) = 2h_0 > h_0, \lambda(0) = -2h_0 < -h_0.
\end{gathered}$$

Using Lemma 4.2, we can get that $h(t) < \sigma(t)$, $g(t) > \lambda(t)$, and $u(t,x) < \overline{u}(t,x)$, $v(t,x) < \overline{v}(t,x)$ for $g(t) \le x \le h(t)$ provided (u,v) exists. Therefore (u,v) exists globally and $g_\infty > -\infty$, $h_\infty < \infty$. $\square$

Remark 4.5. If $\alpha_1\alpha_2 < 1$, Theorem 3.3 shows that the solution is slow for any initial data. If $\alpha_1\alpha_2 > 1$, Theorem 4.1 shows that the solution grows up for h_0 is sufficiently large. Theorem 4.4 implies that the global fast solution is possible if the initial data and h_0 is suitably small.

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