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A SYSTEM OF SCHRÖDINGER EQUATIONS AND THE OSCILLATOR REPRESENTATION

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Abstract. We construct a copy of the oscillator representation of the metaplectic group $Mp(n)$ in the space of solutions to a system of Schrödinger type equations on $\mathbb{R}^n \times \operatorname{Sym}(n, \mathbb{R})$ that has very simple intertwining maps to the realizations given by Kashiwara and Vergne.

1. Introduction

Generalizing results from [23, 24] and using techniques similar to those found in [16], this paper uses Lie symmetry analysis to study the system of partial differential equations

$$\begin{aligned} 4s\partial_{t_{ii}} f(x,t) + \partial_{x_i}^2 f(x,t) = 0, \quad 1 \le i \le n, \\ 2s\partial_{t_{ij}} f(x,t) + \partial_{x_i}\partial_{x_j} f(x,t) = 0, \quad 1 \le i < j \le n, \end{aligned} \tag{1.1}$$

with $s \in i\mathbb{R}^\times$. Here $x = (x_i)$ and $t = (t_{ij})$ are the standard coordinates on $\mathbb{R}^n$ and the space of real symmetric matrices $\operatorname{Sym}(n, \mathbb{R})$, respectively. A brief statement of some of the main results contained in this paper, without proofs, can be found in [15].

A standard application of Lie's prolongation method shows that the infinitesimal symmetries of Equation (1.1) are the Jacobi Lie algebra $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R}) \ltimes \mathfrak{h}_{2n+1}$, where $\mathfrak{sp}(n, \mathbb{R})$ is the symplectic Lie algebra on $\mathbb{R}^{2n}$ and $\mathfrak{h}_{2n+1}$ is the $(2n+1)$-dimensional Heisenberg Lie algebra, plus an infinite dimensional Lie algebra reflecting the fact that Equation (1.1) is linear. It follows that the space of all complex-valued functions $f \in \mathcal{C}^\infty(\mathbb{R}^n \times \operatorname{Sym}(n, \mathbb{R}))$ satisfying (1.1) carries a representation of $\mathfrak{g}$.

While the $\mathfrak{g}$-action on $\mathcal{C}^\infty(\mathbb{R}^n \times \operatorname{Sym}(n, \mathbb{R}))$ does not exponentiate to a global action of the Jacobi group $G^J = Sp(n, \mathbb{R}) \ltimes H_{2n+1}$ or any cover group, we construct canonical $\mathfrak{g}$-invariant subspaces $I'(q, r, s) \subseteq \mathcal{C}^\infty(\mathbb{R}^n \times \operatorname{Sym}(n, \mathbb{R}))$ such that the $\mathfrak{g}$-action on $I'(q, r, s)$ does exponentiate to a global action of the group $G = Mp(n) \ltimes H_{2n+1}$, where $Mp(n)$ is the metaplectic group, i.e., the double cover of $Sp(n, \mathbb{R})$. We then show that the space of solutions to (1.1) in $I'(q, r, s)$ gives a realization of the oscillator representation (or its dual, depending on the sign of σ where $s = i\sigma$) of $Mp(n)$. In addition, we construct very simple intertwining maps to two realizations of the oscillator representation given by Kashiwara and Vergne in

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[18]. One intertwining map is given by evaluation at $t = 0$ (followed by a Fourier transform) and the other is given by either evaluation at $x = 0$ or application of a gradient and then evaluation at $x = 0$.

For a thorough development of the history of the oscillator representation, ω, often called the metaplectic or Segal-Shale-Weil representation, we refer the reader to [4]. In this subsection, we content ourselves by reproducing some of the highlights as we gave them in [15]:

From classical number theory, the invariance properties of Jacobi theta functions [9] are found by lifting such functions to G^J. This lift, in turn, utilizes the oscillator representation [5]. A complete treatment of theta functions appears in [17] and many more results demonstrating the interplay between ω and aspects of number theory can be found in [19, 20, 29].

The quantization procedure in theoretical physics associates classical geometric systems to quantum mechanical systems and is very well studied [1, 12, 26, 27, 28, 30]. For example, the oscillator representation arises in quantum mechanics when one quantizes a single particle structure [22]. The representation ω is constructed and then used to establish results about the inducibility of a field automorphism by a unitary operator in all quantizations [25]. Another application of ω appears in quantum optics. In [2], the tensor product of ω with discrete series representations of $SU(1,1)$ admits squeezed coherent states. The broader role that ω plays in physics can be found in [7, 11].

In representation theory, the oscillator representation is used to construct other important representations. For instance, the representations of G^J $(n = 1)$ with nontrivial central character are realized as products of representations of $Mp(1)$ and the oscillator representation [5]. In the well-known article [18], the k-fold tensor product $\otimes_k\omega$ is decomposed into irreducible unitary representations. First conjectured by Kashiwara and Vergne and later proved by Enright and Parthasarathy [8], all irreducible unitary highest weight representations for which the Verma module $N(\lambda + \rho)$ is reducible (i.e., λ is a reduction point) are found in $\otimes_k\omega$ for some k. In a similar vein, it is shown in [13] that every genuine discrete series representation of $Mp(n)$ appears in $(\otimes_k\omega) \otimes (\otimes_m\omega^*)$, for some k and m. Finally, if F is a finite field, irreducible representations of $GL(2, F)$ can be constructed by using the Weil representation [6], the restriction of ω to $SL(2, F)$. For F a non-Archimedean local field, the same is true of many supercuspidal irreducible representations of $GL(2, F)$.

Given the manifold applications of ω, it may be helpful to identify some canonical realizations. A standard realization of ω arises via the Stone-von Neumann theorem as an intertwining operator between equivalent irreducible unitary representations of H_{2n+1} on $L^2(R^n)$ ([10] and, in more generality, [29]). A second realization is the Fock model, where ω is realized as an integral operator on a reproducing kernel space. Motivated by Lie's prolongation method ([21]), we induce from a subgroup of G and use a system of Schrödinger type equations to find a subspace on which the action irreducible. In [3], a reproducing space of holomorphic functions on $Sp(n, \mathbb{R})/U(n) \times U(n)$ is shown to satisfy analogous differential equations (if one replaces real with complex differentiation), but no unitary action on that space is provided.

Now we turn to a more careful description of the results contained in this paper. For a certain analogue of a parabolic subalgebra $\overline{P}$ of G (see §2.2), we begin with the induced representations

$$I(q,r,s) = \mathrm{Ind}_{\overline{P}}^{G}\, \chi_{q,r,s}$$

(see §2.3) where $\chi_{q,r,s} : \overline{P} \to \mathbb{C}$ index certain characters of $\overline{P}$ with $q \in \mathbb{Z}$ (determined only up to mod 4 when n is odd and up to mod 2 when n is even) and $r, s \in \mathbb{C}$. Looking at the analogue to the noncompact picture provides a realization of $I(q,r,s)$, denoted

$$I'(q,r,s) \subseteq \mathcal{C}^{\infty}(\mathbb{R}^n \times \mathrm{Sym}(n,\mathbb{R}))$$

(see §4). We then look for solutions to Equation (1.1) inside $I'(q,r,s)$. With appropriate parity and initial decay conditions, those solutions are denoted by $\mathcal{D}'_{\pm}$ (see Definition 5.3).

We show that this space of solutions to Equation (1.1) is invariant under G precisely when $r = -1/2$ (Theorem 5.1). Moreover, when s is nonzero and purely imaginary and with appropriate choice of q, the resulting representation is isomorphic to the oscillator representation or its dual, depending on the sign of σ. In the case of the oscillator representation, this realization provides a kind of interpolation between two famous realizations given by Kashiwara and Vergne in [18]. As noted above, the intertwining maps are simply evaluation at $t = 0$ (followed by a Fourier transform) and either evaluation at $x = 0$ or the application of a gradient and then evaluation at $x = 0$.

To be a bit more precise, Kashiwara and Vergne give an embedding of the tensor product of the oscillator representation into a subspace of sections of vector bundles over the Siegel upper half-space, $\mathfrak{H}_n$, and also into a subspace of certain principal series representations. For instance, in the very special case of the even part of the oscillator representation realized on the even Schwartz functions, $\mathcal{S}_+(\mathbb{R}^n)$, they construct the maps

$$\begin{tikzcd} \mathcal{I}'_+ \subseteq \mathcal{C}^{\infty}(\mathrm{Sym}(n,\mathbb{R})) & & \mathcal{S}_+(\mathbb{R}^n) \arrow[ll, "\mathcal{F}_1"'] \arrow[dl, "\mathcal{F}_0"] \\ & \mathcal{O}(\mathfrak{H}_n) \arrow[ul, "\mathrm{BV}"] & \end{tikzcd}$$

where $\mathcal{S}(\mathbb{R}^n)$ denotes the set of Schwartz functions on $\mathbb{R}^n$, $\mathcal{I}'_+$ denotes the image of $\mathcal{S}_+(\mathbb{R}^n)$ under the map $\mathcal{F}_1 = \mathrm{BV} \circ \mathcal{F}_0$ (with $\mathcal{C}^{\infty}(\mathrm{Sym}(n,\mathbb{R}))$ being the noncompact picture of a certain principal series representation of the metaplectic group $Mp(n)$), and the maps are given by

$$\begin{gathered} (\mathcal{F}_0\psi)(Z) = \int_{\mathbb{R}^n} \psi(\xi) e^{\frac{i}{2}\xi Z \xi^T}\, d\xi, \\ (\mathrm{BV}\,\Psi)(t) = \lim_{Y \to 0^+} \Psi(t + iY), \\ (\mathcal{F}_1\psi)(t) = \int_{\mathbb{R}^n} \psi(\xi) e^{\frac{i}{2}\xi t \xi^T}\, d\xi \end{gathered}$$

where $\mathbb{R}^n$ is identified with $M_{1\times n}(\mathbb{R})$, $\psi \in \mathcal{S}_+(\mathbb{R}^n)$, $Z \in \mathfrak{H}_n$, $t \in \mathrm{Sym}(n,\mathbb{R})$, $\Psi \in \mathrm{Im}(\mathcal{F}_0) \subseteq \mathcal{O}(\mathfrak{H}_n)$, and $\lim_{Y\to 0^+}$ denotes the limit as $Y \to 0$ with $Y \in \mathrm{Sym}(n,\mathbb{R})$ and $Y > 0$.

Turning to our realization, with the parameter choice of $r = -1/2$ and $s = -2\pi^2 i$, we have a commutative diagram

$$\begin{tikzcd} & \mathcal{D}'_+ \arrow[dl, "\mathcal{G}"'] \arrow[dr, "\mathcal{E}"] & \\ \mathcal{I}'_+ & & \mathcal{S}_+(\mathbb{R}^n) \arrow[ll, "\mathcal{H}"'] \end{tikzcd}$$

where

$$\mathcal{D}'_+ \subseteq \mathcal{C}^\infty(\mathbb{R}^n \times \mathrm{Sym}(n,\mathbb{R}))$$

is the set of smooth solutions, f, satisfying the system of partial differential equations

$$\begin{aligned} i\partial_{t_{i,j}} f &= \frac{1}{4\pi^2}\partial_{x_i}\partial_{x_j} f \quad (\text{for } i \neq j) \\ i\partial_{t_{ii}} f &= \frac{1}{8\pi^2}\partial_{x_i}^2 f \end{aligned} \tag{1.2}$$

with $f(\cdot,t) \in \mathcal{S}_+(\mathbb{R}^n)$ for each $t \in \mathrm{Sym}(n,\mathbb{R})$ and

$$\mathcal{I}'_+ \subseteq \mathcal{C}^\infty(\mathrm{Sym}(n,\mathbb{R}))$$

is a subspace of the noncompact picture of a certain principal series representation, see §2.3, that essentially consists of the set of Fourier transforms of Schwartz functions pulled back as measures on $\{-y^T y : y \in \mathbb{R}^n\} \subseteq \mathrm{Sym}(n,\mathbb{R})$ (see Corollary 7.4). The maps $\mathcal{E}$ and $\mathcal{G}$ are given by the particularly simple maps

$$(\mathcal{E}f)(x) = \widehat{f}(x,0)$$

(with the Fourier transform given by $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i \xi x^T}\,dx$) and

$$(\mathcal{G}f)(t) = f(0,t).$$

There is an explicit integral formula for $\mathcal{E}^{-1}$ given by

$$(\mathcal{E}^{-1}\psi)(x,t) = \int_{\mathbb{R}^n} f(\xi) e^{\frac{i}{2}\xi t \xi^T} e^{2\pi i \xi x^T}\,d\xi$$

which gives rise to a formula for $\mathcal{H} = \mathcal{F}_1$. An inverse for $\mathcal{G}$ can be given by viewing elements of $\mathcal{I}'_+$ as tempered distributions on $\mathrm{Sym}(n,\mathbb{R})$, applying a Fourier transform, and taking a limit using approximations to a δ-function (see the proof of Theorem 7.2).

The highest weight vector in $\mathcal{D}'_+$ is given by the function $f_+ \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathrm{Sym}(n,\mathbb{R}))$ defined as

$$f_+(x,t) = \det(I_n - it)^{-1/2} e^{-2\pi^2 x (I_n - it)^{-1} x^T}$$

(Theorem 8.1). The corresponding vector in $\mathcal{I}'_+$ is

$$f_+(0,t) = \det(I_n - it)^{-1/2}$$

and in $\mathcal{S}_+(\mathbb{R}^n)$ is

$$\widehat{f_+}(\xi,0) = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}\|\xi\|^2}.$$

Note that the choice of, say, $s = 2\pi^2 i$ gives rise to the dual representation and Schrödinger-like partial differential operators with lowest weight representations.

The above commutative diagram fits on top of the Kashiwara-Vergne picture to give the following commutative diagram.

$$\begin{tikzcd} & \mathcal{D}'_+ \arrow[dl, "\mathcal{G}"'] \arrow[dr, "\mathcal{E}"] & \\ \mathcal{I}'_+ & & \mathcal{S}_+(\mathbb{R}^n) \arrow[ll, "\mathcal{H}=\mathcal{F}_1"'] \arrow[dl, "\mathcal{F}_0"] \\ & \mathcal{O}(\mathfrak{H}_n) \arrow[ul, "\mathrm{BV}"] & \end{tikzcd}$$

There is a similar picture for the odd part of the oscillator representation that fits in with the Kashiwara-Vergne realization in an analogous way. There our diagram looks like

$$\begin{tikzcd} & \mathcal{D}'_- \arrow[dl, "\mathcal{G}_n"'] \arrow[dr, "\mathcal{E}"] & \\ \mathcal{I}'_- & & \mathcal{S}_-(\mathbb{R}^n) \arrow[ll, "\mathcal{H}_n"'] \end{tikzcd}$$

$\mathcal{S}_-(\mathbb{R}^n)$ denotes the odd Schwartz functions,

$$\mathcal{D}'_- \subseteq \mathcal{C}^\infty(\mathbb{R}^n \times \mathrm{Sym}(n,\mathbb{R}))$$

is the set of smooth solutions, f, satisfying the system of partial differential equations from Equation 1.2 with $f(\cdot,t) \in \mathcal{S}_-(\mathbb{R}^n)$ for each $t \in \mathrm{Sym}(n,\mathbb{R})$ and

$$\mathcal{I}'_- \subseteq \mathcal{C}^\infty(\mathrm{Sym}(n,\mathbb{R}),\mathbb{R}^n)$$

is a subspace of the noncompact picture of a certain principal series representation, see §2.3 and Corollary 7.4. Here the maps are given by the same $\mathcal{E}$,

$$(\mathcal{E}f)(x) = \widehat{f}(x,0),$$

and the related gradient to $\mathcal{G}$,

$$(\mathcal{G}_n f)(t) = \nabla_{\mathbb{R}^n} f(0,t).$$

In this case,

$$\begin{aligned}(\mathcal{H}_n f)(t) &= \nabla\Big(\int_{\mathbb{R}^n} f(\xi)e^{\frac{i}{2}\xi t\xi^T}e^{2\pi i\xi x^T}\,d\xi\Big)|_{x=0}\\ &= 2\pi i\Big(\int_{\mathbb{R}^n}\xi_1 f(\xi)e^{\frac{i}{2}\xi t\xi^T}\,d\xi,\dots,\int_{\mathbb{R}^n}\xi_n f(\xi)e^{\frac{i}{2}\xi t\xi^T}\,d\xi\Big)\end{aligned}$$

and $\mathcal{G}_n^{-1}$ can be recovered from certain Fourier transforms (Theorem 7.2).

The highest K-finite vectors of $\mathcal{D}'_-$ consist of the functions f_a given by

$$f_a(x,t) = \det(I_n - it)^{-1/2}\big(x(I_n-it)^{-1}a^T\big)e^{-2\pi^2 x(I_n-it)^{-1}x^T}$$

where $a \in \mathbb{C}^n$ (Theorem 6.3). The corresponding vector in $\mathcal{I}'_-$ is

$$\nabla f_a(0,t) = \det(I_n-it)^{-\frac12}\big(a(I_n-it)^{-1}\big).$$

and in $\mathcal{S}_+(\mathbb{R}^n)$ is

$$\widehat{f_a}(\xi,0) = (2\pi)^{-\frac{n}{2}+1} i(\xi a^T)e^{-\frac12\|\xi\|^2}.$$

2. Notation

2.1. A Double Cover of the Jacobi Group. With respect to the standard symplectic form $J_{n+1} = \begin{pmatrix} 0 & -I_{n+1} \\ I_{n+1} & 0 \end{pmatrix}$, let

$$\begin{aligned} \mathfrak{g} &= \mathfrak{sp}(n+1,\mathbb{R}) \cap \Big\{ \begin{pmatrix} * \\ 0_{1\times(2n+2)} \end{pmatrix} \Big\} \\ &\cong \mathfrak{sp}(n,\mathbb{R}) \ltimes \mathfrak{h}_{2n+1} \end{aligned}$$

where $\mathfrak{h}_{2n+1}$ is the $2n+1$ dimensional real Heisenberg Lie algebra. This is the Lie algebra to the *Jacobi group*

$$\begin{aligned} G^J &= Sp(n+1,\mathbb{R}) \cap \Big\{ \begin{pmatrix} * & * \\ 0_{1\times(2n+1)} & 1 \end{pmatrix} \Big\} \\ &\cong Sp(n,\mathbb{R}) \ltimes H_{2n+1} \end{aligned}$$

where H_{2n+1} is the $2n+1$ dimensional real Heisenberg Lie group. Of course, written in $n \times 1 \times n \times 1$ block form, $Sp(n,\mathbb{R})$ is embedded in G^J as

$$\left\{ \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : C^TA = A^TC,\ D^TB = B^TD,\ A^TD - C^TB = I_n \right\}$$

and H_{2n+1} is embedded as

$$\left\{ \begin{pmatrix} I_n & 0 & 0 & x^T \\ y & 1 & x & z \\ 0 & 0 & I_n & -y^T \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

We write $\mathfrak{H}_n$ for the *Siegel upper half-space*

$$\mathfrak{H}_n = \{ Z = X + iY : X, Y \in \mathrm{Sym}(n,\mathbb{R}) \text{ with } Y > 0 \text{ (positive definite)} \}.$$

The Siegel upper half-space carries a transitive action by $Sp(n,\mathbb{R})$ by linear fractional transformations,

$$g \cdot Z = (AZ+B)(CZ+D)^{-1}.$$

Note that the stabilizer of iI_n in $Sp(n,\mathbb{R})$ is the maximal compact subgroup, $U(n)$, embedded in $Sp(n,\mathbb{R})$ by $A + iB \in U(n) \to \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$.

The main object of study is the double cover of G^J,

$$G = Mp(n) \ltimes H_{2n+1}.$$

Here the action of $Mp(n)$ on H_{2n+1} factors through its projection to $Sp(n,\mathbb{R})$ and we realize the metaplectic group as

$$\begin{aligned} Mp(n) = \Big\{ \Big(g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \varepsilon \Big) &: g \in Sp(n,\mathbb{R}) \text{ with smooth } \varepsilon : \mathfrak{H}_n \to \mathbb{C} \\ &\text{satisfying } \varepsilon(Z)^2 = \det(CZ+D) \Big\}. \end{aligned}$$

The group law on $Mp(n)$ is given by

$$(g_1, \varepsilon_1) \cdot (g_2, \varepsilon_2) = (g_1 g_2, Z \to \varepsilon_1(g_2 \cdot Z)\varepsilon_2(Z)).$$

Note that the identity element is $(I_n, Z \to 1)$ and $(g,\varepsilon)^{-1} = (g^{-1}, Z \to \varepsilon(g^{-1}\cdot Z)^{-1})$. To be explicit, the group law on $Mp(n) \ltimes H_{2n+1}$ is given by

$$((g_1,\varepsilon_1),h_1)\cdot((g_2,\varepsilon_2),h_2) = ((g_1,\varepsilon_1)\cdot(g_2,\varepsilon_2), g_2^{-1}h_1g_2h_2).$$

2.2. **Parabolic Subgroup.** Consider the subalgebra of $\mathfrak{g}$ given, written in $n\times 1\times n\times 1$ block form, by

$$\overline{\mathfrak{p}} = \Big\{ \begin{pmatrix} a & 0 & 0 & 0 \\ y & 0 & 0 & z \\ c & 0 & -a^T & -y^T \\ 0 & 0 & 0 & 0 \end{pmatrix} : c^T = c \Big\}.$$

Then $\overline{\mathfrak{p}}$ is the semidirect product of the maximal parabolic subalgebra

$$\overline{\mathfrak{p}}_{\mathfrak{sp}} = \Big\{ \begin{pmatrix} a & 0 \\ c & -a^T \end{pmatrix} : c^T = c \Big\}$$

of $\mathfrak{sp}(n,\mathbb{R})$ and a copy of $\mathbb{R}^{n+1}$ given by

$$\mathfrak{w} = \Big\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ y & 0 & 0 & z \\ 0 & 0 & 0 & -y^T \\ 0 & 0 & 0 & 0 \end{pmatrix} \Big\}.$$

The Langlands decomposition for $\overline{\mathfrak{p}}_{\mathfrak{sp}}$ is $\overline{\mathfrak{p}}_{\mathfrak{sp}} = \mathfrak{m}\mathfrak{a}\overline{\mathfrak{n}}$ where

$$\mathfrak{a} = \Big\{ \begin{pmatrix} \lambda I_n & 0 \\ 0 & -\lambda I_n \end{pmatrix} : \lambda \in \mathbb{R} \Big\}$$
$$\mathfrak{m} = \Big\{ \begin{pmatrix} a & 0 \\ 0 & -a^T \end{pmatrix} : a \in \mathfrak{sl}(n,\mathbb{R}) \Big\}$$
$$\overline{\mathfrak{n}} = \Big\{ \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} : c^T = c \Big\}.$$

Before turning to the group, first note that the Lie algebra of the maximal compact subgroup of $Sp(n,\mathbb{R})$ is

$$\mathfrak{k} = \Big\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : b^T = b,\ a^T = -a \Big\} \cong \mathfrak{u}(n)$$

and the corresponding maximal compact in $Mp(n)$ is

$$K = \Big\{ (k_{A,B} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \varepsilon) : A + iB \in U(n),\ \varepsilon^2(Z) = \det(-BZ + A) \Big\}.$$

We turn now to the group. Writing $A = \exp\mathfrak{a}$, we see

$$A = \Big\{ a_t = \Big(\begin{pmatrix} e^t I_n & 0 \\ 0 & e^{-t} I_n \end{pmatrix}, Z \to e^{-\frac{n}{2}t} \Big) \Big\}$$

and $\overline{N} = \exp\mathfrak{n}$ is

$$\overline{N} = \Big\{ \overline{n}_C = \Big(\begin{pmatrix} I_n & 0 \\ C & I_n \end{pmatrix}, \varepsilon_C \Big) : C^T = C \Big\}$$

where ε_C is the unique smooth function

$$\varepsilon_C : \mathfrak{H}_n \to \mathbb{C}$$

satisfying $\varepsilon_C(Z)^2 = \det(CZ + I_n)$ determined by the condition that $\varepsilon_C(Z) = \sqrt{\det(CZ + I_n)}$ for sufficiently small $Z \in \mathfrak{H}_n$ (where $\sqrt{\cdot}$ denotes the principal square root).

Now it is easy to check that the centralizer of A in K is

$$\Big\{\Big(\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, Z \to c\Big) : A \in O(n,\mathbb{R}),\ c^2 = \det A\Big\}$$

which has the structure of $SO(n) \times \mathbb{Z}_4$ when n is odd and $SO(n) \rtimes \mathbb{Z}_4$ when n is even. The subgroup M is then defined to be the group generated by this centralizer and $\exp \mathfrak{m}$ so (using the subscript 0 to denote the connected component)

$$\begin{aligned} M_0 &= \Big\{\Big(\begin{pmatrix} A & 0 \\ 0 & A^{-1,T} \end{pmatrix}, Z \to 1\Big) : A \in SL(n,\mathbb{R})\Big\}, \\ M &= \Big\{m_{A,c} = \Big(\begin{pmatrix} A & 0 \\ 0 & A^{-1,T} \end{pmatrix}, Z \to c\Big) : \\ &\qquad A \in GL(n,\mathbb{R}),\ \det A \in \{\pm 1\},\ c^2 = \det A^{-1}\Big\}. \end{aligned}$$

Thus the component group, M/M_0, is isomorphic to $\mathbb{Z}_4$. Finally, writing $W = \exp \mathfrak{w}$, we see

$$W = \Big\{w_{y,z} = \begin{pmatrix} I_n & 0 & 0 & 0 \\ y & 1 & 0 & z \\ 0 & 0 & I_n & -y^T \\ 0 & 0 & 0 & 1 \end{pmatrix}\Big\}.$$

We let $\overline{P}$ be given by

$$\overline{P} = MA\overline{N} \ltimes W.$$

2.3. **Induced Representations.** For $q \in \mathbb{Z}$ (determined only up to mod 4 or mod 2 depending on n), $r \in \mathbb{C}$, and $s \in \mathbb{C}$, we define a character

$$\chi_{q,r,s} : \overline{P} \to \mathbb{C}$$

by

$$\chi_{q,r,s}(m_{A,c} a_t \overline{n}_C w_{y,z}) = c^q e^{rnt} e^{sz}.$$

Note that for $n = 1$, the choice of q in [23] is the negative of the choice here. We study the induced representation

$$\begin{aligned} I(q,r,s) &= \mathrm{Ind}_{\overline{P}}^{G} \chi_{q,r,s} \\ &= \Big\{\text{smooth } \phi : G \to \mathbb{C} : \phi(gp) = \chi_{q,r,s}(p)^{-1}\phi(g) \text{ for } g \in G,\ p \in \overline{P}\Big\} \end{aligned}$$

with action group action $(g \cdot \phi)(g') = \phi(g^{-1}g')$.

We will also have occasion to use two related induced representations of $Mp(n)$. To this end, define a character and an n-dimensional representation of $MA\overline{N}$

$$\begin{gathered} \chi_{q,r} : MA\overline{N} \to \mathbb{C}, \\ \pi_{q,r} : MA\overline{N} \to GL(n,\mathbb{C}) \end{gathered}$$

by

$$\begin{gathered} \chi_{q,r}(m_{A,c} a_t \overline{n}_C) = c^q e^{rnt}, \\ \pi_{q,r}(m_{A,c} a_t \overline{n}_C) \cdot v = c^q e^{rnt} v A^{-1} \end{gathered}$$

for $v \in \mathbb{C}^n$ given as a row vector. The associated induced representations are

$$\begin{aligned} I(q,r) &= \operatorname{Ind}_{MA\overline{N}}^{Mp(n)} \chi_{q,r} \\ &= \{\mathcal{C}^\infty\ \phi : G \to \mathbb{C} : \phi(gp) = \chi_{q,r}(p)^{-1}\phi(g) \text{ for } g \in Mp(n),\ p \in MA\overline{N}\} \\ I_n(q,r) &= \operatorname{Ind}_{MA\overline{N}}^{Mp(n)} \pi_{q,r} \\ &= \{\mathcal{C}^\infty\ \phi : G \to \mathbb{C}^n : \phi(gp) = \pi_{q,r}(p)^{-1}\cdot\phi(g) \text{ for } g \in Mp(n),\ p \in MA\overline{N}\} \end{aligned}$$

with action group action $(g\cdot\phi)(g') = \phi(g^{-1}g')$.

3. Boundary Values of ε

Recall elements of $Mp(n)$ are given by pairs (g,ε) with $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and smooth $\varepsilon : \mathfrak{H}_n \to \mathbb{C}$ satisfying $\varepsilon(Z)^2 = \det(CZ+D)$. If we are in the special case of $\det D \neq 0$, then $\det(CZ+D) = \operatorname{sgn}(\det D)|\det D|\det(D^{-1}CZ+I_n)$. In particular, for all sufficiently small Z,

$$\begin{aligned} \varepsilon(Z) &= i^p|\det D|^{\frac{1}{2}}\sqrt{\det(D^{-1}CZ+I_n)} \\ &= i^p|\det D|^{1/2}\varepsilon_{D^{-1}C}(Z) \end{aligned}$$

where $\sqrt{\cdot}$ denotes the principal square root and $p = p(\varepsilon)$ is one of the two choices (determined precisely by ε) of $p \in \mathbb{Z}_4$ for which $(-1)^p = \operatorname{sgn}(\det D)$. Note that the identity

$$\varepsilon = i^p|\det D|^{1/2}\varepsilon_{D^{-1}C}$$

then holds for all Z since the functions are analytic.

We need to extend the definition of ε from $\mathfrak{H}_n$ to $\operatorname{Sym}(n,\mathbb{R})$ almost everywhere. For this, let $\varepsilon : \operatorname{Sym}(n,\mathbb{R}) \to \mathbb{C}$ be given by

$$\varepsilon(X) = \lim_{Y\to 0^+} \varepsilon(X+iY)$$

(here $Y \to 0^+$ denotes $Y \to 0$ with $Y > 0$) which will be defined when $\det(CX+D) \neq 0$. To see this limit exists when $\det(CX+D) \neq 0$, observe that, for Z with sufficiently small $\operatorname{Im}(Z)$, we can write $\varepsilon(Z) = i^l\sqrt{\operatorname{sgn}(\det(CX+D))\det(CZ+D)}$ where $\sqrt{\cdot}$ denotes the principal square root and $l = l(\varepsilon, X)$ is one of the two choices (determined precisely by ε and X) of $l \in \mathbb{Z}_4$ for which $(-1)^l = \operatorname{sgn}(\det(CX+D))$. In particular, we see $\varepsilon(X)$ exists and is given by

$$\varepsilon(X) = i^l\sqrt{|\det(CX+D)|}. \tag{3.1}$$

In the special case where $X = 0$ and $\det D \neq 0$, there is a useful formula for recovering the p in the formula $\varepsilon = i^p|\det D|^{1/2}\varepsilon_{D^{-1}C}$. Namely,

$$i^p = \frac{\varepsilon(0)}{|\det D|^{1/2}}.$$

Finally, define an almost everywhere action of $Sp(n,\mathbb{R})$ on $\operatorname{Sym}(n,\mathbb{R})$ given by

$$g\cdot X = (AX+B)(CX+D)^{-1}$$

for $X \in \operatorname{Sym}(n,\mathbb{R})$ when $\det(CX+D) \neq 0$ so that $g\cdot X = \lim_{Y\to 0^+} g\cdot(X+iY)$.

4. Noncompact Pictures

Let

$$\mathfrak{x} = \left\{ \begin{pmatrix} 0 & 0 & 0 & x^T \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}$$

so that $X = \exp \mathfrak{x}$ is given by

$$X = \left\{ e_x = \begin{pmatrix} I_n & 0 & 0 & x^T \\ 0 & 1 & x & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \cong \mathbb{R}^n$$

and let

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b^T = b \right\}$$

so that $N = \exp \mathfrak{n}$ is given by

$$N = \left\{ n_B = \left(\begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix}, Z \to 1 \right) : B^T = B \right\}.$$

Restriction to $XN \cong \mathbb{R}^n \times \mathrm{Sym}(n, \mathbb{R})$ gives what would be called the *noncompact realization* of the induced representation if we were in the semisimple category and which we denote by

$$\begin{aligned} &I'(q, r, s) \\ &= \left\{ f : \mathbb{R}^n \times \mathrm{Sym}(n, \mathbb{R}) \to \mathbb{C} : f(x, B) = \phi(e_x n_B) \text{ for some } \phi \in I(q, r, s) \right\}. \end{aligned}$$

We make $I'(q, r, s)$ into a G-module so that the restriction map $\phi \to f$ is an intertwining map. When necessary, we will coordinatize $\mathrm{Sym}(n, \mathbb{R})$ as $\mathbb{R}^{\frac{n(n+1)}{2}}$ by writing

$$B = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{12} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{1n} & t_{2n} & \cdots & t_{nn} \end{pmatrix}.$$

Theorem 4.1. *For $f \in I'(q, r, s)$, the action of $g = \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \varepsilon \right) \in Mp(n)$ on f is given by*

$$\begin{aligned} ((g, \varepsilon) \cdot f)(x, t) = i^{lq} |\det(A - tC)|^r e^{-sxC(A-tC)^{-1}x^T} \\ \times f(x(-C^T t + A^T)^{-1}, (A - tC)^{-1}(tD - B)) \end{aligned}$$

when $\det(A - tC) \neq 0$ and $l \in \mathbb{Z}_4$ satisfies $\varepsilon(g^{-1} \cdot t) = i^l |\det(A - tC)|^{-1/2}$.

The action of $h = \begin{pmatrix} I_n & 0 & 0 & y_0^T \\ x_0 & 1 & y_0 & z_0 \\ 0 & 0 & I_n & -x_0^T \\ 0 & 0 & 0 & 1 \end{pmatrix} \in H_{2n+1}$ on f is given by

$$(h \cdot f)(x, t) = e^{s(2xx_0^T + z_0 - x_0 t x_0^T - y_0 x_0^T)} f(x - y_0 - x_0 t, t).$$

Proof. When $\det D \neq 0$, write $\varepsilon(Z) = i^p |\det D|^{1/2}\sqrt{\det(D^{-1}CZ + I_n)}$ for all sufficiently small Z and recall that $i^p = \varepsilon(0)|\det D|^{-1/2}$. It is straightforward to verify that

$$(g,\varepsilon) = n_{BD^{-1}}\, m_{|\det D|^{-\frac{1}{n}} D^{-1,T}, i^p}\, a_{\ln(|\det D|^{-\frac{1}{n}})}\, \overline{n}_{D^{-1}C} \tag{4.1}$$

and

$$(g,\varepsilon)\, e_x = n_{BD^{-1}}\, e_{xD^{-1}}\, (\begin{pmatrix} D^{-1,T} & 0 \\ C & D \end{pmatrix}, \varepsilon)\, w_{-xD^{-1}C, -xD^{-1}Cx^T}. \tag{4.2}$$

Suppose $f \in I'(q,r,x)$ corresponds to $\varphi \in I(q,r,s)$. Then

$$\begin{aligned} ((g,\varepsilon)\cdot f)(x,t) &= \phi(g^{-1}e_x n_t) \\ &= \phi((\begin{pmatrix} D^T & D^T t - B^T \\ -C^T & -C^T t + A^T \end{pmatrix}, Z \to \varepsilon(g^{-1}\cdot (Z+t))^{-1})e_x). \end{aligned}$$

Using Equations 4.2 and 4.1 when $\det(A - tC) \neq 0$, it follows that

$$\begin{aligned} &((g,\varepsilon)\cdot f)(x,t) \\ &= \Big(\frac{\varepsilon(g^{-1}\cdot t)^{-1}}{|\det(-tC + A)|^{1/2}}\Big)^{-q} \big(|\det(-tC+A)|^{-\frac{1}{n}}\big)^{-rn} \\ &\quad \times \cdot e^{-sx(-C^T t + A^T)^{-1}C^T x^T}\, \phi(n_{(D^T t - B^T)(-C^T t + A^T)^{-1}}, e_{x(-C^T t + A^T)^{-1}}). \end{aligned}$$

Finally, it is easy to see that $C(g^{-1}\cdot t) + D = (A^T - C^T t)^{-1}$. Looking at Equation 3.1, there is an $l \in \mathbb{Z}_4$ so that $\varepsilon(g^{-1}\cdot t) = i^l |\det(A^T - C^T t)|^{-1/2}$ and the result follows. The calculation for H_{2n+1} is similar and omitted. □

A straightforward calculation yields:

Corollary 4.2. *Let $f \in I'(q,r,s)$. The element $h = (x_0, y_0, z_0) \in \mathfrak{h}_{2n+1}$ acts on f by*

$$h\cdot f(x,t) = s(2x_0 x^T + z_0) f(x,t) - \sum_{i=1}^n (x_0 t + y_0)_i \partial_{x_i} f(x,t).$$

The element $a_\lambda \in \mathfrak{a}$, $\lambda \in \mathbb{R}$, acts on f by

$$(a_\lambda \cdot f)(x,t) = nr\lambda f(x,t) - \lambda \sum_{i=1}^n x_i \partial_{x_i} f(x,t) - 2\lambda \sum_{i\le j} t_{i,j}\partial_{t_{i,j}} f(x,t).$$

The element $n_c \in \overline{\mathfrak{n}}$, $c^T = c$, acts on f by

$$\begin{aligned} (n_c\cdot f)(x,t) &= -rTr(tc)f(x,t) - sxcx^T f(x,t) + \sum_{i=1}^n (xct)_i \partial_{x_i} f(x,t) \\ &\quad + \sum_{i\le j} (tct)_{i,j}\partial_{t_{i,j}} f(x,t). \end{aligned}$$

If $k_{a,b} \in \mathfrak{k}$, $b^T = b$, $a^T = -a$, then $k_{a,0}$ acts on f by

$$(k_{a,0}\cdot f)(x,t) = \sum_{i=1}^n (xa)_i \partial_{x_i} f(x,t) + \sum_{i\le j} (ta - at)_{i,j}\partial_{t_{i,j}} f(x,t)$$

and $k_{0,b}$ acts by

$$(k_{0,b}\cdot f)(x,t) = rTr(tb)f(x,t) + sxbx^T f(x,t)$$

$$- \sum_{i=1}^{n} (xbt)_i \partial_{x_i} f(x,t) - \sum_{i \le j} (Tr(tb)t + b)_{i,j} \partial_{t_{i,j}} f(x,t).$$

In a similar fashion, we also have the noncompact realizations of $I(q,r)$ and $I_n(q,r)$ given by restriction to $N \cong \mathrm{Sym}(n,\mathbb{R})$. We denote these realizations by

$$I'(q,r) = \{ f : \mathrm{Sym}(n,\mathbb{R}) \to \mathbb{C} : f(B) = \phi(n_B) \text{ for some } \phi \in I(q,r) \}$$
$$I'_n(q,r) = \{ f : \mathrm{Sym}(n,\mathbb{R}) \to \mathbb{C}^n : f(B) = \phi(n_B) \text{ for some } \phi \in I_n(q,r) \}.$$

Simple modifications of the proof Theorem 4.1 give the following result.

Corollary 4.3. *For $f \in I'(q,r)$, the action of $\big(g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \varepsilon\big) \in Mp(n)$ on f is given by*

$$((g,\varepsilon)\cdot f)(t) = i^{lq} |\det(A - tC)|^r \, f((A - tC)^{-1}(tD - B))$$

when $\det(A - tC) \neq 0$ *and* $l \in \mathbb{Z}_4$ *satisfies* $\varepsilon(g^{-1}\cdot t) = i^l |\det(A - tC)|^{-1/2}$.

For $f_n \in I'_n(q,r)$, the action of $\big(g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \varepsilon\big) \in Mp(n)$ on f_n is given by

$$\big((g,\varepsilon)\cdot f\big)(t) = i^{lq} |\det(A - tC)|^{r - \frac{1}{n}} \, f_n((A - tC)^{-1}(tD - B))(-tC + A)^{-1}.$$

We also see that:

Corollary 4.4. *There is an $Mp(n)$-intertwining map*

$$\mathcal{G} : I'(q,r,s) \to I'(q,r)$$

given by the mapping $f \to f(0,\cdot)$.

The corresponding map from $I(q,r,s) \to I(q,r)$ is given by $\phi \to \phi|_{Mp(n)}$.

There is also an $Mp(n)$-intertwining map

$$\mathcal{G}_n : I'(q,r,s) \to I'_n(q, r - \frac{1}{n})$$

given by mapping $f \to \nabla f(0,\cdot)$.

The corresponding map from $I(q,r,s) \to I_n(q, r - \frac{1}{n})$ is given by $\phi \to \nabla(\phi(\cdot e_x))|_{x=0}$.

Proof. The first statement is obvious since

$$((g,\varepsilon)\cdot f)(0,t) = i^{lq} |\det(A - tC)|^r \, f(0,(A - tC)^{-1}(tD - B)).$$

It also follows trivially from the definitions that the map $f \to f(0,\cdot)$ on $I'(q,r,s) \to I'(q,r)$ corresponds to the map $\phi \to \phi|_{Mp(n)}$ on $I(q,r,s) \to I(q,r)$.

For the second statement, observe that

$$\begin{aligned} &\Big(\frac{\partial}{\partial x_i}((g,\varepsilon)\cdot f)\Big)(0,t) \\ &= i^{lq} |\det(A - tC)|^r \sum_j \big((-C^T t + A^T)^{-1}\big)_{ij} \frac{\partial f}{\partial x_j}(0,(A - tC)^{-1}(tD - B)). \end{aligned}$$

Thus

$$\begin{aligned} &\nabla\big((g,\varepsilon)\cdot f\big)(0,\cdot) \\ &= i^{lq} |\det(A - tC)|^r \nabla f(0,(A - tC)^{-1}(tD - B))(-C^T t + A^T)^{-1,T} \end{aligned}$$

and the map intertwines. Finally, we claim that the map given by $f \to \nabla f(0,\cdot)$ on $I'(q,r,s) \to I'_n(q,r-\frac{1}{n})$ is induced by the map $\varphi \to \nabla(\varphi(\cdot e_x))|_{x=0}$ on $I(q,r,s) \to I_n(q,r-\frac{1}{n})$. To check this, note that it is easy to verify that

$$(g,\varepsilon)e_x = e_{xD^{-1}}\,(g,\varepsilon)\,w_{-xD^{-1}C,-xD^{-1}Cx^T}$$

when D is invertible.

Then, for $\gamma \in Mp(n)$ and $p \in MA\overline{N}$ written as $p = m_{A,c}a_t\overline{n}_C$,

$$\begin{aligned}
\nabla(\phi(\gamma p e_x))|_{x=0} &= \nabla(\phi(\gamma m_{A,c}a_t\overline{n}_C e_x))|_{x=0}\\
&= \nabla(\phi(\gamma e_{e^t xA^T} m_{A,c}a_t\overline{n}_C w_{-xA^TA^{-1}C,-xA^TA^{-1}Cx^T}))|_{x=0}\\
&= \nabla(c^{-q}e^{-rnt}e^{sxA^TA^{-1}Cx^T}\phi(\gamma e_{e^t xA^T}))|_{x=0}\\
&= c^{-q}e^{-rnt}\nabla(\phi(\gamma e_x))|_{x=0}\,e^tA\\
&= c^{-q}e^{-(r-\frac{1}{n})nt}\nabla(\phi(\gamma e_x))|_{x=0}\,A\\
&= \pi_{q,r-\frac{1}{n}}(p)^{-1}\cdot\nabla(\phi(\gamma e_x))|_{x=0}.
\end{aligned}$$

Thus $\nabla(\phi(\cdot e_x))|_{x=0} \in I_n(q,r-1/n)$. Moreover, noting that $n_Be_x = e_xn_B$, we have $\nabla(\phi(e_Ce_x))|_{x=0} = \nabla f(0,C)$ so that $\nabla(\phi(\cdot e_x))|_{x=0} \in I_n(q,r)$ corresponds to $\nabla f(0,\cdot) \in I'_n(q,r)$. □

5. An Invariant Subspace

Theorem 5.1. *For $r=-1/2$, the set of functions $f \in I'(q,r,s)$ satisfying the system of partial differential equations (from Equation* (1.1)*)*

$$\begin{gathered}
2s\partial_{t_{i,j}}f + \partial_{x_i}\partial_{x_j}f = 0, \quad i \neq j\\
4s\partial_{t_{ii}}f + \partial_{x_i}^2 f = 0
\end{gathered}$$

is G-invariant.

Proof. Temporarily write $D = \{2s\partial_{t_{i,j}} + \partial_{x_i}\partial_{x_j}, 4s\partial_{t_{ii}} + \partial^2_{x_i} : 1 \le i \neq j \le n\}$. First observe that the differential operators in D commute with the Heisenberg group action. This is clear for $(0,y,z) \in H_{2n+1}$ since D consists of constant coefficient differential operators and $((0,y,z).f)(x,t) = e^{sz}\,f(x-y,t)$ by Theorem 4.1. Checking commutivity for $(x,0,0) \in H_{2n+1}$ is a straightforward application of the chain rule and is omitted. The invariance of D under $Mp(n)$ follows by a Lie algebra calculation showing that $[X,D_i]$ lies in the $\mathcal{C}^\infty(\mathbb{R}^n \times \mathrm{Sym}(n,\mathbb{R}))$-span of D for any $X \in \mathfrak{g}$ and $D_i \in D$. As the details are straightforward and all similar, we give the particulars only for the element $X = E_{n+1,1} \in \mathfrak{sp}(n,\mathbb{R})$ as representative of the most interesting case. By Corollary 4.2,

$$E_{n+1,1}\cdot f = -rt_{11}f - sx_1^2 f + \sum_{i=1}^n x_1t_{1,i}\partial_{x_i}f + \sum_{i\le j} t_{1,i}t_{1,j}\partial_{t_{i,j}}f.$$

Then

$$\begin{aligned}
&[-rt_{11} - sx_1^2 + \sum_{i=1}^n x_1t_{1,i}\partial_{x_i} + \sum_{i\le j} t_{i,1}t_{1,j}\partial_{t_{i,j}},\, 4s\partial_{t_{11}} + \partial^2_{x_1}]\\
&= -4s(-r + x_1\partial_{x_1} + 2t_{1,1}\partial_{t_{1,1}} + \sum_{j=2}^n t_{1,j}\partial_{t_{1,j}}) - (-2s - 4sx_1\partial_{x_1} + 2\sum_{i=1}^n t_{1,i}\partial_{x_1}\partial_{x_i})
\end{aligned}$$

$$= 2s(1+2r) - 2t_{1,1}(4s\partial_{t_{1,1}} + \partial_{x_1}^2) - 2\sum_{j=2}^{n} t_{1,j}(2s\partial_{t_{1,j}} + \partial_{x_1}\partial_{x_j}).$$

The result follows. □

It is helpful to be able to write down explicit formulas for solutions to Equation (1.1).

Theorem 5.2. *Let $s \neq 0$ be purely imaginary. If $f \in \mathcal{C}^2(\mathbb{R}^n \times \mathrm{Sym}(n,\mathbb{R}))$ satisfying $f(\cdot,0)$, $\widehat{f(\cdot,0)} \in L^1(\mathbb{R}^n)$ and the system of partial differential equations from Equation* (1.1)*, then*

$$f(x,t) = \int_{\mathbb{R}^n} \widehat{f}(\xi,0) e^{\frac{\pi^2}{s}\xi t \xi^T} e^{2\pi i \xi x^T}\, d\xi.$$

Proof. By standard Fourier techniques, when $f(\cdot,0)$ is a tempered distribution, there is a unique solution to the Cauchy problem in the space of $\mathcal{C}(\mathrm{Sym}(n,\mathbb{R}), \mathcal{S}'(\mathbb{R}^n))$– i.e., $f(x,t)$ is continuous in t and takes values in the set of tempered distributions on $\mathbb{R}^n$. In fact, if $\int_{\mathbb{R}^n}(1+\|x\|^2) f(x,0)\, dx < \infty$, the solution is classical in the sense that it has continuous derivatives with respect to each $t_{i,j}$ and continuous second order derivatives with respect to each x_i. Alternately, if $f(\cdot,0) \in L^2(\mathbb{R}^n)$, then $f \in \mathcal{C}(\mathrm{Sym}(n,\mathbb{R}), L^2(\mathbb{R}^n))$ with $\|f(\cdot,t)\|_{L^2(\mathbb{R}^n)} = \|f(\cdot,0)\|_{L^2(\mathbb{R}^n)}$.

The calculation goes as follows: take the Fourier transform with respect to x of the partial differential equations from Equation (1.1) to get

$$\begin{gathered}(2s\partial_{t_{i,j}} - 4\pi^2\xi_i\xi_j)\widehat{f} = 0, \quad i \neq j \\ (4s\partial_{t_{ii}} - 4\pi^2\xi_i^2)\widehat{f} = 0.\end{gathered}$$

Thus

$$\widehat{f}(\xi,t) = \widehat{f}(\xi,0) e^{\frac{\pi^2}{s}(\sum_{i=1}^n \xi_i^2 t_{ii} + 2\sum_{i<j}\xi_i\xi_j t_{i,j})} = \widehat{f}(\xi,0) e^{\frac{\pi^2}{s}\xi t \xi^T}. \tag{5.1}$$

Therefore,

$$f(x,t) = \int_{\mathbb{R}^n} \widehat{f}(\xi,0) e^{\frac{\pi^2}{s}\xi t \xi^T} e^{2\pi i \xi x^T}\, d\xi.$$

□

Definition 5.3. Let $s \neq 0$ be purely imaginary and $r = -1/2$. Define

$$\mathcal{D}' \subseteq I'(q,r,s) \subseteq \mathcal{C}^\infty(\mathbb{R}^n \times \mathrm{Sym}(n,\mathbb{R}))$$

to be the space of functions $f \in I'(q,r,s)$ that satisfy the system of partial differential equations from Equation (1.1) with $f(\cdot,0) \in \mathcal{S}(\mathbb{R}^n)$. Write $\mathcal{D}'_+$ and $\mathcal{D}'_-$ for the functions in $\mathcal{D}'$ that are even (respectively, odd) in x for each $t \in \mathrm{Sym}(n,\mathbb{R})$.

Remark 5.4. For the rest of the paper, we will assume $r = -1/2$ and that s is nonzero and purely imaginary. We write $s = i\sigma$ with $\sigma \in \mathbb{R}^\times$. We will also write

$$\varepsilon_\sigma = \mathrm{sgn}(\sigma)$$

so that $\sigma = \varepsilon_\sigma|\sigma|$.

Theorem 5.5. *The space $\mathcal{D}'$ is G-invariant.*

Proof. Since σ is purely imaginary, the invariance of $\mathcal{D}'$ under H_{2n+1} follows from the action given in Theorem 4.1. Let $(g,\varepsilon)\in Mp(n)$ and $f\in\mathcal{D}'$ and let $h=(g,\varepsilon)\cdot f$. By Theorem 5.1, it suffices to show $h(\cdot,0)\in\mathcal{S}(\mathbb{R}^n)$. Fix $t_0\in\mathrm{Sym}(n,\mathbb{R})$ so that $\det(A-t_0C)\neq 0$ and let $\widetilde{t_0}=(A-t_0C)^{-1}(t_0D-B)$. Theorem 4.1 shows that

$$h(x,t_0)=i^{lq}|\det(A-t_0C)|^r e^{-sxC(A-t_0C)^{-1}x^T}\,f(x(-C^Tt_0+A^T)^{-1},\widetilde{t_0})$$

where $\varepsilon(g^{-1}\cdot t_0)=i^l|\det(A-t_0C)|^{-1/2}$. Since Equation 5.1 shows

$$f(x,\widetilde{t_0})=(\widehat{f}(\cdot,0)e^{\frac{\pi^2}{s}(\cdot)\widetilde{t_0}(\cdot)^T})^\vee(x),$$

it follows that $f(\cdot,\widetilde{t_0})\in\mathcal{S}(\mathbb{R}^n)$ and therefore that $h(\cdot,t_0)\in\mathcal{S}(\mathbb{R}^n)$. Finally, since $h(x,0)=(\widehat{h}(\cdot,t_0)e^{-\pi^2/s(\cdot)t_0(\cdot)^T})^\vee(x)$, it follows that $h(\cdot,0)\in\mathcal{S}(\mathbb{R}^n)$. □

Definition 5.6. Write $\widetilde{J}\in Mp(n)$ for the element $\widetilde{J}=(J_n,\varepsilon_{\widetilde{J}})$ where $\varepsilon_{\widetilde{J}}^2(Z)=\det Z$ with $\varepsilon_{\widetilde{J}}(Z)=\sqrt{\det Z}$ for $Z=(\lambda+i\mu)I_n$ for $\lambda,\mu>0$ with $\arctan\frac{\mu}{\lambda}<\frac{\pi}{n}$. The *Cartan involution* $\theta: Mp(n)\to Mp(n)$ is the anti-involution $\theta(g,\varepsilon)=(g^T,\varepsilon^T)$ where

$$(g^T,\varepsilon^T)=\widetilde{J}(g,\varepsilon)^{-1}\widetilde{J}^{-1}.$$

Notice that

$$\begin{aligned}
(g^T,\varepsilon^T)&=\widetilde{J}(g,\varepsilon)^{-1}\widetilde{J}^{-1}\\
&=\Big(J_ng^{-1}J_n^{-1},Z\to\varepsilon_{\widetilde{J}}(g^{-1}J_n^{-1}\cdot Z)\varepsilon(g^{-1}J_n^{-1}\cdot Z)^{-1}\varepsilon_{\widetilde{J}}(-Z^{-1})^{-1}\Big)\\
&=(g^T,Z\to\varepsilon(-(B^TZ+D^T)(A^TZ+C^T)^{-1})^{-1}\\
&\quad\times\varepsilon_{\widetilde{J}}\Big(-(B^TZ+D^T)(A^TZ+C^T)^{-1}\Big)\varepsilon_{\widetilde{J}}(-Z^{-1})^{-1})
\end{aligned}$$

so that

$$\begin{aligned}
\varepsilon^T(Z)&=\varepsilon(-(B^TZ+D^T)(A^TZ+C^T)^{-1})^{-1}\\
&\quad\times\varepsilon_{\widetilde{J}}(-(B^TZ+D^T)(A^TZ+C^T)^{-1})\varepsilon_{\widetilde{J}}(-Z^{-1})^{-1}.
\end{aligned}$$

Of course,

$$\begin{aligned}
\varepsilon^T(Z)^2&=\frac{\det(-(B^TZ+D^T)(A^TZ+C^T)^{-1})}{\det(-C(B^TZ+D^T)(A^TZ+C^T)^{-1}+D)\det(-Z^{-1})}\\
&=\frac{\det(B^TZ+D^T)}{\det(C(B^TZ+D^T)-D(A^TZ+C^T))\det(-Z^{-1})}\\
&=\det(B^TZ+D^T)
\end{aligned}$$

as required.

Theorem 5.7. *When $\sigma>0$ and $q\equiv -1$, we can define $\phi_+,\phi_{+,\alpha}\in I(q,r,s)$ with $\alpha\in\mathbb{C}^n$ by*

$$\phi_+((g,\varepsilon)\,h_{x,y,z})=\frac{e^{i\sigma(-z-xy^T+x(g^T\cdot iI_n)x^T)}}{\varepsilon^T(iI_n)},$$

$$\phi_{+,\alpha}((g,\varepsilon)\,h_{x,y,z})=\frac{(x(Bi+D)^{-1}\alpha^T)e^{i\sigma(-z-xy^T+x(g^T\cdot iI_n)x^T)}}{\varepsilon^T(iI_n)}$$

(recall $\varepsilon^T(Z)^2=\det(ZB+D)$). The corresponding elements $f_+,f_{+,\alpha}\in\mathcal{D}'$ are

$$f_+(x,t)=\varepsilon_t(iI_n)^{-1}e^{-\sigma x(I_n+it)^{-1}x^T}$$

$$f_{+,\alpha}(x,t) = \varepsilon_t(iI_n)^{-1}(x(I_n+it)^{-1}\alpha^T)e^{-\sigma x(I_n+it)^{-1}x^T}$$

where, recall, $\varepsilon_t(Z)$ is the analytic continuation to $Z \in \mathfrak{H}_n$ of the function $Z \to \sqrt{\det(I_n+tZ)}$ for sufficiently small Z.

When $\sigma < 0$ and $q \equiv 1$, we can define $\phi_-, \phi_{-,\alpha} \in I(q,r,s)$ with $\alpha \in \mathbb{C}^n$ by

$$\phi_-((g,\varepsilon)\,h_{x,y,z}) = \frac{e^{i\sigma(-z-xy^T+x(g^T\cdot(-iI_n))x^T)}}{\overline{\varepsilon^T(iI_n)}}$$

$$\phi_{-,\alpha}((g,\varepsilon)\,h_{x,y,z}) = \frac{(x(-Bi+D)^{-1}\alpha^T)e^{i\sigma(-z-xy^T+x(g^T\cdot(-iI_n))x^T)}}{\overline{\varepsilon^T(iI_n)}}.$$

The corresponding elements $f_-, f_{-,\alpha} \in \mathcal{D}'_+$ are

$$f_-(x,t) = \overline{\varepsilon_t(iI_n)}^{-1}e^{\sigma x(I_n-it)^{-1}x^T}$$

$$f_{-,\alpha}(x,t) = \overline{\varepsilon_t(iI_n)}^{-1}(x(I_n-it)^{-1}\alpha^T)e^{\sigma x(I_n-it)^{-1}x^T}.$$

Proof. To determine when $\phi_+ \in I(q,r,s)$, first write $\overline{p} = m_{A_0,c_0}a_{t_0}\overline{n}_{C_0} = (\overline{p}_0, \varepsilon_{p_0})$ so that

$$\overline{p}_0 = \begin{pmatrix} e^{t_0}A_0 & 0 \\ e^{-t_0}A_0^{-1,T}C_0 & e^{-t_0}A_0^{-1,T} \end{pmatrix},$$

$$\varepsilon_{p_0}(Z) = c_0e^{-\frac{n}{2}t_0}\varepsilon_{C_0}(Z).$$

Since $\varepsilon_{p_0}^T(Z)^2 = \det(e^{-t_0}A_0^{-1}) = e^{-nt_0}\det A_0^{-1}$ and $c_0^2 = \det A_0^{-1}$, it follows that $\varepsilon_{p_0}^T(Z) = \pm c_0e^{-\frac{n}{2}t_0}$. The exact answer can be determined by using the continuity of the Cartan involution and its evaluation on the central elements, $Z = (\pm I_n, c)$ with $c^2 = (\pm 1)^{-n}$:

$$\varepsilon^T(Z) = c^{-1}\varepsilon_{\widetilde{J}}(-Z^{-1})\varepsilon_{\widetilde{J}}(-Z^{-1})^{-1} = c^{-1}.$$

It follows that

$$\varepsilon_{p_0}^T(Z) = c_0^{-1}e^{-\frac{n}{2}t_0}.$$

In particular, it we see that

$$((g,\varepsilon)\overline{p})^T = (\overline{p}_0^Tg^T, c_0^{-1}e^{-\frac{n}{2}t_0}\varepsilon^T).$$

Turning to ϕ_+, a straightforward calculation shows that

$$\begin{aligned}
&\phi_+((g,\varepsilon)h_{x,y,z}\,\overline{p}w_{y_0,z_0}) \\
&= \phi_+((g,\varepsilon)\overline{p}h_{e^{-t_0}xA_0^{-1,T},e^{t_0}yA_0+e^{-t_0}xA_0^{-1,T}C_0,z}w_{y_0,z_0}) \\
&= \phi_+((g,\varepsilon)\overline{p}h_{e^{-t_0}xA_0^{-1,T},e^{t_0}yA_0+e^{-t_0}xA_0^{-1,T}C_0+y_0,z+z_0-e^{-t_0}xA_0^{-1,T}y_0^T}) \\
&= e^{-i\sigma(z+z_0-e^{-t_0}xA_0^{-1,T}y_0^T)}e^{-i\sigma e^{-t_0}xA_0^{-1,T}(e^{t_0}yA_0+e^{-t_0}xA_0^{-1,T}C_0+y_0)^T} \\
&\quad\times e^{i\sigma(e^{-t_0}xA_0^{-1,T})(e^{2t_0}A_0^T(g^T\cdot iI_n)A_0+C_0)(e^{-t_0}A_0^{-1}x^T)}/[c_0^{-1}e^{-\frac{n}{2}t_0}\varepsilon^T(iI_n)] \\
&= c_0e^{\frac{n}{2}t_0}e^{-i\sigma z_0}\phi_+((g,\varepsilon)\,h_{x,y,z}).
\end{aligned}$$

It follows that $\phi_+ \in I(-1,-1/2,\iota\sigma)$.

Next observe that the ε for

$$n_t^T = \Big(\begin{pmatrix} I_n & 0 \\ t & I_n \end{pmatrix}, Z \to \varepsilon_{\widetilde{J}}(-(tZ+I_n)Z^{-1})\varepsilon_{\widetilde{J}}(-Z^{-1})^{-1}\Big).$$

Now for $Z = \rho e^{i\theta} I_n$, $\det(-Z^{-1}) = \rho^{-n} e^{in(\pi-\theta)}$ so that $\varepsilon_{\widetilde{J}}(-Z^{-1}) = \rho^{-\frac{n}{2}} e^{i\frac{n(\pi-\theta)}{2}}$ for $\pi-\theta$ sufficiently positively small and $\rho > 0$. Therefore $\varepsilon_{\widetilde{J}}(-Z^{-1}) = \rho^{-\frac{n}{2}} e^{i\frac{n(\pi-\theta)}{2}}$ for all $0 < \theta < \pi$. Similarly, $\det(-(tZ + I_n)Z^{-1}) = \det(tZ + I_n)\rho^{-n} e^{in(\pi-\theta)}$ so that $\varepsilon_{\widetilde{J}}(-(tZ + I_n)Z^{-1}) = \sqrt{\det(tZ + I_n)}\rho^{-\frac{n}{2}} e^{i\frac{n(\pi-\theta)}{2}}$ for $\pi - \theta$ and ρ sufficiently positively small. It follows that $\varepsilon_{\widetilde{J}}(-(tZ + I_n)Z^{-1})\varepsilon_{\widetilde{J}}(-Z^{-1})^{-1} = \varepsilon_t(Z)$ for all $Z \in \mathfrak{H}_n$. In particular, we see that $n_t^T = \overline{n}_t$.

Thus

$$\phi_+(n_t h_{x,0,0}) = \frac{e^{i\sigma x \frac{\varepsilon_\sigma i}{\varepsilon_\sigma it + I_n} x^T}}{\varepsilon_t(\varepsilon_\sigma i I_n)} = \varepsilon_t(\varepsilon_\sigma i I_n)^{-1} e^{-\sigma x(I_n+it)^{-1}x^T}.$$

Finally, we must show $f_+ \in \mathcal{D}_+$. As $f_+(\cdot, 0)$ is clearly Schwartz when $\sigma > 0$, it remains only to show that f_+ satisfies the system given in Equation (1.1). For the sake of brevity, we will only show $4s\partial_{t_{ii}} f_+ + \partial_{x_i}^2 f_+ = 0$ and omit the similar calculation that $2s\partial_{t_{i,j}} f + \partial_{x_i}\partial_{x_j} f = 0$, $i \neq j$. For $X \in M_n(\mathbb{C})$, write $X_{(i,j)}$ for the (i, j) minor of X. Then

$$\begin{aligned}
\partial_{t_{i,i}} f_+ &= -i\frac{1}{2}\det(I_n + it)^{-1}\det(I_n + it)_{(i,i)} f_+ \\
&\quad + i\sigma x(I_n + it)^{-1} E_{i,i}(I_n + it)^{-1} x^T f_+ \\
&= -i\frac{1}{2}((I_n + it)^{-1})_{i,i} f_+ + i\sigma x(I_n + it)^{-1} E_{i,i}(I_n + it)^{-1} x^T f_+
\end{aligned}$$

while

$$\begin{aligned}
\partial_{x_i}^2 f_+ &= \partial_{x_i}\big(-2\sigma e_i(I_n + it)^{-1} x^T f_+\big) \\
&= -2\sigma e_i(I_n + it)^{-1} e_i^T f_+ + 4\sigma^2\big(e_i(I_n + it)^{-1} x^T\big)^2 f_+ \\
&= -2\sigma\big((I_n + it)^{-1}\big)_{i,i} f_+ + 4\sigma^2 x(I_n + it)^{-1} e_i^T e_i (I_n + it)^{-1} x^T f_+ \\
&= -2\sigma\big((I_n + it)^{-1}\big)_{i,i} f_+ + 4\sigma^2 x(I_n + it)^{-1} E_{i,i}(I_n + it)^{-1} x^T f_+
\end{aligned}$$

which finishes the claim.

Turn now to the second part of the Theorem. Taking conjugates, it follows that $\phi_- = \overline{\phi}_+ \in I(1, -1/2, -\iota\sigma)$, $f_-(\cdot, 0)$ is Schwartz, and f_- satisfies the system given in Equation (1.1) (with σ replaced by $-\sigma$). Renaming σ, the result follows. The calculations for ϕ_α are trivial modifications of the above argument. $\square$

Corollary 5.8. *For $q = -\operatorname{sgn}\sigma$, $\mathcal{D}'_\pm$ is nonzero.*

6. Restriction to $t = 0$

By Theorem 5.2, the map from $\mathcal{D}'$ to $\mathcal{S}(\mathbb{R}^n)$ given by restriction to $t = 0$ is injective. Following this map by the Fourier transform gives the following injective map. Recall that $\mathcal{D}'_\pm$ is nonzero when $q = -\operatorname{sgn}\sigma$ and we assume this is so for the rest of the paper.

Definition 6.1. Let $\mathcal{E} : \mathcal{D}' \to \mathcal{S}(\mathbb{R}^n)$ be given by

$$(\mathcal{E}f)(x) = \widehat{f}(x, 0).$$

We also write $\mathcal{S} = \operatorname{Im}(\mathcal{E})$ and $\mathcal{S}_+$ and $\mathcal{S}_-$ for the images of $\mathcal{D}'_+$ and $\mathcal{D}'_-$, respectively. We make $\mathcal{S}$ into a G-module by requiring $\mathcal{E}$ to be an intertwining isomorphism

$$\mathcal{E} : \mathcal{D}' \to \mathcal{S}.$$

Theorem 6.2. *For $f \in \mathcal{S}$ and $(g,\varepsilon) \in Mp(n)$, $((g,\varepsilon)\cdot f)(x)$ is given by*

(1) *For* $m_{A,a} = \Big(\begin{pmatrix} A & 0 \\ 0 & A^{-1,T} \end{pmatrix}, Z \to a \Big)$ *with* $a^2 = \det A^{-1}$ *(so* $(a|\det A|^{1/2})^2 = \operatorname{sgn}(\det A)$*),*
$$(m_{A,a}\cdot f)(x) = \big(a|\det A|^{1/2}\big)^q |\det A|^{1/2} f(xA).$$

(2) *For* $n_{B,\varepsilon} = \Big(\begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix}, Z \to \varepsilon \Big)$ *with* $\varepsilon^2 = 1$,
$$(n_{B,\varepsilon}\cdot f)(x) = \varepsilon^q e^{-\frac{\pi^2}{s} xBx^T} f(x).$$

(3) *For* $\overline{n}_C = \Big(\begin{pmatrix} I_n & 0 \\ C & I_n \end{pmatrix}, \varepsilon_C(Z) \Big)$,
$$(\overline{n}_C \cdot f)(x) = \big(e^{-s(\cdot)C(\cdot)^T} f^\vee(\cdot)\big)^\wedge(x) = (\widehat{e^{-s(\cdot)C(\cdot)^T}} * f)(x).$$

(4) *Let* $\omega = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ *and* $\varepsilon_\omega(Z)$ *satisfy* $\varepsilon_\omega(Z)^2 = \det(Z)$ *with* $\varepsilon_\omega((\lambda + i\mu)I_n) = \sqrt{(\lambda+i\mu)^n}$ *for* $\lambda,\mu \in \mathbb{R}^+$ *with* $\arctan(\frac{\mu}{\lambda}) < \frac{\pi}{n}$. *Then*
$$((\omega,\varepsilon_\omega)\cdot f)(x) = e^{-\varepsilon_\sigma \frac{i\pi n}{4}} |\frac{\pi}{\sigma}|^{n/2} \widehat{f}(\frac{\pi}{\sigma}x).$$

Proof. For $f \in \mathcal{S}$ and $(g,\varepsilon) \in Mp(n)$,
$$\big((g,\varepsilon)\cdot f\big)(x) = \mathcal{E}((g,\varepsilon)\cdot(\mathcal{E}^{-1}(f)))(x) = ((g,\varepsilon)\cdot(\mathcal{E}^{-1}(f)))^\wedge(x,0).$$
Since
$$(\mathcal{E}^{-1}(f))(x,t) = \int_{\mathbb{R}^n} f(\xi) e^{\frac{\pi^2}{s}\xi t \xi^T} e^{2\pi i \xi x^T}\, d\xi,$$
we use Theorem 4.1 to calculate the new action.

In the first case, $(m_{A,a}\cdot f)(x,t) = i^{lq}|\det A|^r f(xA^{-1,T}, A^{-1}tA^{-1,T})$ with $i^l = a|\det A|^{1/2}$. Therefore
$$\begin{aligned} (m_{A,a}\cdot(\mathcal{E}^{-1}(f)))^\vee(x,0) &= i^{lq}|\det A|^r (\mathcal{E}^{-1}(f))(xA^{-1,T},0) \\ &= i^{lq}|\det A|^r f^\vee(xA^{-1,T}) \end{aligned}$$
so that
$$(m_{A,a}\cdot(\mathcal{E}^{-1}(f)))(x,0) = i^{lq}|\det A|^{r+1} f(xA).$$
In the second case, $(n_B\cdot f)(x,t) = \varepsilon^q f(x, t-B)$ so
$$\begin{aligned} (n_B\cdot(\mathcal{E}^{-1}(f)))^\vee(x,0) &= \varepsilon^q (\mathcal{E}^{-1}(f))(x,-B) \\ &= \varepsilon^q \int_{\mathbb{R}^n} f(\xi) e^{-\frac{\pi^2}{s}\xi B\xi^T} e^{2\pi i\xi x^T}\, d\xi \end{aligned}$$
so that
$$(n_B\cdot(\mathcal{E}^{-1}(f)))(x,0) = \varepsilon^q e^{-\frac{\pi^2}{s}xBx^T} f(x).$$
For the third case,
$$\begin{aligned} (\overline{n}_C\cdot f)(x,t) &= i^{lq}|\det(I_n - tC)|^r e^{-sxC(I_n-tC)^{-1}x^T} \\ &\quad\times f(x(-Ct+I_n)^{-1}, (I_n-tC)^{-1}t) \end{aligned}$$
with $i^l|\det(I_n-tC)|^{-\frac12} = \sqrt{\det(I_n-tC)^{-1}}$ for small t. Therefore
$$(\overline{n}_C\cdot(\mathcal{E}^{-1}(f)))^\vee(x,0) = e^{-sxCx^T}(\mathcal{E}^{-1}(f))(x,0)$$

$$= e^{-sxCx^T} \int_{\mathbb{R}^n} f(\xi) e^{2\pi i \xi x^T} \, d\xi$$

so

$$\begin{aligned}(\overline{n}_C \cdot (\mathcal{E}^{-1}(f)))(x,0) &= (e^{-s(\cdot)C(\cdot)^T} f^\vee(\cdot))^\wedge(x) \\ &= (\widehat{e^{-s(\cdot)C(\cdot)^T}} * f)(x).\end{aligned}$$

Finally, when t is invertible,

$$((\omega,\varepsilon_\omega)\cdot f)(x,t) = i^{lq} |\det t|^r e^{sxt^{-1}x^T} f(-xt^{-1}, -t^{-1})$$

where $\varepsilon_\omega(-t^{-1}) = i^l |\det t|^{-1/2}$. In the case of $t = \lambda I_n$ with $\lambda < 0$,

$$\varepsilon_\omega(-t^{-1}) = \lim_{\mu\to 0^+} \varepsilon_\omega((-\lambda^{-1} + i\mu)I_n) = \sqrt{(-\lambda^{-1}+i\mu)^n} = |\lambda|^{-n/2}$$

so that $i^l = 1$ and $((\omega,\varepsilon_\omega)\cdot f)(x,\lambda I_n) = |\lambda|^{nr} e^{s\lambda^{-1}\|x\|^2} f(-\lambda^{-1}x, -\lambda^{-1}I_n)$. We now will calculate the action of $(\omega,\varepsilon_\omega)$ on $\mathcal{S}(\mathbb{R}^n)$ using

$$\begin{aligned}((\omega,\varepsilon_\omega)\cdot f)(x) &= ((\omega,\varepsilon_\omega)\cdot(\mathcal{E}^{-1}(f)))^\wedge(x,0) \\ &= \lim_{\lambda\to 0^-} ((\omega,\varepsilon_\omega)\cdot(\mathcal{E}^{-1}(f)))^\wedge(x,\lambda I_n).\end{aligned}$$

Now

$$((\omega,\varepsilon_\omega)\cdot(\mathcal{E}^{-1}(f)))^\vee(x,\lambda I_n) = |\lambda|^{nr} e^{s\lambda^{-1}\|x\|^2} (\mathcal{E}^{-1}(f))(-\lambda^{-1}x, -\lambda^{-1}I_n).$$

We first rewrite $(\mathcal{E}^{-1}(f))(w, -\lambda^{-1}I_n)$ using the identity

$$\int_{\mathbb{R}^n} e^{-2\pi i \xi x^T} e^{-\pi\alpha\|\xi\|^2} \, d\xi = \alpha^{-n/2} e^{-\frac{\pi}{\alpha}\|x\|^2}$$

for $\operatorname{Re}\alpha > 0$. We get (taking $\alpha = \varepsilon + \pi/(s\lambda)$), using Dominated Convergence and Fubini,

$$\begin{aligned}(\mathcal{E}^{-1}(f))(w,-\lambda^{-1}I_n) &= \int_{\mathbb{R}^n} f(\xi) e^{-\frac{\pi^2}{s\lambda}\|\xi\|^2} e^{2\pi i \xi w^T} \, d\xi \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widehat{f}(y) e^{2\pi i \xi y^T} e^{-\frac{\pi^2}{s\lambda}\|\xi\|^2} e^{2\pi i \xi w^T} \, dy d\xi \\ &= \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widehat{f}(y) e^{-\pi(\varepsilon+\pi/s\lambda)\|\xi\|^2} e^{2\pi i \xi (y+w)^T} \, dy d\xi \\ &= \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widehat{f}(y) e^{-\pi(\varepsilon+\pi/s\lambda)\|\xi\|^2} e^{-2\pi i \xi (-y-w)^T} \, d\xi dy \\ &= \lim_{\epsilon\to 0^+} (\varepsilon+\pi/s\lambda)^{-n/2} \int_{\mathbb{R}^n} \widehat{f}(y) e^{-\frac{\pi}{\varepsilon+\pi/s\lambda}\|y+w\|^2} \, dy.\end{aligned}$$

Now write $s = i\sigma$ (and recall $\lambda < 0$) so that analytic continuation of $\alpha^{-n/2}$ on $\mathbb{R}^+$ gives

$$\lim_{\epsilon\to 0^+} (\varepsilon+\pi/s\lambda)^{-n/2} = \begin{cases} |\frac{\pi}{s\lambda}|^{-\frac{n}{2}} e^{-\frac{i\pi n}{4}}, & \sigma > 0 \\ |\frac{\pi}{s\lambda}|^{-\frac{n}{2}} e^{\frac{i\pi n}{4}}, & \sigma < 0. \end{cases}$$

Thus

$$(\mathcal{E}^{-1}(f))(w,-\lambda^{-1}I_n) = |\frac{\pi}{s\lambda}|^{-\frac{n}{2}} e^{-\varepsilon_\sigma \frac{i\pi n}{4}} \int_{\mathbb{R}^n} \widehat{f}(y) e^{-s\lambda\|y+w\|^2} \, dy.$$

Therefore,

$$((\omega,\varepsilon_\omega)\cdot(\mathcal{E}^{-1}(f)))^\vee(x,\lambda I_n)$$

$$
\begin{aligned}
&= |\lambda|^{nr} e^{s\lambda^{-1}\|x\|^2} \, (\mathcal{E}^{-1}(f))(-\lambda^{-1}x, -\lambda^{-1}I_n) \\
&= |\lambda|^{nr} |\frac{\pi}{s\lambda}|^{-\frac{n}{2}} e^{-\varepsilon_\sigma \frac{i\pi n}{4}} e^{s\lambda^{-1}\|x\|^2} \int_{\mathbb{R}^n} \widehat{f}(y) e^{-s\lambda\|y-\lambda^{-1}x\|^2} \, dy \\
&= |\frac{\pi}{s}|^{-\frac{n}{2}} e^{-\varepsilon_\sigma \frac{i\pi n}{4}} e^{s\lambda^{-1}\|x\|^2} \int_{\mathbb{R}^n} \widehat{f}(y) e^{-s\lambda\|y-\lambda^{-1}x\|^2} \, dy \\
&= |\frac{\pi}{s}|^{-\frac{n}{2}} e^{-\varepsilon_\sigma \frac{i\pi n}{4}} \int_{\mathbb{R}^n} \widehat{f}(y) e^{-s\lambda\|y\|^2} e^{2syx^T} \, dy \\
&= |\frac{\pi}{\sigma}|^{-\frac{n}{2}} e^{-\varepsilon_\sigma \frac{i\pi n}{4}} \int_{\mathbb{R}^n} \widehat{f}(y) e^{-s\lambda\|y\|^2} e^{2\pi i \frac{\sigma}{\pi} yx^T} \, dy \\
&= |\frac{\pi}{\sigma}|^{n/2} e^{-\varepsilon_\sigma \frac{i\pi n}{4}} \int_{\mathbb{R}^n} \widehat{f}(\frac{\pi}{\sigma}y) e^{-\frac{i\lambda\pi^2}{\sigma}\|y\|^2} e^{2\pi i yx^T} \, dy \\
&= |\frac{\pi}{\sigma}|^{-\frac{n}{2}} e^{-\varepsilon_\sigma \frac{i\pi n}{4}} \int_{\mathbb{R}^n} \widehat{f \circ M_{\frac{\sigma}{\pi}}}(y) e^{-\frac{i\lambda\pi^2}{\sigma}\|y\|^2} e^{2\pi i yx^T} \, dy
\end{aligned}
$$

where $M_{\sigma/\pi}$ is the multiplication map given by $M_{\sigma/\pi}(x) = \sigma x/\pi$. As a result,

$$
\begin{aligned}
&((\omega, \varepsilon_\omega) \cdot f)(x) \\
&= \lim_{\lambda \to 0^-} ((\omega, \varepsilon_\omega) \cdot (\mathcal{E}^{-1}(f)))^\wedge(x, \lambda I_n) \\
&= \lim_{\lambda \to 0^-} \int_{\mathbb{R}^n} ((\omega, \varepsilon_\omega) \cdot (\mathcal{E}^{-1}(f)))(\xi, \lambda I_n) e^{-2\pi i \xi x^T} \, d\xi \\
&= e^{-\varepsilon_\sigma \frac{i\pi n}{4}} |\frac{\pi}{\sigma}|^{-\frac{n}{2}} \lim_{\lambda \to 0^-} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widehat{f \circ M_{\frac{\sigma}{\pi}}}(y) e^{-s\lambda\|y\|^2} e^{2sy\xi^T} e^{-2\pi i \xi x^T} \, dy d\xi \\
&= e^{-\varepsilon_\sigma \frac{i\pi n}{4}} |\frac{\pi}{\sigma}|^{-\frac{n}{2}} \lim_{\lambda \to 0^-} \widehat{f \circ M_{\frac{\sigma}{\pi}}}(x) e^{-s\lambda\|x\|^2} \\
&= e^{-\varepsilon_\sigma \frac{i\pi n}{4}} |\frac{\pi}{\sigma}|^{-\frac{n}{2}} \widehat{f \circ M_{\frac{\sigma}{\pi}}}(x) \\
&= e^{-\varepsilon_\sigma \frac{i\pi n}{4}} |\frac{\pi}{\sigma}|^{n/2} \widehat{f}(\frac{\pi}{\sigma}x).
\end{aligned}
$$

□

To match these formulas with the realization of the oscillator representation in, say, Kashiwara and Vergne, consider the dilation operator defined by

$$
(Tf)(x) = f(\frac{|\sigma|^{1/2}}{\pi\sqrt{2}}x).
$$

Making T into an intertwining map, Theorem 6.2 gives an equivalent action on $T(\mathcal{S}) \subseteq \mathcal{S}(\mathbb{R}^n)$. Note, of course, that the map T can be modified by multiplying by the scalar $(|\sigma|^{1/2}/(\pi\sqrt{2}))^{n/2}$ to make it a unitary map with respect to $L^2(\mathbb{R}^n)$. This modification will not change the theorem below.

Theorem 6.3. *The action of $Mp(n)$ on $T(\mathcal{S})$ is given by*

$$
\begin{gathered}
(m_{A,a} \cdot f)(x) = |\det A|^{1/2} \, f(xA), \ \textit{for } a > 0 \\
(n_B \cdot f)(x) = e^{\varepsilon_\sigma \frac{i}{2} xBx^T} \, f(x), \\
(\overline{n}_C \cdot f)(x) = (e^{-\varepsilon_\sigma \widehat{2i\pi^2(\cdot)C(\cdot)^T}} * f)(x) \\
((\omega, \varepsilon_\omega) \cdot f)(x) = e^{-\varepsilon_\sigma \frac{i\pi n}{4}} (\frac{1}{2\pi})^{n/2} \int_{\mathbb{R}^n} f(\xi) e^{-\varepsilon_\sigma i\xi x^T} \, d\xi.
\end{gathered}
$$

In particular, when $s = i\sigma$ with $\sigma < 0$, this is a dense $Mp(n)$-invariant subspace in the oscillator representation. When $\sigma > 0$, this representation is isomorphic to the dual to the oscillator representation.

In either case, this action completes to a unitary representation on $L^2(\mathbb{R}^n)$ and decomposes as a direct sum of irreducible representation via the set of odd and even function,

$$L^2(\mathbb{R}^n) = L^2(\mathbb{R}^n)_+ \oplus L^2(\mathbb{R}^n)_-.$$

Proof. For $a > 0$,

$$\begin{aligned}
(m_{A,a} \cdot f)(x) &= (T(m_{A,a} \cdot T^{-1}f))(x) \\
&= (m_{A,a} \cdot T^{-1}f)(\frac{|\sigma|^{1/2}}{\pi\sqrt{2}}x) \\
&= |\det A|^{1/2}(T^{-1}f)(\frac{|\sigma|^{1/2}}{\pi\sqrt{2}}xA) \\
&= |\det A|^{1/2}\, f(xA),
\end{aligned}$$

and

$$\begin{aligned}
(n_B \cdot f)(x) &= (T(n_B \cdot T^{-1}f))(x) \\
&= (n_B \cdot T^{-1}f)(\frac{|\sigma|^{1/2}}{\pi\sqrt{2}}x) \\
&= e^{-\frac{\pi^2}{i\sigma}\frac{|\sigma|}{2\pi^2}xBx^T}\,(T^{-1}f)(\frac{|\sigma|^{1/2}}{\pi\sqrt{2}}x) \\
&= e^{(\varepsilon_\sigma)\frac{i}{2}xBx^T}\, f(x),
\end{aligned}$$

and

$$\begin{aligned}
(\overline{n}_C \cdot f)(x) &= (T(\overline{n}_C \cdot T^{-1}f))(x) \\
&= (\overline{n}_C \cdot T^{-1}f)(\frac{|\sigma|^{1/2}}{\pi\sqrt{2}}x) \\
&= (e^{-\widehat{s(\cdot)C(\cdot)^T}} * T^{-1}f)(x)(\frac{|\sigma|^{1/2}}{\pi\sqrt{2}}x) \\
&= (\frac{|\sigma|^{1/2}}{\pi\sqrt{2}})^n(Te^{-\widehat{s(\cdot)C(\cdot)^T}} * f)(x) \\
&= (T^{-1}\widehat{e^{-s(\cdot)}C(\cdot)^T} * f)(x) \\
&= (e^{-\widehat{\frac{2\pi^2 s}{|\sigma|}(\cdot)C(\cdot)^T}} * f)(x)
\end{aligned}$$

and

$$\begin{aligned}
((\omega,\varepsilon_\omega) \cdot f)(x) &= (T((\omega,\varepsilon_\omega) \cdot T^{-1}f))(x) \\
&= ((\omega,\varepsilon_\omega) \cdot T^{-1}f)(\frac{|\sigma|^{1/2}}{\pi\sqrt{2}}x) \\
&= e^{-\varepsilon_\sigma\frac{i\pi n}{4}}|\frac{\pi}{\sigma}|^{n/2} f \circ \widehat{M_{\frac{\pi\sqrt{2}}{|\sigma|^{1/2}}}}(\frac{\pi}{\sigma}\frac{|\sigma|^{1/2}}{\pi\sqrt{2}}x) \\
&= e^{-\varepsilon_\sigma\frac{i\pi n}{4}}|\frac{\pi}{\sigma}|^{n/2}|\frac{|\sigma|^{1/2}}{\pi\sqrt{2}}|^n \widehat{f}(\frac{\pi}{\sigma}\frac{|\sigma|}{2\pi^2}x)
\end{aligned}$$

$$\begin{aligned}&= e^{-\varepsilon_\sigma \frac{i\pi n}{4}}(\frac{1}{2\pi})^{n/2}\widehat{f}(\frac{\varepsilon_\sigma}{2\pi}x)\\&= e^{-\varepsilon_\sigma \frac{i\pi n}{4}}(\frac{1}{2\pi})^{n/2}\int_{\mathbb{R}^n} f(\xi)e^{-\varepsilon_\sigma i\xi x^T}\,d\xi.\end{aligned}$$

□

7. Restriction to $x = 0$

Recall from Corollary 4.4 that there is an $Mp(n)$-intertwining map $\mathcal{G} : I'(q, r, s) \to I'(q, r)$ given by

$$(\mathcal{G}f)(t) = f(0, t)$$

and an intertwining map $\mathcal{G}_n : I'(q, r, s) \to I'_n(q, r - \frac{1}{n})$ given by

$$(\mathcal{G}_n f)(t) = \nabla f(0, t).$$

By the definitions and Theorem 5.2, restricting to $\mathcal{D}'$ and pre-composing with $\mathcal{E}^{-1}$ gives $Mp(n)$-maps $\mathcal{H} : \mathcal{S} \to I'(q, r)$ and $\mathcal{H}_n : \mathcal{S} \to I'_n(q, r - \frac{1}{n})$ given by

$$(\mathcal{H}f)(t) = \int_{\mathbb{R}^n} f(\xi)e^{\frac{\pi^2}{s}\xi t\xi^T}\,d\xi$$

and

$$\begin{aligned}(\mathcal{H}_n f)(t) &= \nabla\Big(\int_{\mathbb{R}^n} f(\xi)e^{\frac{\pi^2}{s}\xi t\xi^T}e^{2\pi i\xi x^T}\,d\xi\Big)\Big|_{x=0}\\&= 2\pi i\Big(\int_{\mathbb{R}^n}\xi_1 f(\xi)e^{\frac{\pi^2}{s}\xi t\xi^T}\,d\xi, \dots, \int_{\mathbb{R}^n}\xi_n f(\xi)e^{\frac{\pi^2}{s}\xi t\xi^T}\,d\xi\Big).\end{aligned}$$

Clearly $\mathcal{S}_- \subseteq \ker\mathcal{H}$ and $\mathcal{S}_+ \subseteq \ker\mathcal{H}_n$ (equivalently, $\mathcal{D}'_- \subseteq \ker\mathcal{G}$ and $\mathcal{D}'_+ \subseteq \ker\mathcal{G}_n$). To show these are the entire kernels involves inverting $\mathcal{H}|_{\mathcal{S}_+}$ and $\mathcal{H}_n|_{\mathcal{S}_-}$ (equivalently, $\mathcal{G}|_{\mathcal{D}'_+}$ and $\mathcal{G}_n|_{\mathcal{D}'_-}$). Straightforward Fourier analysis requires a bit more care due to the fact that the images usually do not have sufficient decay properties to be L^1 or L^2 functions (unless $n = 1$, see [23]). In fact, if we could view $f \in \mathcal{D}' \subseteq I'(q, r, s)$ as a tempered distribution $f(x, \cdot) \in \mathcal{S}'(\mathrm{Sym}(n, \mathbb{R}) \cong \mathbb{R}^{n(n+1)/2})$ and writing $\mathcal{F}$ for the Fourier transform on $\mathcal{S}(\mathrm{Sym}(n, \mathbb{R}))$ given by

$$(\mathcal{F}f)(\tau) = \int_{\mathrm{Sym}(n,\mathbb{R})} f(t)e^{-2\pi i\,\mathrm{tr}(t\tau)}\,dt,$$

we would have

$$\begin{gathered}8\pi is\tau_{i,j}\mathcal{F}f + \partial_{x_i}\partial_{x_j}\mathcal{F}f = 0, \quad i \neq j,\\8\pi is\tau_{i,i}\mathcal{F}f + \partial^2_{x_i}\mathcal{F}f = 0.\end{gathered}$$

Looking at $\partial^2_{x_i}\partial^2_{x_j}\mathcal{F}f$ written in two ways for $i \neq j$, we would get

$$(\tau_{i,i}\tau_{j,j} - \tau^2_{i,j})\mathcal{F}f = 0$$

so that $\mathcal{F}f$ would be supported on $\{\tau \in \mathrm{Sym}(n, \mathbb{R}) : \tau_{i,i}\tau_{j,j} = \tau^2_{i,j} \text{ all } i \neq j\}$. This is, of course a rank of at most one condition on $\mathrm{Sym}(n, \mathbb{R})$. As a result, it will be useful to consider the cone defined by the function $\theta : \mathbb{R}^n \to \mathrm{Sym}(n, \mathbb{R})$ given by

$$\theta(y) = \frac{\pi}{2\sigma}\,y^T y.$$

Lemma 7.1. *(1) For $f \in \mathcal{D}' \subseteq I'(q,r,s)$ and each $x \in \mathbb{R}^n$, $f(x,\cdot)$ may be viewed as a tempered distribution on* $\mathrm{Sym}(n,\mathbb{R})$ *given by*

$$\langle f(x,\cdot),\phi\rangle = \int_{\mathrm{Sym}(n,\mathbb{R})} f(x,t)\phi(t)\,dt$$

for each $\phi \in \mathcal{S}(\mathrm{Sym}(n,\mathbb{R}))$. Its Fourier transform $\mathcal{F}f(x,\cdot) \in \mathcal{S}'(\mathrm{Sym}(n,\mathbb{R}))$ is given by

$$\langle \mathcal{F}f(x,\cdot),\phi\rangle = \int_{\mathbb{R}^n} \widehat{f}(\xi,0)(\phi\circ\theta)(\xi)e^{2\pi i\xi x^T}\,d\xi = (f(\cdot,0)*(\phi\circ\theta))(x)$$

and is supported on $\mathrm{Im}\,\theta$.

(2) For each $1 \le j \le n$, $\partial_{x_j} f(x,\cdot)$ may be viewed as a tempered distribution on $\mathrm{Sym}(n,\mathbb{R})$ *given by*

$$\langle \partial_{x_j} f(x,\cdot),\phi\rangle = \int_{\mathrm{Sym}(n,\mathbb{R})} \partial_{x_j} f(x,t)\phi(t)\,dt$$

for each $\phi \in \mathcal{S}(\mathrm{Sym}(n,\mathbb{R}))$. Its Fourier transform $\mathcal{F}(\partial_{x_j} f)(x,\cdot) \in \mathcal{S}'(\mathrm{Sym}(n,\mathbb{R}))$ is given by

$$\begin{aligned}\langle \mathcal{F}(\partial_{x_j} f)(x,\cdot),\phi\rangle &= 2\pi i\int_{\mathbb{R}^n} \xi_j\widehat{f}(\xi,0)(\phi\circ\theta)(\xi)e^{2\pi i\xi x^T}\,d\xi\\ &= 2\pi i((\partial_{x_j} f)(\cdot,0)*(\phi\circ\theta))(x)\end{aligned}$$

and is supported on $\mathrm{Im}\,\theta$.

Proof. First of all, since

$$|f(x,t)| \le \int_{\mathbb{R}^n} |\widehat{f}(\xi,0)e^{\frac{\pi^2}{s}\xi t\xi^T}e^{2\pi i\xi x^T}|\,d\xi = \|\widehat{f}(\cdot,0)\|_{L^1(\mathbb{R}^n)} < \infty,$$

$f(x,\cdot)$ is bounded. As it is also continuous, it is clearly locally integrable and therefore gives rise to an element of $\mathcal{S}'(\mathrm{Sym}(n,\mathbb{R}))$. To calculate its Fourier transform, use Fubini to see that

$$\begin{aligned}\langle \mathcal{F}f(x,\cdot),\phi\rangle &= \langle f(x,\cdot),\mathcal{F}\phi\rangle\\ &= \int_{\mathrm{Sym}(n,\mathbb{R})} f(x,t)\mathcal{F}\phi(t)\,dt\\ &= \int_{\mathrm{Sym}(n,\mathbb{R})}\int_{\mathbb{R}^n} \widehat{f}(\xi,0)e^{\frac{\pi^2}{s}\xi t\xi^T}e^{2\pi i\xi x^T}\mathcal{F}\phi(t)\,d\xi dt\\ &= \int_{\mathbb{R}^n}\int_{\mathrm{Sym}(n,\mathbb{R})} \widehat{f}(\xi,0)e^{2\pi i\xi x^T}\mathcal{F}\phi(t)e^{\frac{\pi^2}{s}\xi t\xi^T}\,dtd\xi\\ &= \int_{\mathbb{R}^n}\int_{\mathrm{Sym}(n,\mathbb{R})} \widehat{f}(\xi,0)e^{2\pi i\xi x^T}\mathcal{F}\phi(t)e^{2\pi i(-\frac{\pi}{2\sigma})\,\mathrm{tr}(t\xi^T\xi)}\,dtd\xi\\ &= \int_{\mathbb{R}^n} \widehat{f}(\xi,0)e^{2\pi i\xi x^T}\mathcal{F}^2\phi(-\frac{\pi}{2\sigma}\xi^T\xi)\,d\xi\\ &= \int_{\mathbb{R}^n} \widehat{f}(\xi,0)e^{2\pi i\xi x^T}\phi(\theta(\xi))\,d\xi.\end{aligned}$$

Finally,

$$\langle \mathcal{F}f(x,\cdot),\phi\rangle = \int_{\mathbb{R}^n} \widehat{f}(\xi,0)(\phi\circ\theta)(\xi)e^{2\pi i\xi x^T}\,d\xi$$

$$= (\widehat{f}(\cdot,0)(\phi\circ\theta)(\cdot))^{\vee}(x)$$
$$= (f(\cdot,0)*(\phi\circ\theta))(x).$$

Turning to $\partial_{x_j} f$,

$$|\partial_{x_j} f(x,t)| \le \int_{\mathbb{R}^n} |2\pi i\xi_j \widehat{f}(\xi,0) e^{\frac{\pi^2}{s}\xi t\xi^T} e^{2\pi i \xi x^T}|\, d\xi = 2\pi \|(\cdot)_j \widehat{f}(\cdot,0)\|_{L^1(\mathbb{R}^n)} < \infty$$

so that $\partial_{x_j} f(x,\cdot)$ gives rise to an element of $\mathcal{S}'(\mathrm{Sym}(n,\mathbb{R}))$. The rest of the Lemma is a simple modification of the above argument and is omitted. □

Theorem 7.2. *$\mathcal{H}|_{\mathcal{S}_+}$ is injective and $\mathcal{H}_n|_{\mathcal{S}_-}$ is injective. Equivalently, $\mathcal{G}|_{\mathcal{D}'_+}$ is injective and $\mathcal{G}_n|_{\mathcal{D}'_-}$ is injective.*

Proof. We show how to construct the inverse maps. Let $f \in \mathcal{S}$. By the definitions and Lemma 7.1,

$$\langle \mathcal{F}\mathcal{H}f, \phi\rangle = (f^{\vee} * (\phi\circ\theta))(0)$$

for $\phi \in \mathcal{S}(\mathrm{Sym}(n,\mathbb{R}))$. Fix $\psi \in \mathcal{S}(\mathrm{Sym}(n,\mathbb{R}))$ with $\int_{\mathrm{Sym}(n,\mathbb{R})} \psi(t)\, dt = 1$ and let $\psi_\epsilon(t) = \varepsilon^{-n(n+1)/2}\psi(\varepsilon^{-1}t)$ for $\varepsilon > 0$ so that $\psi_\epsilon \to \delta_0$ as an element of $\mathcal{S}'(\mathrm{Sym}(n,\mathbb{R}))$ as $\epsilon \to 0^+$. Then, for any $x \in \mathbb{R}^n$, $\tau_{\theta(x)}\psi_\epsilon \to \delta_{\theta(x)}$ as $\epsilon \to 0^+$. As $\theta(y) = \frac{\pi}{2\sigma} y^T y$, it is trivial to check that $(\tau_{\theta(x)}\psi_\epsilon)\circ\theta \to \delta_x + \delta_{-x}$ as elements of $\mathcal{S}'(\mathbb{R}^n)$ as $\epsilon \to 0^+$. If $f \in \mathcal{S}_+$, then

$$\lim_{\epsilon\to 0^+} \langle \mathcal{F}\mathcal{H}f, \tau_{\theta(x)}\psi_\epsilon\rangle = \lim_{\epsilon\to 0^+} (f^{\vee} * ((\tau_{\theta(x)}\psi_\epsilon)\circ\theta))(0) = f^{\vee}(x) + f^{\vee}(-x) = 2f^{\vee}(x).$$

In particular, $f^{\vee} \in \mathcal{S}_+$ (and therefore f) can be recovered from $\mathcal{H}f$ by taking the Fourier transform and looking at approximations to translations of the delta distribution.

Next, view the image of $\mathcal{H}_n$ as landing in $\oplus_{j=1}^n \mathcal{S}'(\mathrm{Sym}(n,\mathbb{R}))$. Evaluating via the diagonal map (so viewing the image as landing in $\mathcal{S}'(\mathrm{Sym}(n,\mathbb{R}),\mathbb{R}^n)$) and applying the Fourier transform in each coordinate, it follows that

$$\langle \mathcal{F}\mathcal{H}_n f, \phi\rangle = 2\pi i((\partial_{x_1} f^{\vee} * (\phi\circ\theta))(0), \dots, (\partial_{x_n} f^{\vee} * (\phi\circ\theta))(0)).$$

As above, when $f \in \mathcal{S}_-$,

$$\lim_{\epsilon\to 0^+} \langle \mathcal{F}\mathcal{H}_n f^{\vee}, \tau_{\theta(x)}\psi_\epsilon\rangle = 4\pi i(\partial_{x_1} f^{\vee}(x), \dots, \partial_{x_n} f^{\vee}(x)).$$

In particular $f^{\vee} \in \mathcal{S}(\mathbb{R}^n)_-$ (and therefore $f^{\vee}$) can also be recovered from $\mathcal{H}_n f$ by taking the Fourier transform and looking at approximations to translations of the delta distribution. □

Definition 7.3. Let $\mathcal{I}'_\pm$ be the image of $\mathcal{D}'_\pm$ under $\mathcal{G}$ and $\mathcal{G}_n$, respectively (alternately, the image of $\mathcal{S}_\pm$ under $\mathcal{H}$ and $\mathcal{H}_n$, respectively).

From Corollary 4.4 and Theorem 7.2, we see $\mathcal{I}'_\pm$ is isomorphic to $\mathcal{D}'_\pm$ (and $\mathcal{S}_\pm$) as $Mp(n)$-representations. In particular, they complete to unitary highest ($\sigma < 0$) or lowest ($\sigma > 0$) weight representations isomorphic to the oscillator representation or its dual.

The next corollary identifies $\mathcal{I}'_\pm$ by viewing the Schwartz space as tempered distributions supported on $\mathrm{Im}\,\theta$, taking their Fourier transform, and implicitly identifying the resulting tempered distribution with the smooth function it generates.

Corollary 7.4. *(1) Embed $\mathcal{S} \hookrightarrow \mathcal{S}'(\mathrm{Sym}(n,\mathbb{R}))$ via θ by mapping $\psi \to \langle \psi, \cdot \rangle$ where*

$$\langle \psi, \phi \rangle = \int_{\mathbb{R}^n} \psi(\xi)(\phi \circ \theta)(\xi)\, d\xi$$

for $\phi \in \mathcal{S}(\mathrm{Sym}(n,\mathbb{R}))$. Then $\mathcal{I}'_+ \subseteq I'(q,r)$ is given explicitly by

$$\mathcal{I}'_+ = \Big\{ \mathcal{F}\psi : \psi \in \mathcal{S} \subseteq \mathcal{S}'(\mathrm{Sym}(n,\mathbb{R})) \Big\}.$$

(2) Embed $\mathcal{S} \hookrightarrow \mathcal{S}'(\mathrm{Sym}(n,\mathbb{R}),\mathbb{R}^n)$ via θ by mapping $\psi \to \langle \psi, \cdot \rangle$ where

$$\langle \psi, \phi \rangle = \Big(\int_{\mathbb{R}^n} \xi_1 \psi(\xi)(\phi \circ \theta)(\xi)\, d\xi, \dots, \int_{\mathbb{R}^n} \xi_n \psi(\xi)(\phi \circ \theta)(\xi)\, d\xi \Big)$$

for $\phi \in \mathcal{S}(\mathrm{Sym}(n,\mathbb{R}))$. Then $\mathcal{I}'_- \subseteq I'_n(q, r - \frac{1}{n})$ is given explicitly by

$$\mathcal{I}'_- = \Big\{ \mathcal{F}\psi : \psi \in \mathcal{S} \subseteq \mathcal{S}'(\mathrm{Sym}(n,\mathbb{R}),\mathbb{R}^n) \Big\}.$$

Proof. Part (1) follows immediately from the formula $\langle \mathcal{F}\mathcal{H}f, \phi \rangle = \int_{\mathbb{R}^n} f(\xi)(\phi \circ \theta)(\xi)\, d\xi$ and Lemma 7.1 and Theorem 7.2. Similarly, part (2) follows from the formula $\langle \mathcal{F}\mathcal{H}_n f, \phi \rangle = (\int_{\mathbb{R}^n} \xi_1 \psi(\xi)(\phi \circ \theta)(\xi)\, d\xi, \dots, \int_{\mathbb{R}^n} \xi_n \psi(\xi)(\phi \circ \theta)(\xi)\, d\xi)$. □

8. K-finite Vectors

If $M \in M_n(\mathbb{C})$ and p is a complex valued polynomial on $\mathbb{R}^n$, define $\widetilde{p}(x,M)$ by

$$\widetilde{p}(x,M) = e^{|\sigma| x M x^T} p(\partial_x)\Big(e^{-|\sigma| x M x^T} \Big)$$

with $p(\partial_x)$ representing the constant coefficient differential operator obtained by replacing x_j by ∂_{x_j}. For p of the form x^α, $\widetilde{p}$ defines a generalization of the Hermite polynomials.

Theorem 8.1. *The highest ($\sigma < 0$) and lowest ($\sigma > 0$) K-finite vector of $(\mathcal{D}'_+)_K$, up to a constant multiple, is given by the function f_- and f_+, respectively (see Theorem 5.7).*

The highest and lowest K-type vectors of $(\mathcal{D}'_-)_K$ consist of the functions $f_{-,a}$ and $f_{+,a}$, respectively, for $a \in \mathbb{C}^n$.

In general, the K-finite vectors in $\mathcal{D}'$ consists of the functions $f_{-,p}$ and $f_{+,p}$ where

$$\begin{aligned} f_{-,p}(x,t) &= \overline{\varepsilon_t(iI_n)}^{-1} \widetilde{p}(x, (I_n - it)^{-1}) e^{\sigma x (I_n - it)^{-1} x^T} \\ f_{+,p}(x,t) &= \varepsilon_t(iI_n)^{-1} \widetilde{p}(x, (I_n + it)^{-1}) e^{-\sigma x (I_n + it)^{-1} x^T} \end{aligned}$$

where p is a complex valued polynomial on $\mathbb{R}^n$.

Proof. It is well known that the K-finite vectors in the oscillator representation (see, e.g., [18] or [14]) are spanned by functions of the form $p(x)e^{-\|x\|^2/2}$ with p a polynomial on $\mathbb{R}^n$. Pulling back this standard picture by $Tf = M_{|\sigma|^{1/2}/\pi\sqrt{2}} f$, we see that the K-finite vectors in the image of $\mathcal{E}$, $\mathcal{S}(\mathbb{R}^n)_K$, are spanned by functions of the form $p(x)e^{-\frac{\pi^2}{|\sigma|}\|x\|^2}$ (a different p of the same degree). Pulling these functions back to $\mathcal{D}'$ involves solving a system of partial differential equations with initial condition at $t = 0$ given by the inverse Fourier transform of $p(x)e^{-\frac{\pi^2}{|\sigma|}\|x\|^2}$, that is, functions of the form $\widetilde{p}(x)e^{-|\sigma|\|x\|^2}$ for some polynomial $\widetilde{p}$ determined by p. By Theorem 5.2, the solution of this system is given by

$$\begin{aligned}
f(x,t) &= \int_{\mathbb{R}^n} p(\xi)e^{-\frac{\pi^2}{|\sigma|}\|\xi\|^2} e^{\frac{\pi^2}{s}\xi t\xi^T} e^{2\pi i\xi x^T}\, d\xi \\
&= \int_{\mathbb{R}^n} p(\xi)e^{-\frac{\pi^2}{|\sigma|}\xi(1+i\varepsilon_\sigma t)\xi^T} e^{2\pi i\xi x^T}\, d\xi \\
&= \Big(p(\cdot)e^{-\frac{\pi^2}{|\sigma|}(\cdot)(1+i\varepsilon_\sigma t)(\cdot)^T}\Big)^\vee(x) \\
&= p(-2\pi i\partial_x)\Big(e^{-\frac{\pi^2}{|\sigma|}(\cdot)(1+i\varepsilon_\sigma t)(\cdot)^T}\Big)^\vee(x).
\end{aligned}$$

As a result, the problem comes down to finding the function defined by

$$\begin{aligned}
F(x,t) &= (\frac{\pi}{|\sigma|})^{n/2}\Big(e^{-\frac{\pi^2}{|\sigma|}(\cdot)(1+i\varepsilon_\sigma t)(\cdot)^T}\Big)^\vee(x) \\
&= (\frac{\pi}{|\sigma|})^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{\pi^2}{|\sigma|}\|\xi\|^2} e^{\frac{\pi^2}{s}\xi t\xi^T} e^{2\pi i\xi x^T}\, d\xi.
\end{aligned}$$

We claim that this function is given exactly by $F = f_{\operatorname{sgn}\sigma}$ from Theorem 5.7.

To verify this claim, note that, by definition, F is the unique solution to the system given in Equation (1.1) satisfying the initial condition of $\widehat{F}(\xi,0) = (\pi/|\sigma|)^{n/2}e^{-\frac{\pi^2}{|\sigma|}\|\xi\|^2}$ or, equivalently, that $F(x,0) = e^{-|\sigma|\|x\|^2}$. Obviously, our proposed solution, $f_{\operatorname{sgn}\sigma}$, satisfies that initial condition. By the proof of Theorem 5.7, it also satisfies the system of differential operators which finishes the claim.

Since the highest/lowest K-type space in the oscillator representation is spanned by $e^{-\|x\|^2/2}$ (for the even functions) and $x_ie^{-\|x\|^2/2}$ (for the odd functions), the above discussion shows that the corresponding functions (up to a multiple) in $\mathcal{D}'$ are $f_{\operatorname{sgn}\sigma}$ and $\partial_{x_i}f_{\operatorname{sgn}\sigma}$. Since $f_{\operatorname{sgn}\sigma}$ has been calculated, consider $\partial_{x_i}f_{\operatorname{sgn}\sigma}$:

$$\begin{aligned}
\partial_{x_i}f_- &= 2|\sigma|\overline{\varepsilon_t(iI_n)}^{-1}(x(I_n-it)^{-1}e_i)e^{\sigma x(I_n-it)^{-1}x^T} \\
\partial_{x_i}f_+ &= -2\sigma\varepsilon_t(iI_n)^{-1}(x(I_n+it)^{-1}e_i)e^{-\sigma x(I_n+it)^{-1}x^T}.
\end{aligned}$$

Finally, the last statement follows from the fact that the element of $\mathcal{D}'$ corresponding to the function $p(x)e^{-\frac{\pi^2}{|\sigma|}\|x\|^2}$ in the image of $\mathcal{E}$ is $p(-2\pi i\partial_x)f_+(x)$. $\square$

Corollary 8.2. *The highest ($\sigma < 0$) and lowest ($\sigma > 0$), respectively, K-finite vector of $(\mathcal{I}'_+)_K$ is spanned by the function $f_{\operatorname{sgn}\sigma}$ given by*

$$f_-(0,t) = \overline{\varepsilon_t(iI_n)}^{-1}, \quad f_+(0,t) = \varepsilon_t(iI_n)^{-1}.$$

The highest ($\sigma < 0$) and lowest ($\sigma > 0$), respectively, K-type vectors of $(\mathcal{I}'_-)_K$ is given by the functions $f_{\operatorname{sgn}\sigma,a}$ where

$$\begin{aligned}
f_{-,a}(t) &= \overline{\varepsilon_t(iI_n)}^{-1}a(I_n-it)^{-1} \\
f_{+,a}(t) &= \varepsilon_t(iI_n)^{-1}a(I_n+it)^{-1}
\end{aligned}$$

for $a \in \mathbb{R}^n$.

It is possible to describe the general K-finite vector, though the details are more involved. For instance, it is straightforward to check that the K-finite vectors of

$(\mathcal{I}'_+)_K$ are spanned by functions of the form

$$f(t) = \det(I_n + i\varepsilon_\sigma t)^{-1/2} \sum_{\sigma\in \widetilde{S}_{2k}} \prod_{l=1}^{k} ((I_n + i\varepsilon_\sigma t)^{-1})_{j_{\sigma(2l-1)}, j_{\sigma(2l)}}$$

where $k \in \mathbb{N}$, $j_1, \dots, j_{2k} \in \{1, \dots, n\}$ and$\widetilde{S}_{2k}$ denotes the set elements of the symmetric group S_{2k} satisfying $\sigma(2l-1) < \sigma(2l)$ and $\sigma(1) < \sigma(3) < \dots < \sigma(2k-1)$. Notice that each term in the summand is the k-fold product of the determinant of a minor of $(I_n + i\varepsilon_\sigma t)$ divided by $\det(I_n + i\varepsilon_\sigma t)$.

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