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EXACT SOLUTIONS OF A NONCONSERVATIVE SYSTEM IN ELASTODYNAMICS

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Abstract. In this article we find an explicit formula for solutions of a nonconservative system when the initial data lies in the level set of one of the Riemann invariants. Also for nonconservative shock waves in the sense of Volpert we derive an explicit formula for the viscous shock profile.

1. Introduction

One of the systems of equations that comes in modelling propagation of elastic waves, is the nonconservative system

$$\begin{gathered} u_t + uu_x - \sigma_x = 0, \\ \sigma_t + u\sigma_x - k^2 u_x = 0, \end{gathered} \tag{1.1}$$

which was introduced in [4]. Here u is the velocity, σ is the stress and $k > 0$ is the speed of propagation of the elastic waves. The system (1.1) is strictly hyperbolic with characteristic speeds

$$\lambda_1(u,\sigma) = u - k, \quad \lambda_2(u,\sigma) = u + k \tag{1.2}$$

and corresponding Riemann invariants

$$w_1(u,\sigma) = \sigma - ku, \quad w_2(u,\sigma) = \sigma + ku. \tag{1.3}$$

It is well known that smooth global in time solutions do not exist even if the initial data is smooth, then the term $u\sigma_x$ appearing in equations, does not make sense in the theory of distributions, and classical theory of Lax [11] does not work. There are many approaches starting with Volpert [14], and subsequently by Colombeau [1, 3, 4], Dal Maso, LeFloch and Murat [5] and LeFloch and Tzavaras [12] to define such products. They are not equivalent but are related and have some common features.

They consider systems of N equations of the form

$$U_t + A(U)U_x = 0, \tag{1.4}$$

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where $A(U)U_x$, is not in conservative form $F(U)_x$. Here $A(U)$ an $N \times N$ matrix, depending smoothly on $U \in \Omega$, and Ω is an open connected set in $\mathbb{R}^N$. Assume that U has a discontinuity along $x = st$ and of the form

$$U(x,t) = \begin{cases} U_-, & \text{if } x < st, \\ U_+, & \text{if } x > st. \end{cases} \tag{1.5}$$

where U_- and U_+ are constant vectors in Ω. Volpert [14] defined $A(U)U_x$ as a measure

$$A(U)U_x = \frac{1}{2}(A(U_+ + A(U_-)(U_+ - U_-)\delta_{x=st}. \tag{1.6}$$

As this definition is inadequate for many applications, Dal Maso,LeFloch, Murat [5] generalized this definition by

$$A(U(x,t))U_x(x,t) = \Big(\int_0^1 A(\phi(s,U_-,U_+))\partial_s\phi(s;U_-,U_+)\Big)\delta_{x=st} \tag{1.7}$$

where ϕ is a family of Lipschitz paths, $\phi : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$, with $\phi(0,U_-,U_+) = U_-$ and $\phi(1,U_-,U_+) = U_+$, with some natural conditions. Volpert product corresponds to taking ϕ the straight line path connecting U_- and U_+. Further they solved Riemann problem for (1.4) with Riemann data

$$U(x,t) = \begin{cases} U_-, & \text{if } x < 0, \\ U_+,, & \text{if } x > 0. \end{cases} \tag{1.8}$$

when the system is strictly hyperbolic and $|U_+ - U_-|$ is small. Choudhury [2] has recently shown that Riemann problem for (1.1) with $k = 0$, in which case the system is not strictly hyperbolic, do have a solution in the class of shock waves and rarefaction waves if one uses the product in [5], with special choice of paths but not for straight line paths. This example shows advantages of the product in [5] over the Volpert product. Different paths give different solutions. So as pointed out in [5, 12, 13] any discussion of well-posedness of solution for nonconservative systems, should be based on a given nonconservative product in addition to admissibility criterion for shock discontinuities. As the system of the type (1.4), is an approximation and is obtained when one ignores higher order derivative terms, which give smoothing effects with small parameters as coefficients . So a natural way to construct the physical solution is, by the limit of a given regularization as these small parameters goes to zero. Different regularizations correspond to different nonconservative product and admissibility condition, see [5, 12] and the references there more details.

Another method to handle the nonconservative product is using Colombeau algebra. Initial value problem for the system (1.1) was solved in this space first in [1, 4] using numerical approximation for a restricted class of initial data. More general class of initial data including the L^∞ space was treated in [7] by parabolic approximations with out any conditions on the smallness of data. Dafermos regularization and the approach of [12] was used [9, 10] to study Riemann problem.

In this paper we take a parabolic regularization and explain its connection with the Volpert nonconservative product and Lax admissibility conditions. Also we give explicit formula for the solution when the initial data lie in the level set of one of the Riemann invariants of the system (1.1).

2. Viscous shocks profile for Volpert shock

First we recall some known facts about the Riemann problem for (1.1). Here the initial data takes the form

$$(u(x,0),\sigma(x,0)) = \begin{cases} (u_-,\sigma_-), & \text{if } x<0,\\ (u_+,\sigma_+), & \text{if } x>0. \end{cases} \tag{2.1}$$

A shock wave is a weak solution of (1.1), with speed s is of the form

$$(u(x,t),\sigma(x,0)) = \begin{cases} (u_-,\sigma_-), & \text{if } x<st,\\ (u_+,\sigma_+), & \text{if } x>st. \end{cases} \tag{2.2}$$

When Volpert product is used the Rankine Hugoniot condition takes the form

$$\begin{gathered} -s(u_+-u_-)+\frac{u_+^2-u_-^2}{2}-(\sigma_+-\sigma_-)=0\\ -s(\sigma_+-\sigma_-)+\frac{u_++u_-}{2}(\sigma_+-\sigma_-)-k^2(u_+-u_-) \end{gathered} \tag{2.3}$$

In [8], it was shown that the Riemann problem can be solved without any smallness assumptions on the Riemann data when the nonconservative product is understood in the sense of Volpert [14] with Lax's admissibility conditions for shock speed. Indeed, corresponding to each characteristic family $\lambda_j, j=1,2$ we can define shock waves and rarefaction waves. Fix a state (u_-,σ_-) the set of states (u_+,σ_+) which can be connected by a single j-shock wave is a straight line called j-shock curve and is denoted by S_j and the states which can be connected by a single j-rarefaction wave is a straight line is called j-rarefaction curve and is denoted by R_j. These wave curves are given by

$$\begin{aligned} R_1(u_-,\sigma_-) &: \sigma=\sigma_-+k(u-u_-), u>u_-\\ S_1(u_-,\sigma_-) &: \sigma=\sigma_-+k(u-u_-), u<u_-\\ R_2(u_-,\sigma_-) &: \sigma=\sigma_--k(u-u_-), u>u_-\\ S_2(u_-,\sigma_-) &: \sigma=\sigma_--k(u-u_-), u<u_-. \end{aligned} \tag{2.4}$$

Further j-shock speed s_j is given by

$$s_j=\frac{u_++u_-}{2}+(-1)^jk, \quad j=1,2 \tag{2.5}$$

and the Lax entropy condition requires that the j- shock speed satisfies inequality

$$\lambda_j(u_+,\sigma_+)\le s_j\le \lambda_j(u_-,\sigma_-). \tag{2.6}$$

This curves fill in the $u-\sigma$ plane and the Riemann problem can be solved uniquely for arbitrary initial states (u_-,σ_-) and (u_+,σ_+) in the class of self similar functions consisting of solutions of shock waves and rarefaction waves separated by constant states. These constant states are obtained from the shock curves and rarefaction curves corresponding to the two families of the characteristic fields.

The nonconservative product and the selection criteria is associated with a regularization. In this paper we analyze shock wave solution of (1.1) with respect to the Volpert product and Lax's shock inequalities and its relation to parabolic approximation. To analyze this connection, first we ask the question, does there exists a traveling wave profile solution $(u(\xi),\sigma(\xi))$ with $\xi=\frac{x-s_j t}{\epsilon}$, of the system the corresponding parabolic approximation

$$\begin{aligned} u_t+uu_x-\sigma_x&=\epsilon u_{xx},\\ \sigma_t+u\sigma_x-k^2u_x&=\epsilon\sigma_{xx}. \end{aligned} \tag{2.7}$$

connecting (u_-,σ_-) to (u_+,σ_+) when (u_+,σ_+) lies on the shock curve $S_j, j=1,2$ passing through (u_-,σ_-).

This amounts to solving the boundary-value problem, for a system of nonlinear ordinary differential equations,

$$-s_j\frac{du}{d\xi}+u\frac{du}{d\xi}-\frac{d\sigma}{d\xi}=\frac{d^2u}{d\xi^2},\quad -s_j\frac{d\sigma}{d\xi}+u\frac{d\sigma}{d\xi}-k^2\frac{du}{d\xi}=\frac{d^2\sigma}{d\xi^2} \tag{2.8}$$

for $\xi\in(-\infty,\infty)$ with boundary conditions

$$u(-\infty)=u_-,\quad u(\infty)=u_+,\quad \sigma(-\infty)=\sigma_-,\quad \sigma(\infty)=\sigma_+. \tag{2.9}$$

Theorem 2.1. *If $(u_+,\sigma_+)\in S_j(u_-,\sigma_-)$, then the viscous shock profile $(u(\xi),\sigma(\xi)$ of (2.8)-(2.9) with speed $s_j=\frac{u_++u_-}{2}+(-1)^jk$ is given by*

$$(u(\xi),\sigma(\xi))=(\phi(\xi),(-1)^{j+1}k\phi(\xi)+\sigma_-+(-1)^jku_-), \tag{2.10}$$

where

$$\phi(\xi)=u_++\frac{(u_--u_+)}{1+\exp(\frac{u_--u_+}{2}\xi-\xi_0)}, \tag{2.11}$$

where ξ_0 is a constant.

Proof. Writing (2.8) in Riemann invariants

$$\begin{aligned} \frac{d^2w_1}{d\xi^2}&=(\lambda_1(r,s)-s)\frac{dw_1}{d\xi},\\ \frac{d^2w_2}{d\xi^2}&=(\lambda_2(r,s)-s)\frac{dw_2}{d\xi}. \end{aligned} \tag{2.12}$$

It follows from the uniqueness of solutions of ODE, any solution w_j of (2.12) with first derivative is zero at a point ξ_0 must be a constant equal to $w_j(\xi_0)$. So either solutions w_j are constants or strictly monotone. Suppose $u_+\in S_1(u_-)$, then $w_1(u_-,\sigma_-)=w_1(u_+,\sigma_+)$ and so $w_1(u(\xi),\sigma(\xi))$ is a constant. Thus u and σ are related by

$$\sigma(\xi)=ku+(\sigma_--ku_-). \tag{2.13}$$

Substituting this relation in any one of the equations in (2.8), we get the same single equation for u. Indeed for 1-shock, u is given by the equation

$$-s_1u'+uu'-ku'=u",\quad u(-\infty)=u_-,u(\infty)=u_+.$$

Once u is known, σ is obtained from (2.13). In terms of the new variable $v=u-k$, this problem reduces to

$$-s_1v'+vv'=v",\quad v(-\infty)=u_--k,u(\infty)=u_+-k,$$

whose solution is

$$v(\xi)=u_++\frac{(u_--u_+)}{1+\exp(\frac{u_--u_+}{2}\xi-\xi_0)}-k.$$

Since $u = v + k$, and $\sigma(\xi) = ku + (\sigma_- - ku_-)$, we have traveling wave corresponding to 1-shock wave is given by

$$u(\xi) = u_+ + \frac{(u_- - u_+)}{1 + \exp(\frac{u_- - u_+}{2}\xi - \xi_0)}, \quad \sigma(\xi) = ku(\xi) + (\sigma_- - ku_-)$$

where $\xi = \frac{x - s_1 t}{\epsilon}$, and the formula (2.10)-(2.11) follows for the case $j = 1$. The analysis for the case $j = 2$ is similar and is omitted. □

3. Explicit formula for initial data lying on level sets of Riemann invariants

Now we consider initial value problem for the viscous system

$$\begin{aligned} u_t + uu_x - \sigma_x &= \epsilon u_{xx}, \\ \sigma_t + u\sigma_x - k^2 u_x &= \epsilon \sigma_{xx}. \end{aligned} \tag{3.1}$$

with initial data

$$u(x,0) = u_0(x), \quad \sigma_0(x,0) = \sigma_0(x) \tag{3.2}$$

When $u_0(x)$ and $\sigma_0(x)$ are functions of bounded variation and continuously differentiable, existence of classical solution satisfying initial data was shown in [7]. Additionally if we assume $(u_0(x), \sigma_0(x))$ lies on the level set on one of the j-Riemann invariants, $w_j(u,\sigma) = \sigma + (-1)^j ku$, $j = 1, 2$, the system can be reduced to the Burgers equation and an explicit formula can be derived for the corresponding initial value problem.

Theorem 3.1. *Assume that the initial data (u_0, σ_0) is function of bounded variation and there exists c a constant such that $w_j(u_0(x), \sigma_0(x)) = c$ for all x, for fixed $j = 1$ or $j = 2$.*
(a). Then the viscous system (3.1)-(3.2), has a solution of the form

$$\begin{aligned} u^\epsilon(x,t) &= \frac{\int_{-\infty}^{\infty} \frac{x-y}{t} e^{-\frac{1}{2\epsilon}\theta(x,y,t)dy}}{\int_{-\infty}^{\infty} e^{-\frac{1}{2\epsilon}\theta(x,y,t)}dy} + (-1)^{j+1}k \\ \sigma^\epsilon(x,t) &= (-1)^{j+1}k[\frac{\int_{-\infty}^{\infty} \frac{x-y}{t} e^{-\frac{1}{2\epsilon}\theta(x,y,t)}}{\int_{-\infty}^{\infty} e^{\theta(x,y,t)}} + (-1)^j k] + c, \end{aligned} \tag{3.3}$$

where

$$\theta(x,y,t) = \frac{(x - y - (-1)^j kt)^2}{2t} + \int_0^y u_0(z)dz. \tag{3.4}$$

(b). For each fixed $t > 0$, except on a countable points $x \in \mathbb{R}^1$, there exits a unique minimizer $y(x,t)$ for

$$\min_{y\in\mathbb{R}^1}[\frac{(x - y - (-1)^j kt)^2}{2t} + \int_0^y u_0(z)dz]. \tag{3.5}$$

At these points the point wise limit $\lim_{\epsilon\to 0}(u^\epsilon(x,t), \sigma^\epsilon(x,t)) = (u(x,t), \sigma(x,t))$ exits and is given by

$$\begin{aligned} u(x,t) &= (-1)^{j+1}k + \frac{(x - y(x,t))}{t}, \\ \sigma(x,t) &= (-1)^{j+1}k[(-1)^j k + \frac{(x - y(x,t))}{t}] + c. \end{aligned} \tag{3.6}$$

Further (u, σ) given by (3.6) is a weak solution to (1.1) with initial condition (3.2).

Proof. Since the initial data in the level set of j-Riemann invariant, we seek a solution lying in the same invariant set. So we seek (u, σ) satisfying

$$\sigma = (-1)^{j+1} k u + c. \tag{3.7}$$

The an easy computation shows that the system become a single Burgers equation for u,

$$u_t + u u_x - (-1)^{j+1} k u_x = \epsilon u_{xx}$$

Once u, is known then formula for σ follows. To find u we make a substitution

$$v = u - (-1)^{j+} k \tag{3.8}$$

and then the equation for v can be written as

$$v_t + v v_x = \epsilon v_{xx}$$

with initial conditions

$$v(x, 0) = u_0(x) - (-1)^{j+1} k.$$

Applying Hopf-Cole transformation [6]

$$v = -2\epsilon \frac{w_x}{w} \tag{3.9}$$

the problem is reduced to

$$w_t = \epsilon w_{xx}$$

with initial conditions

$$w(x, 0) = e^{\frac{-1}{2\epsilon}(\int_0^x u_0(z)dz - (-1)^{j+1} kx)}.$$

Solving this system, we get

$$w(x,t) = \frac{1}{(4\pi t\epsilon)^{1/2}} \int_{-\infty}^{\infty} e^{\frac{-1}{2\epsilon}[\frac{(x-y)^2}{2t} + \int_0^y u_0(z)dz - (-1)^{j+1} ky]} dy. \tag{3.10}$$

An easy computation shows that

$$w_x(x,t) = \frac{-1}{2\epsilon} . \frac{1}{(4\pi t\epsilon)^{1/2}} \int_{-\infty}^{\infty} \frac{(x-y)}{t} e^{\frac{-1}{2\epsilon}[\frac{(x-y)^2}{2t} + \int_0^y u_0(z)dz - (-1)^{j+1} ky]} dy. \tag{3.11}$$

Notice that

$$(x-y)^2 - (-1)^{j+1} 2tky = (x - y - (-1)^j kt)^2 + (-1)^j 2tkx - t^2 k^2. \tag{3.12}$$

Using (3.12) in (3.10) and (3.11), substituting the resulting expressions in (3.9), and using $u = v + (-1)^{j+1} k$, from (3.8) we get the formula for u in (3.3). Then the formula for σ is obtained from the relation (3.7).

The formula for vanishing viscosity limit follows from analysis of Hopf [6] and Lax [11]. Indeed for each fixed (x, t), there is at least one minimizer for (3.5). There may be many minimizers, take $y(x, t)_-$ is the smallest such minimizer and $y(x, t)_+$ is the largest one. Hopf has proved that, for each fixed $t > 0$, $y(x, t)_\pm$ is a nondecreasing function of x and so has at most countable points of discontinuities and except these points, these minimizer is unique and $y(x, t) = y(x, t)_- = y(x, t)_+$. Then formula (3.6) holds at these points (x, t).

Now to show that the limit satisfies (1.1), we just notice that

$$\begin{aligned} u_t + u u_x - \sigma_x - \epsilon u_{xx} &= u_t + \frac{(u^2)_x}{2} - (-1)^{j+1} k u_x - \epsilon u_{xx} \\ \sigma_t + u\sigma_x - k^2 u_x - \epsilon \sigma_{xx} &= (-1)^{j+1} k [u_t + \frac{(u^2)_x}{2} - (-1)^{j+1} k u_x - \epsilon u_{xx}]. \end{aligned} \tag{3.13}$$

which is conservative, and by standard theory of conservation laws works [6, 11], and we can pass to the limit in the equation in the weak sense. Also from [6, 11] it follows that the solution satisfies the initial data in weak sense. □

In the above theorem the solution of the inviscid system (1.1), that we have constructed lie in the level set of a Riemann invariant. Assume that the solution is on the j-Riemann invariant. Then σ and u are related by (3.7) and then $u\sigma_x = (-1)^{j+1}k(\frac{u^2}{2})_x$, a conservative product. A computation as in (3.13) show that the system (1.1) becomes a single equation in conservation form for u, namely

$$u_t + (\frac{(u^2)}{2} - (-1)^{j+1}ku)_x = 0.$$

Then all paths give the same Rankine- Hugoniot conditions for the shocks, see [5].

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