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WELL-POSEDNESS AND BLOWUP CRITERION OF GENERALIZED POROUS MEDIUM EQUATION IN BESOV SPACES

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Abstract. We study the generalized porous medium equation of the form $u_t + \nu\Lambda^\beta u = \nabla\cdot(u\nabla Pu)$ where P is an abstract operator. We obtain the local well-posedness in Besov spaces for large initial data, and show the solution becomes global if the initial data is small. Also, we prove a blowup criterion for the solution.

1. Introduction

In this article, we study the equation in $\mathbb{R}^n$ of the form

$$\begin{gathered} u_t + \nu\Lambda^\beta u = \nabla\cdot(u\nabla Pu); \\ u(0,x) = u_0. \end{gathered} \tag{1.1}$$

Here $u = u(t,x)$ ia a real-valued function, represents a density or concentration. The dissipative coefficient $\nu > 0$ corresponds to the viscous case, while $\nu = 0$ corresponds to the inviscid case. The fractional operator Λ^β is defined by Fourier transform as $(\Lambda^\beta u)^\wedge = |\xi|^\beta \hat{u}$, and P is an abstract operator.

The general form of equation (1.1) has good suitability in many cases. The simplest case $\nu = 0$, $Pu = u$ comes from a model in groundwater infiltration, that is, $u_t = \Delta u^2$ (see [2, 33]). We call (1.1) generalized porous medium equation (GPME) inspired by Caffarelli and Vázquez [12], in which they introduced the fractional porous medium flow (FPME) when $\nu = 0$ and $Pu = \Lambda^{-2s}u, 0 < s < 1$. When $Pu = \Lambda^{-2}u$, it is the mean field equation, which is first studied by Lin and Zhang [28]. Some results on the well-posedness and regularity on those equations can be seen [7, 8, 13, 14, 30, 31, 34, 39] and the references therein.

Another similar model occurs in the aggregation equation, which is an important equation arising in physics, biology, chemistry, population dynamics, etc.([15, 10, 21, 32]). In this model, the operator P is a convolution operator with kernel K; that is, $Pu = K * u$. The typical kernels are $|x|^\gamma$, see [9, 22, 27], and $-e^{-|x|}$, see [3, 6, 25, 26]. For more results on this equation, we refer to [4, 5, 11, 23, 24] and

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the references therein. Besides, if we rewrite the equation (1.1) with same initial data as

$$\begin{gathered} u_t + \nu\Lambda^\beta u + v \cdot \nabla u = -u(\nabla \cdot v); \\ v = -\nabla P u, \end{gathered} \tag{1.2}$$

then it is a kind of special transport type equation. Furthermore, if we assume that v is divergence-free vector function ($\nabla \cdot v = 0$), the form (1.2) can contain the $2-D$ quasi-geostrophic (Q-G) equation [17, 18, 19, 35].

In this article we study the well-posedness of equation (1.1) in homogenerous Besov spaces under a general condition

$$\|\nabla P u\|_{\dot{B}^s_{p,q}} \le C\|u\|_{\dot{B}^{s+\sigma}_{p,q}}. \tag{1.3}$$

It is widespread adopted in the case of FPME, Q-G equation, or aggregation equation with its usual kernel $|x|^\gamma$ and it plays somewhat key role in the well-posedness and regularity of those equations in Besov spaces or Sobolev spaces. Based on the ideas used in [29, 36, 37, 38], we prove the following theorem.

Theorem 1.1. *Assume P satisfies (1.3), $\beta \in (0, 2]$, $p \ge 1$ and $\sigma + 1 < \beta < \sigma + n\min(2/p, 1)$. Then for any initial data $u_0 \in \dot{B}^{\frac{n}{p}+\sigma-\beta}_{p,1} \cap \dot{B}^{\frac{n}{p}+\sigma-\beta+1}_{p,1}$, the Cauchy problem (1.1) admits a unique solution*

$$u \in C([0,T); \dot{B}^{\frac{n}{p}+\sigma-\beta}_{p,1} \cap \dot{B}^{\frac{n}{p}+\sigma-\beta+1}_{p,1}) \cap L^1([0,T); \dot{B}^{\frac{n}{p}+\sigma}_{p,1} \cap \dot{B}^{\frac{n}{p}+\sigma+1}_{p,1}).$$

Moreover, if T^* denotes the maximal time of existence of u,

(i) *there exists a constant $C_0 > 0$ such that if $\|u_0\|_{\dot{B}^{\frac{n}{p}+\sigma-\beta+1}_{p,1}} < C_0$, then $T^* = \infty$;*
(ii) *if $T^* < \infty$, then $\int_0^{T^*} \|u(t)\|_{\dot{B}^{n/p+\sigma+1}_{p,1}} dt = \infty$.*

Remark 1.2. In the case of aggregation equation, Wu and Zhang [36] proved a similar result under the condition $\nabla K \in W^{1,1}$, $\beta \in (0, 1)$. Corresponding to their case we prove same result for $\sigma = 0$, that is $\nabla K \in L^1$, $\beta \in (1, 2)$, and also a similar result for $\sigma = -1$; that is, $\nabla K \in \dot{W}^{1,1}$, $\beta \in (0, 1)$.

Throughout this article, C denotes a positive constant that may differ line by line, the notation $A \lesssim B$ means $A \le CB$, and $A \sim B$ denotes $A \lesssim B$ and $B \lesssim A$.

2. Preliminaries

Let us recall some basic knowledge on Littlewood-Paley theory and Besov spaces. Let $\mathscr{S}(\mathbb{R}^n)$ be the Schwartz class and $\mathscr{S}'$ be its dual space. Given $f \in \mathscr{S}(\mathbb{R}^n)$, we use its Fourier transform $\mathcal{F}f = \hat{f}$ as

$$\hat{f} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a radial real-valued smooth function such that $0 \le \varphi(\xi) \le 1$ and

$$\operatorname{supp}\varphi \subset \{\xi \in \mathbb{R}^n : \frac{3}{4} \le |\xi| \le \frac{8}{3}\}, \quad \sum_{j\in Z} \varphi(2^{-j}\xi) = 1 \text{ for any } \xi \neq 0.$$

We denote $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ and $\mathbb{P}$ the set of all polynomials. Setting $h = \mathcal{F}^{-1}\varphi$, we define the frequency localization operators as follows:

$$\Delta_j u = \varphi(2^{-j}D)u = 2^{jn}\int_{\mathbb{R}^n} h(2^j y)u(x-y)dy, \quad S_j f = \sum_{k\le j-1}\Delta_k u.$$

Definition 2.1. For $s\in\mathbb{R}, p,q\in[1,\infty]$, we define the homogeneous Besov space $\dot{B}^s_{p,q}$ as

$$\dot{B}^s_{p,q} = \{f\in\mathscr{S}'/\mathbb{P} : \|f\|_{\dot{B}^s_{p,q}} = \Big(\sum_{j\in\mathbb{Z}} 2^{jsq}\|\Delta_j f\|^q_{L^p}\Big)^{1/q} < \infty\}.$$

Here the norm changes normally when $p=\infty$ or $q=\infty$.

Definition 2.2. In this paper, we need two kinds of mixed time-space norm defined as follows: For $s\in\mathbb{R}, 1\le p,q\le\infty, I=[0,T), T\in(0,\infty]$, and X a Banach space with norm $\|\cdot\|_X$ where

$$\|f(t,x)\|_{L^r(I;X)} := \Big(\int_I \|f(\tau,\cdot)\|^r_X d\tau\Big)^{1/r},$$
$$\|f(t,x)\|_{\mathcal{L}^r(I;\dot{B}^s_{p,q})} := \Big(\sum_{j\in\mathbb{Z}} 2^{jsq}\|\Delta_j f\|^q_{L^r(I;L^p)}\Big)^{1/q}.$$

By Minkowski' inequality, there holds

$$\begin{gathered} L^r(I;\dot{B}^s_{p,q}) \hookrightarrow \mathcal{L}^r(I;\dot{B}^s_{p,q}), \quad \text{if } r\le q, \\ \mathcal{L}^r(I;\dot{B}^s_{p,q}) \hookrightarrow L^r(I;\dot{B}^s_{p,q}), \quad \text{if } r\ge q. \end{gathered} \tag{2.1}$$

Now we state some basic properties about the homogeneous Besov spaces.

Proposition 2.3 ([1])**.** *For $s\in\mathbb{R}, 1\le p,q\le\infty$, the following hold:*

(i) *Let $\beta\in\mathbb{R}$, we have the equivalence of norms: $\|\Lambda^\beta f\|_{\dot{B}^s_{p,q}} \sim \|f\|_{\dot{B}^{s+\beta}_{p,q}}$.*
(ii) *If $p_1\le p_2, q_1\le q_2$, then $\dot{B}^s_{p_1,q_1} \hookrightarrow \dot{B}^{s-n(1/p_1-1/p_2)}_{p_2,q_2}$.*
(iii) *Let $1\le p,q\le\infty, s_1,s_2<\frac{n}{p}$ when $q>1$ (or $s_1,s_2\le\frac{n}{p}$ when $q=1$), and $s_1+s_2>0$, there holds*

$$\|uv\|_{\dot{B}^{s_1+s_2-\frac{n}{p}}_{p,q}} \le C\|u\|_{\dot{B}^{s_1}_{p,q}}\|v\|_{\dot{B}^{s_2}_{p,q}},$$

where $C>0$ be a constant depending on s_1,s_2,p,q,n.

Lemma 2.4 (Bernstein's inequalities [17])**.** *Set $\mathcal{B}$ to be a ball and $\mathcal{C}$ to be an annulus, and let $1\le p\le q\le\infty$, $\alpha\in(\{0\}\cup\mathbb{N})^n$, then the following estimates hold:*

(i) *If $\operatorname{supp}\widehat{f}\in 2^j\mathcal{B}$, $\beta+|\alpha|\ge 0$, then*

$$\|\Lambda^\beta D^\alpha f\|_{L^q} \le C2^{j(\beta+|\alpha|+n(1/p-1/q))}\|f\|_{L^p}.$$

(ii) *If $\operatorname{supp}\widehat{f}\in 2^j\mathcal{C}$, then*

$$C2^{j(\beta+|\alpha|)}\|f\|_{L^p} \le \|\Lambda^\beta D^\alpha f\|_{L^p} \le C'2^{j(\beta+|\alpha|)}\|f\|_{L^p},$$

where $C\le C'$ are positive constants independent of j.

Lemma 2.5 ([1])**.** *Let $\mathcal{C}$ ba an annulus. If $\operatorname{supp}\widehat{f}\subset 2^j\mathcal{C}$, then positive constant $c>0$ exists such that for any $t>0$, there holds*

$$\|e^{-t\Lambda^\beta}f\|_{L^p} \le Ce^{-ct2^{j\beta}}\|f\|_{L^p}.$$

Lemma 2.6 ([29]). *Let $s \in \mathbb{R}$ and $1 \le p \le p_1 \le \infty$. Set $R_j := (S_{j-1}v - v) \cdot \nabla \Delta_j u - [\Delta_j, v \cdot \nabla]u$. There exists a constant $C = C(n,s)$ such that*

$$\begin{aligned}
2^{js}\|R_j\|_{L^p} \le C\Big(&\sum_{|j-j'|\le 4} \|S_{j'-1}\nabla v\|_{L^\infty} 2^{j's}\|\Delta_{j'}u\|_{L^p} \\
&+ \sum_{j'\ge j-3} 2^{j-j'}\|\Delta_{j'}\nabla v\|_{L^\infty} 2^{js}\|\Delta_j u\|_{L^p} \\
&+ \sum_{\substack{|j'-j|\le 4\\ j''\le j'-2}} 2^{(j-j'')(s-1-\frac{n}{p_1})} 2^{j'\frac{n}{p_1}}\|\Delta_{j'}\nabla v\|_{L^{p_1}} 2^{j''s}\|\Delta_{j''}u\|_{L^p} \\
&+ \sum_{\substack{j'\ge j-3\\ |j''-j'|\le 1}} 2^{(j-j')(s+n\min(\frac{1}{p_1},\frac{1}{p'}))} 2^{j'\frac{n}{p_1}} (2^{j-j'}\|\Delta_{j'}\nabla v\|_{L^{p_1}} \\
&+ \|\Delta_{j'}\nabla \cdot v\|_{L^{p_1}}) 2^{j''s}\|\Delta_{j''}u\|_{L^p}\Big).
\end{aligned}$$

Now we recall a priori estimates needed in our proof. Consider the transport-diffusion equation

$$\partial_t u + v \cdot \nabla u + \nu \Lambda^\beta u = f, \quad u(0,x) = u_0(x). \tag{2.2}$$

Lemma 2.7 ([29]). *Let $1 \le r_1 \le r \le \infty, 1 \le p \le p_1 \le \infty$ and $1 \le q \le \infty$. Assume $s \in \mathbb{R}$ satisfies the following conditions:*

$$\begin{gathered}
s < 1 + \frac{n}{p_1} \quad (\text{or } s \le \frac{n}{p_1}, \text{if } q = 1), \\
s > -\min(\frac{n}{p_1}, \frac{n}{p'}) \quad (\text{or } s > -1 - \min(\frac{n}{p_1}, \frac{n}{p'}), \text{if} \operatorname{div} u = 0).
\end{gathered}$$

There exists a positive constant $C = C(n, \beta, s, p, p_1, q)$ such that for any smooth solution u of (2.2) with $\nu \ge 0$, the following a priori estimate holds:

$$\nu^{1/r}\|u\|_{\mathcal{L}^r_T \dot{B}^{s+\beta/r}_{p,q}} \le Ce^{CZ(T)}\big(\|u_0\|_{\dot{B}^s_{p,q}} + \nu^{1/r_1-1}\|f\|_{\mathcal{L}^{r_1}_T \dot{B}^{s-\beta+\beta/r_1}_{p,q}}\big),$$

where $Z(T) = \int_0^T \|\nabla v(t)\|_{\dot{B}^{n/p_1}_{p_1,\infty}\cap L^\infty} dt$.

3. Local and global well-posedness

In this section we prove our main theorem. We first rewrite (1.1) as

$$\begin{gathered}
u_t + \Lambda^\beta u + v \cdot \nabla u = u(\Delta P u); \\
v = -\nabla P u; \\
u(0,x) = u_0.
\end{gathered}$$

Step 1: Approximate solutions. In this step we construct approximate equations, and prove the boundedness of the approximate solutions. Set $u^0 = e^{-t\Lambda^\beta}u_0(x)$ and let u^{m+1} be the solution of the linear equation

$$\begin{gathered}
u^{m+1}_t + \Lambda^\beta u^{m+1} + v^m \cdot \nabla u^{m+1} = u^m(\Delta P u^m); \\
v^m = -\nabla P u^m; \\
u^{m+1}(0,x) = u_0.
\end{gathered} \tag{3.1}$$

Set $X=\dot B^{\frac{n}{p}+\sigma-\beta}_{p,1}\cap \dot B^{\frac{n}{p}+\sigma-\beta+1}_{p,1}$ and $Y=\dot B^{\frac{n}{p}+\sigma}_{p,1}\cap \dot B^{\frac{n}{p}+\sigma+1}_{p,1}$. It is easy to show

$$u^0\in \mathcal{L}^\infty(\mathbb{R}^+;X)\cap \mathcal{L}^1(\mathbb{R}^+;Y).$$

Now by induction, we deduce u^m belongs to the same spaces. In fact, by Lemma 2.7

$$\begin{aligned}
&\|u^{m+1}\|_{\mathcal{L}^\infty_T\dot B^{n/p+\sigma-\beta+1}_{p,1}}+\|u^{m+1}\|_{\mathcal{L}^1_T\dot B^{n/p+\sigma+1}_{p,1}}\\
&\lesssim e^{c\int_0^T\|\nabla v^m(t)\|_{\dot B^{n/p}_{p,\infty}\cap L^\infty}dt}\big(\|u_0\|_{\dot B^{n/p+\sigma-\beta+1}_{p,1}}\\
&\quad+\|u^m(\Delta Pu^m)\|_{\mathcal{L}^1_T\dot B^{n/p+\sigma-\beta+1}_{p,1}}\big)\\
&\lesssim e^{c\int_0^T\|\nabla v^m(t)\|_{\dot B^{n/p}_{p,1}}dt}\big(\|u_0\|_{\dot B^{n/p+\sigma-\beta+1}_{p,1}}\\
&\quad+\|u^m\|_{L^\infty_T\dot B^{n/p+\sigma-\beta+1}_{p,1}}\|\Delta Pu^m\|_{L^1_T\dot B^{n/p}_{p,1}}\big)\\
&\lesssim e^{c\|u^m\|_{L^1_T\dot B^{n/p+\sigma+1}_{p,1}}}\big(\|u_0\|_{\dot B^{n/p+\sigma-\beta+1}_{p,1}}\\
&\quad+\|u^m\|_{\mathcal{L}^\infty_T\dot B^{n/p+\sigma-\beta+1}_{p,1}}\|u^m\|_{\mathcal{L}^1_T\dot B^{n/p+\sigma+1}_{p,1}}\big).
\end{aligned}\tag{3.2}$$

Similarly, we conclude that

$$\begin{aligned}
&\|u^{m+1}\|_{\mathcal{L}^\infty_T\dot B^{n/p+\sigma-\beta}_{p,1}}+\|u^{m+1}\|_{\mathcal{L}^1_T\dot B^{n/p+\sigma}_{p,1}}\\
&\lesssim e^{c\|u^m\|_{L^1_T\dot B^{n/p+\sigma+1}_{p,1}}}\big(\|u_0\|_{\dot B^{n/p+\sigma-\beta}_{p,1}}+\|u^m\|_{\mathcal{L}^\infty_T\dot B^{n/p+\sigma-\beta}_{p,1}}\|u^m\|_{\mathcal{L}^1_T\dot B^{n/p+\sigma+1}_{p,1}}\big).
\end{aligned}\tag{3.3}$$

Thus for all $m\in N$, we have $u^m\in\mathcal{L}^\infty(\mathbb{R}^+;X)\cap\mathcal{L}^1(\mathbb{R}^+;Y)$.

Step 2: Uniform estimates. We prove the key uniform estimates during this step. Set $u_j:=\Delta_j u,\ f_j:=\Delta_j(u^m\Delta Pu^m)$. Then we can obtain

Claim 3.1. *There exists $T_1\le T$, such that for all $t\in[0,T_1]$*

$$\begin{aligned}
\|u^{m+1}\|_{\mathcal{L}^r_t\dot B^{s+\beta/r}_{p,1}}&\lesssim\sum_{j\in\mathbb{Z}}\big(1-e^{-ctr2^{j\beta}}\big)^{1/r}2^{js}\|u_{0,j}\|_{L^p}\\
&\quad+\sum_{j\in\mathbb{Z}}\int_0^t 2^{js}\big(\|f_j\|_{L^p}+\|R_j\|_{L^p}\big)d\tau,
\end{aligned}$$

where $R_j:=(S_{j-1}v^m-v^m)\cdot\nabla u_j^{m+1}-[\Delta_j,v^m\cdot\nabla]u^{m+1}$.

We postpone the proof of this claim to the appendix. Taking $s=\frac{n}{p}+\sigma-\beta+1$ and ρ large enough such that $\sigma+1+\beta/\rho\le\beta$. Then by Proposition 2.3 with $s_1=n/p+\sigma+1-\beta+\beta/r, s_2=n/p-\beta/r$, there holds

$$\sum_{j\in\mathbb{Z}}\int_0^t 2^{j(n/p+\sigma+1-\beta)}\|f_j\|_{L^p}d\tau\lesssim\|u^m\|_{L^\rho_t\dot B^{n/p+\sigma+1-\beta+\beta/\rho}_{p,1}}\|u^m\|_{L^{\rho'}_t\dot B^{n/p+\sigma+1-\beta/\rho}_{p,1}}.$$

Taking ρ large enough and using Lemma 2.6 with $s=n/p+1+\sigma-\beta, p_1=p$,

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\int_0^t 2^{j(n/p+\sigma+1-\beta)}\|R_j\|_{L^p}d\tau\\
&\lesssim\|u^{m+1}\|_{L^\rho_t\dot B^{n/p+\sigma+1-\beta+\beta/\rho}_{p,1}}\|u^m\|_{L^{\rho'}_t\dot B^{n/p+\sigma+1-\beta/\rho}_{p,1}}.
\end{aligned}$$

Hence for $r \geq 1$ and ρ large enough, we have

$$
\begin{aligned}
&\|u^{m+1}\|_{\mathcal{L}_t^r \dot{B}_{p,1}^{n/p+\sigma+1-\beta+\beta/r}} \\
&\lesssim \sum_{j\in\mathbb{Z}} \big(1-e^{-ctr2^{j\beta}}\big)^{1/r} 2^{j(n/p+\sigma+1-\beta)} \|u_{0,j}\|_{L^p} \\
&\quad + \|u^m\|_{L_t^\rho \dot{B}_{p,1}^{n/p+\sigma+1-\beta+\beta/\rho}} \|u^m\|_{L_t^{\rho'} \dot{B}_{p,1}^{n/p+\sigma+1-\beta/\rho}} \\
&\quad + \|u^{m+1}\|_{L_t^\rho \dot{B}_{p,1}^{n/p+\sigma+1-\beta+\beta/\rho}} \|u^m\|_{L_t^{\rho'} \dot{B}_{p,1}^{n/p+\sigma+1-\beta/\rho}}.
\end{aligned} \tag{3.4}
$$

Now by (3.4) with $r=\rho$ and the fact that $\big(1-e^{-ct\rho' 2^{j\beta}}\big)^{1/\rho'} \leq \big(1-e^{-ct\rho 2^{j\beta}}\big)^{1/\rho}$ for ρ large,

$$
\begin{aligned}
&\|u^{m+1}\|_{\mathcal{L}_t^\rho \dot{B}_{p,1}^{n/p+\sigma+1-\beta+\beta/\rho}} + \|u^{m+1}\|_{\mathcal{L}_t^{\rho'} \dot{B}_{p,1}^{n/p+\sigma+1-\beta/\rho}} \\
&\lesssim \sum_{j\in\mathbb{Z}} \big(1-e^{-ct\rho 2^{j\beta}}\big)^{1/\rho} 2^{j(n/p+\sigma+1-\beta)} \|u_{0,j}\|_{L^p} \\
&\quad + \|u^m\|_{\mathcal{L}_t^\rho \dot{B}_{p,1}^{n/p+\sigma+1-\beta+\beta/\rho}} \|u^m\|_{\mathcal{L}_t^{\rho'} \dot{B}_{p,1}^{n/p+\sigma+1-\beta/\rho}} \\
&\quad + \|u^{m+1}\|_{\mathcal{L}_t^\rho \dot{B}_{p,1}^{n/p+\sigma+1-\beta+\beta/\rho}} \|u^m\|_{\mathcal{L}_t^{\rho'} \dot{B}_{p,1}^{n/p+\sigma+1-\beta/\rho}}.
\end{aligned}
$$

By Lebesgue dominated convergence theorem, for $\rho < \infty$, we have

$$
\lim_{t\mapsto 0^+} \sum_{j\in\mathbb{Z}} \big(1-e^{-ct\rho 2^{j\beta}}\big)^{1/\rho} 2^{j(n/p+\sigma+1-\beta)} \|u_{0,j}\|_{L^p} = 0.
$$

Now we set

$$
T = \sup\Big\{t>0 : c\sum_{j\in\mathbb{Z}} \big(1-e^{-ct\rho 2^{j\beta}}\big)^{1/\rho} 2^{j(n/p+\sigma+1-\beta)} \|u_{0,j}\|_{L^p} \leq \eta\Big\},
$$

for some $\eta > 0$ sufficiently small. By definition of $u^0, \forall t \leq T$, we have

$$
\|u^0\|_{\mathcal{L}_t^\rho \dot{B}_{p,1}^{n/p+\sigma+1-\beta+\beta/\rho}} \leq \eta, \quad \|u^0\|_{\mathcal{L}_t^{\rho'} \dot{B}_{p,1}^{n/p+\sigma+1-\beta/\rho}} \leq \eta.
$$

Choosing η small enough such that $c\eta \leq 1/2$,

$$
\|u^1\|_{\mathcal{L}_t^\rho \dot{B}_{p,1}^{n/p+\sigma+1-\beta+\beta/\rho}} + \|u^1\|_{\mathcal{L}_t^{\rho'} \dot{B}_{p,1}^{n/p+\sigma+1-\beta/\rho}} \leq 3\eta. \tag{3.5}
$$

If we assume that (3.5) holds for u^m and further take η small enough such that $3c\eta \leq 1/3$, we obtain

$$
\|u^{m+1}\|_{\mathcal{L}_t^\rho \dot{B}_{p,1}^{n/p+\sigma+1-\beta+\beta/\rho}} + \|u^{m+1}\|_{\mathcal{L}_t^{\rho'} \dot{B}_{p,1}^{n/p+\sigma+1-\beta/\rho}} \leq 3\eta. \tag{3.6}
$$

Thus by induction, we prove the uniform boundedness (3.6) for some suitable η and $\forall t \leq T$. Let $r=1$ in (3.4), since $\big(1-e^{-ct2^{j\beta}}\big) \leq \big(1-e^{-ct\rho 2^{j\beta}}\big)^{1/\rho}$, we have

$$
\begin{aligned}
\|u^{m+1}\|_{\mathcal{L}_t^1 \dot{B}_{p,1}^{n/p+\sigma+1}} &\lesssim \sum_{j\in\mathbb{Z}} \big(1-e^{-ct\rho 2^{j\beta}}\big)^{1/\rho} 2^{j(n/p+\sigma+1-\beta)} \|u_{0,j}\|_{L^p} \\
&\quad + \|u^m\|_{\mathcal{L}_t^\rho \dot{B}_{p,1}^{n/p+\sigma+1-\beta+\beta/\rho}} \|u^m\|_{\mathcal{L}_t^{\rho'} \dot{B}_{p,1}^{n/p+\sigma+1-\beta/\rho}} \\
&\quad + \|u^{m+1}\|_{\mathcal{L}_t^\rho \dot{B}_{p,1}^{n/p+\sigma+1-\beta+\beta/\rho}} \|u^m\|_{\mathcal{L}_t^{\rho'} \dot{B}_{p,1}^{n/p+\sigma+1-\beta/\rho}}.
\end{aligned}
$$

This and (3.6) imply

$$
\|u^{m+1}\|_{\mathcal{L}_t^1 \dot{B}_{p,1}^{n/p+\sigma+1}} \leq \eta + 18c\eta^2 \leq 3\eta. \tag{3.7}
$$

Next we prove that u^m are uniformly bounded in $\mathcal{L}_t^\infty \dot B_{p,1}^{n/p+\sigma+1-\beta}$. In fact, by (3.2)

$$\|u^{m+1}\|_{\mathcal{L}_t^\infty \dot B_{p,1}^{n/p+\sigma+1-\beta}} \le ce^{1/3}\|u_0\|_{\dot B_{p,1}^{n/p+\sigma+1-\beta}} + \frac{1}{3}e^{1/3}\|u^m\|_{\mathcal{L}_t^\infty \dot B_{p,1}^{n/p+\sigma+1-\beta}}.$$

By induction, we conclude

$$\begin{aligned} &\|u^m\|_{\mathcal{L}_t^\infty \dot B_{p,1}^{n/p+\sigma+1-\beta}} \\ &\le \frac{1-(\frac{1}{3}e^{1/3})^m}{1-\frac{1}{3}e^{1/3}} ce^{1/3}\|u_0\|_{\dot B_{p,1}^{n/p+\sigma+1-\beta}} + \frac{e^{m/3}}{3^m}\|u^0\|_{\mathcal{L}_t^\infty \dot B_{p,1}^{n/p+\sigma+1-\beta}} \\ &\le (3ce^{1/3}+1)\|u_0\|_{\dot B_{p,1}^{n/p+\sigma+1-\beta}}. \end{aligned} \tag{3.8}$$

Thus approximate solutions are uniformly bounded in the space $\mathcal{L}_T^\infty \dot B_{p,1}^{n/p+\sigma+1-\beta} \cap \mathcal{L}_T^1 \dot B_{p,1}^{n/p+\sigma+1}$. Now we return to (3.3) and by the uniform estimate (3.7)

$$\begin{aligned} &\|u^{m+1}\|_{\mathcal{L}_T^\infty \dot B_{p,1}^{n/p+\sigma-\beta}} + \|u^{m+1}\|_{\mathcal{L}_T^1 \dot B_{p,1}^{n/p+\sigma}} \\ &\lesssim e^{c\|u^m\|_{L_T^1 \dot B_{p,1}^{n/p+\sigma+1}}}\big(\|u_0\|_{\dot B_{p,1}^{n/p+\sigma-\beta}} + \|u^m\|_{\mathcal{L}_T^\infty \dot B_{p,1}^{n/p+\sigma-\beta}}\|u^m\|_{\mathcal{L}_T^1 \dot B_{p,1}^{n/p+\sigma+1}}\big) \\ &\le ce^{1/3}\|u_0\|_{\dot B_{p,1}^{n/p+\sigma-\beta}} + \frac{1}{3}e^{1/3}\|u^m\|_{\mathcal{L}_t^\infty \dot B_{p,1}^{n/p+\sigma-\beta}}. \end{aligned} \tag{3.9}$$

Hence, similarly to (3.8), by induction again,

$$\|u^m\|_{\mathcal{L}_t^\infty \dot B_{p,1}^{n/p+\sigma-\beta}} \le (3ce^{1/3}+1)\|u_0\|_{\dot B_{p,1}^{n/p+\sigma-\beta}}.$$

Substituting this into (3.9) we conclude

$$\|u^m\|_{\mathcal{L}_t^1 \dot B_{p,1}^{n/p+\sigma}} \le (4ce^{1/3}+1)\|u_0\|_{\dot B_{p,1}^{n/p+\sigma-\beta}}.$$

Thus we conclude that $(u^m)_{m\in\mathbb{N}}$ is uniformly bounded in $\mathcal{L}_T^\infty X \cap \mathcal{L}_T^1 Y$.

Step 3: Strong convergence. Let $(m,k) \in \mathbb{N}^2, m > k$ and $u^{m,k} = u^m - u^k, v^{m,k} = v^m - v^k$. Then $u^{m,k}$ satisfies the equation

$$\begin{gathered} u_t^{m+1,k+1} + \Lambda^\beta u^{m+1,k+1} + v^k \cdot \nabla u^{m+1,k+1} \\ = u^{m,k}(\Delta P u^m) + u^k(\Delta P u^{m,k}) - v^{m,k} \cdot \nabla u^{m+1}; \\ v^{m,k} = -\nabla P u^{m,k}; \\ u^{m+1,k+1}(0,x) = 0. \end{gathered}$$

Set $U^{m+1,k+1}(T) = \|u^{m+1,k+1}\|_{\mathcal{L}_T^\rho \dot B_{p,1}^{n/p+\sigma-\beta/\rho'}} + \|u^{m+1,k+1}\|_{\mathcal{L}_T^{\rho'} \dot B_{p,1}^{n/p+\sigma-\beta/\rho}}$. Applying Lemma 2.7 with $s = n/p+\sigma-\beta$, there holds

$$\begin{aligned} &U^{m+1,k+1}(T) \\ &\lesssim e^{c\|\nabla v^k\|_{L_T^1 \dot B_{p,1}^{n/p}}}\big(\|u^{m,k}(\Delta P u^m)\|_{\mathcal{L}_T^1 \dot B_{p,1}^{n/p+\sigma-\beta}} \\ &\quad + \|u^k(\Delta P u^{m,k})\|_{\mathcal{L}_T^1 \dot B_{p,1}^{n/p+\sigma-\beta}} + \|v^{m,k}\cdot\nabla u^{m+1}\|_{\mathcal{L}_T^1 \dot B_{p,1}^{n/p+\sigma-\beta}}\big). \end{aligned} \tag{3.10}$$

Now applying Proposition 2.3 with $s_1 = n/p - \beta/\rho, s_2 = n/p+\sigma-\beta/\rho'$,

$$\|v^{m,k}\cdot\nabla u^{m+1}\|_{\mathcal{L}_T^1 \dot B_{p,1}^{n/p+\sigma-\beta}} \lesssim \|u^{m,k}\|_{\mathcal{L}_T^{\rho'} \dot B_{p,1}^{n/p+\sigma-\beta/\rho}}\|u^{m+1}\|_{\mathcal{L}_T^\rho \dot B_{p,1}^{n/p+\sigma+1-\beta/\rho'}}.$$

Similarly, for ρ large enough such that $\sigma+1+\beta/\rho\le\beta$, we conclude

$$
\begin{aligned}
U^{m+1,k+1}(T) &\lesssim e^{c\|\nabla v^k\|_{L^1_T\dot B^{n/p}_{p,1}}}\big(\|u^{m,k}\|_{\mathcal{L}^{\rho'}_T\dot B^{n/p+\sigma-\beta/\rho}_{p,1}}\|u^{m+1}\|_{\mathcal{L}^{\rho}_T\dot B^{n/p+\sigma+1-\beta/\rho'}_{p,1}}\\
&\quad+\|u^k\|_{\mathcal{L}^{\rho}_T\dot B^{n/p+\sigma+1-\beta/\rho'}_{p,1}}\|u^{m,k}\|_{\mathcal{L}^{\rho'}_T\dot B^{n/p+\sigma-\beta/\rho}_{p,1}}\\
&\quad+\|u^{m,k}\|_{\mathcal{L}^{\rho'}_T\dot B^{n/p+\sigma-\beta/\rho}_{p,1}}\|u^{m}\|_{\mathcal{L}^{\rho}_T\dot B^{n/p+\sigma+1-\beta/\rho'}_{p,1}}\big)\\
&\le c\eta U^{m,k}(T).
\end{aligned}
$$

Choosing η small enough such that $c\eta<1$ and by induction, we conclude that $\{u^m\}$ is a Cauchy sequence in $\mathcal{L}^{\rho}_T\dot B^{n/p+\sigma-\beta/\rho'}_{p,1}\cap\mathcal{L}^{\rho'}_T\dot B^{n/p+\sigma-\beta/\rho}_{p,1}$. So u^m hence converges strongly to some u in it. Now by taking $r=1$ and $r=\infty$ in (3.4), respectively, and by passing to the limit into the approximation equation, we can get a solution to in $\mathcal{L}^\infty_T X\cap\mathcal{L}^1_T Y$.

Step 4: Uniqueness. Let $u_1,u_2\in\mathcal{L}^\infty_T X\cap\mathcal{L}^1_T Y$ be two solutions of (1.1) with the same initial data. Let $u_{1,2}=u_1-u_2$, then

$$
\begin{gathered}
\partial_t u_{1,2}+\Lambda^\beta u_{1,2}+v_2\cdot\nabla u_{1,2}=u_{1,2}(\Delta Pu_1)+u_2(\Delta Pu_{1,2})-v_{1,2}\cdot\nabla u_1;\\
v_{1,2}=-\nabla Pu_{1,2};\\
u_{1,2}(0,x)=0.
\end{gathered}
$$

According to Lemma 2.7,

$$
\begin{aligned}
&\|u_{1,2}\|_{\mathcal{L}^\infty_t\dot B^{n/p+\sigma-\beta}_{p,\rho}}\\
&\lesssim e^{c\|u_2\|_{L^1_t\dot B^{n/p+\sigma+1}_{p,1}}}\big(\|u_{1,2}(\Delta Pu_1)\|_{\mathcal{L}^\rho_t\dot B^{n/p+\sigma-2\beta+\beta/\rho}_{p,\rho}}\\
&\quad+\|u_2(\Delta Pu_{1,2})\|_{\mathcal{L}^\rho_t\dot B^{n/p+\sigma-2\beta+\beta/\rho}_{p,\rho}}+\|v_{1,2}\cdot\nabla u_1\|_{\mathcal{L}^\rho_t\dot B^{n/p+\sigma-2\beta+\beta/\rho}_{p,\rho}}\big).
\end{aligned}
$$

By a similar argument as in Step 3, we have

$$
\begin{aligned}
&\|u_{1,2}\|^\rho_{\mathcal{L}^\infty_t\dot B^{n/p+\sigma-\beta}_{p,\rho}}\\
&\lesssim e^{c\rho\|u_2\|_{L^1_T\dot B^{n/p+\sigma+1}_{p,1}}}\int_0^t\|u_{1,2}\|^\rho_{L^\infty_\tau\dot B^{n/p+\sigma-\beta}_{p,\rho}}\big(\|u_1(\tau)\|^\rho_{\dot B^{n/p+\sigma+1-\beta+\beta/\rho}_{p,\rho}}\\
&\quad+\|u_2(\tau)\|^\rho_{\dot B^{n/p+\sigma+1-\beta+\beta/\rho}_{p,\rho}}\big)d\tau.
\end{aligned}
$$

Since the inclusion $\dot B^s_{p,1}\subset\dot B^s_{p,\rho}$ holds for any $\rho\in[1,\infty]$. Thus the Minkowski's inequality and Gronwall's inequality give that $u_1=u_2,\forall t\in[0,T]$.

Step 5: Continuity in time. For all $t,t'\in[0,T)$, we have

$$
\begin{aligned}
&\|u(t)-u(t')\|_{\dot B^{n/p+\sigma-\beta}_{p,1}}\\
&\le\sum_{j\le N}2^{j(n/p+\sigma-\beta)}\|u_j(t)-u_j(t')\|_{L^p}+2\sum_{j>N}2^{j(n/p+\sigma-\beta)}\|u_j\|_{L^\infty_T L^p}.
\end{aligned}
$$

Since $u\in L^\infty_T\dot B^{n/p+\sigma-\beta}_{p,1}$, for any $\epsilon>0$, we can choose N large enough such that

$$
\sum_{j>N}2^{j(n/p+\sigma-\beta)}\|u_j\|_{L^\infty_T L^p}\le\frac{\epsilon}{4}.
$$

On the other hand,

$$\begin{aligned}\sum_{j\le N} 2^{j(n/p+\sigma-\beta)}\|u_j(t)-u_j(t')\|_{L^p} &\le |t-t'|\sum_{j\le N} 2^{j(n/p+\sigma-\beta)}\|\partial_t u_j\|_{L^\infty_T L^p}\\ &\le |t-t'|2^{N\beta}\|\partial_t u\|_{\mathcal{L}^\infty_T \dot B^{n/p+\sigma-2\beta}_{p,1}}.\end{aligned}$$

Now write

$$u_t = -\Lambda^\beta u - v\cdot\nabla u + u(\Delta P u) \quad\text{and}\quad v=-\nabla P u.$$

Obviously, we have

$$\|\Lambda^\beta u\|_{\mathcal{L}^\infty_T \dot B^{n/p+\sigma-2\beta}_{p,1}} \le \|u\|_{\mathcal{L}^\infty_T \dot B^{n/p+\sigma-\beta}_{p,1}}.$$

Applying Proposition 2.3 with $s_1=n/p-\beta, s_2=n/p+\sigma-\beta$,

$$\|v\cdot\nabla u\|_{\mathcal{L}^\infty_T \dot B^{n/p+\sigma-2\beta}_{p,1}} \le \|u\|_{\mathcal{L}^\infty_T \dot B^{n/p+\sigma-\beta}_{p,1}}\|u\|_{\mathcal{L}^\infty_T \dot B^{n/p+\sigma+1-\beta}_{p,1}}.$$

Similarly, we have

$$\|u(\Delta P u)\|_{\mathcal{L}^\infty_T \dot B^{n/p+\sigma-2\beta}_{p,1}} \le \|u\|_{\mathcal{L}^\infty_T \dot B^{n/p+\sigma-\beta}_{p,1}}\|u\|_{\mathcal{L}^\infty_T \dot B^{n/p+\sigma+1-\beta}_{p,1}}.$$

Thus for $|t-t'|\le (2^{N\beta}\|\partial_t u\|_{\mathcal{L}^\infty_T \dot B^{n/p+\sigma-2\beta}_{p,1}})^{-1}\frac{\epsilon}{2}$, we conclude

$$\|u(t)-u(t')\|_{\dot B^{n/p+\sigma-\beta}_{p,1}} \le \epsilon.$$

Hence $u\in C([0,T);\dot B^{n/p+\sigma-\beta}_{p,1})$. Similarly we obtain $u\in C([0,T);\dot B^{n/p+\sigma+1-\beta}_{p,1})$.

Step 6: Blowup criterion. We give a blowup criterion as follows:

Proposition 3.2. *Let T^* denote the maximal time of existence of a solution u in $C([0,T^*);X)\cap L^1([0,T^*);Y)$. If $T^*<\infty$, then*

$$\int_0^{T^*}\|u(t)\|_{\dot B^{n/p+\sigma+1}_{p,1}}dt=\infty.$$

Proof. Supposing $T^*<\infty$ and $\int_0^{T^*}\|u(t)\|_{\dot B^{n/p+\sigma+1}_{p,1}}dt<\infty$, and using Lemma 2.7 with $\rho=\infty$, we have

$$\begin{aligned}\|u\|_{\mathcal{L}^\infty_{T^*}\dot B^{n/p+\sigma+1-\beta}_{p,1}} &\lesssim e^{c\int_0^{T^*}\|u(t)\|_{\dot B^{n/p+\sigma+1}_{p,1}}dt}\big(\|u_0\|_{\dot B^{n/p+\sigma+1-\beta}_{p,1}}\\ &\quad+\int_0^{T^*}\|u(t)\|_{\dot B^{n/p+\sigma+1-\beta}_{p,1}}\|u(t)\|_{\dot B^{n/p+\sigma+1}_{p,1}}dt\big).\end{aligned}$$

Hence by Gronwall's inequality we have

$$\|u\|_{\mathcal{L}^\infty_{T^*}\dot B^{n/p+\sigma+1-\beta}_{p,1}} \lesssim e^{c\int_0^{T^*}\|u(t)\|_{\dot B^{n/p+\sigma+1}_{p,1}}dt}\|u_0\|_{\dot B^{n/p+\sigma+1-\beta}_{p,1}}<\infty. \tag{3.11}$$

By a similar argument there also holds

$$\|u\|_{\mathcal{L}^\infty_{T^*}\dot B^{n/p+\sigma-\beta}_{p,1}} \lesssim e^{c\int_0^{T^*}\|u(t)\|_{\dot B^{n/p+\sigma+1}_{p,1}}dt}\|u_0\|_{\dot B^{n/p+\sigma-\beta}_{p,1}}<\infty. \tag{3.12}$$

From Step 5, for all $t,t'\in[0,T^*)$, we have

$$\|u(t)-u(t')\|_X\to 0 \quad\text{as}\quad t\to t'.$$

This means that $u(t)$ satisfies the Cauchy criterion at T^*. So there exists an element u^* in X such that $u(t)\to u^*$ in X as $t\to T^*$. Now set $u(T^*)=u^*$ and consider the equation with initial data u^*. By the well-posedness we obtain a solution existing on a larger time interval than $[0,T^*)$, which is a contradiction. $\square$

Step 7: Global solution. To obtain global well-posedness for small initial data, it is sufficient to bound

$$F(t) := \int_0^t \|u(\tau)\|_{\dot B^{n/p+\sigma+1}_{p,1}} d\tau.$$

Lemma 2.7 gives

$$\|u\|_{\mathcal{L}^1_t \dot B^{n/p+\sigma+1}_{p,1}} \lesssim e^{cF(t)}\big(\|u_0\|_{\dot B^{n/p+\sigma+1-\beta}_{p,1}} + \int_0^t \|u(\tau)\|_{\dot B^{n/p+\sigma+1-\beta}_{p,1}} \|u(\tau)\|_{\dot B^{n/p+\sigma+1}_{p,1}} d\tau\big).$$

A similar argument with (3.11) gives

$$\|u\|_{\mathcal{L}^\infty_t \dot B^{n/p+\sigma+1-\beta}_{p,1}} \lesssim e^{cF(t)} \|u_0\|_{\dot B^{n/p+\sigma+1-\beta}_{p,1}}.$$

Hence we conclude

$$F(t) \le Ce^{CF(t)}(1+F(t))\|u_0\|_{\dot B^{n/p+\sigma+1-\beta}_{p,1}}.$$

Since $F(t)$ is continuous and $F(0)=0$, we obtain that: if the initial data satisfies $\|u_0\|_{\dot B^{n/p+\sigma+1-\beta}_{p,1}} < \frac{e^{-C}}{1+C}$, then

$$F(t) \le Ce^{C^2}(1+C)\|u_0\|_{\dot B^{n/p+\sigma+1-\beta}_{p,1}}.$$

By the blow-up criterion, the solution is global.

4. Appendix

We now give the proof of Claim 3.1. We list some lemmas which will be used in our proof.

Lemma 4.1 ([20])**.** *Let v be a smooth vector field, and ψ_t be the solution to*

$$\psi_t(x) = x + \int_0^t v(\tau, \psi_\tau(x)) d\tau.$$

Then for all $t \in \mathbb{R}^+$, the flow ψ_t is a C^1 diffeomorphism over $\mathbb{R}^n$ and there holds

$$\begin{gathered}
\|\nabla \psi_t^{\pm 1}\|_{L^\infty} \le e^{V(t)}, \\
\|\nabla \psi_t^{\pm 1} - Id\|_{L^\infty} \le e^{V(t)} - 1, \\
\|\nabla^2 \psi_t^{\pm 1}\|_{L^\infty} \le e^{V(t)} \int_0^t \|\nabla^2 v(\tau)\|_{L^\infty} e^{V(\tau)} d\tau,
\end{gathered}$$

where $V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau$.

Lemma 4.2 ([20])**.** *Let $\chi \in \mathscr{S}(\mathbb{R}^n)$. There exists a constant $C = C(\chi, n)$ such that for all C^2 diffeomorphism ψ over $\mathbb{R}^n$ with inverse ϕ, and for all $u \in \mathscr{S}', p \in [1,+\infty], (j,j') \in \mathbb{Z}^2$,*

$$\begin{aligned}
&\|\chi(2^{-j'}D)(\Delta_j u \circ \psi)\|_{L^p} \\
&\le C\|J_\phi\|_{L^\infty}^{1/p} \|\Delta_j u\|_{L^p} (2^{-j}\|DJ_\phi\|_{L^\infty}\|J_\psi\|_{L^\infty} + 2^{j'-j}\|D\phi\|_{L^\infty}).
\end{aligned}$$

Lemma 4.3 ([16])**.** *Let $v \in L^1_{loc}(\mathbb{R}^+; Lip)$ be a fixed vector field. For $j \in \mathbb{Z}$, set $u_j = \Delta_j u$, ψ_j be the flow of the regularized vector field $S_{j-1}v$. Then for $u \in \dot{B}^{\beta}_{p,\infty}$ with $\beta \in [0,2)$, $p \in [1,\infty]$, there holds*

$$\|\Lambda^\beta(u_j \circ \psi_j) - (\Lambda^\beta u_j) \circ \psi_j\|_{L^p} \le Ce^{CV(t)} V^{1-\frac{\beta}{2}}(t) 2^{j\beta} \|u_j\|_{L^p},$$

and when $\beta = 2$,

$$\|\Lambda^2(u_j \circ \psi_j) - (\Lambda^2 u_j) \circ \psi_j\|_{L^p} \le Ce^{CV(t)} V(t) 2^{2j} \|u_j\|_{L^p},$$

where $V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau$ and $C = C(\beta, p) > 0$.

Proof of Claim 3.1. Applying Δ_j to (3.1) we have

$$\partial_t u_j^{m+1} + S_{j-1} v^m \cdot \nabla u_j^{m+1} + \Lambda^\beta u_j^{m+1} = f_j + R_j, \tag{4.1}$$

where $R_j := (S_{j-1}v^m - v^m) \cdot \nabla u_j^{m+1} - [\Delta_j, v^m \cdot \nabla] u^{m+1}$. Let ψ_j be the flow of the regularized vector field $S_{j-1}v^m$. Denote $\bar{u}_j := u_j \circ \psi_j$, then (4.1) becomes

$$\partial_t \bar{u}_j^{m+1} + \Lambda^\beta \bar{u}_j^{m+1} = \bar{f}_j + \bar{R}_j + G_j, \tag{4.2}$$

where $G_j := \Lambda^\beta(u_j^{m+1} \circ \psi_j) - (\Lambda^\beta u_j^{m+1}) \circ \psi_j$.

Applying Δ_k on the equivalent integral equation of (4.2), we have

$$\begin{aligned} &\|\Delta_k \bar{u}_j^{m+1}(t)\|_{L^p} \\ &\lesssim e^{-ct2^{k\beta}} \|\Delta_k u_{0,j}\|_{L^p} \\ &\quad + \int_0^t e^{-c(t-\tau)2^{k\beta}} \big(\|\Delta_k \bar{f}_j\|_{L^p} + \|\Delta_k \bar{R}_j\|_{L^p} + \|\Delta_k G_j\|_{L^p}\big) d\tau. \end{aligned} \tag{4.3}$$

Lemma 4.3 implies

$$\|\Delta_k G_j(t)\|_{L^p} \lesssim e^{cV(t)} V^{1-\beta/2}(t) 2^{j\beta} \|u_j^{m+1}\|_{L^p},$$

with $V(t) = \int_0^t \|\nabla v^m(\tau)\|_{L^\infty} d\tau$. From Bernstein inequality and Lemma 4.1

$$\|\Delta_k \bar{f}_j\|_{L^p} \lesssim 2^{-k} \|\nabla \Delta_k \bar{f}_j\|_{L^p} \lesssim 2^{-k} \|(\nabla f_j) \circ \psi_j\|_{L^p} \|\nabla \psi_j\|_{L^\infty} \lesssim 2^{j-k} e^{cV(t)} \|f_j\|_{L^p}.$$

A similarly argument implies

$$\|\Delta_k \bar{R}_j(t)\|_{L^p} \lesssim 2^{j-k} e^{cV(t)} \|R_j\|_{L^p}.$$

Taking the L^r norm over $[0,t]$ on (4.3) and plugging the above estimates give

$$\begin{aligned} 2^{j(s+\beta/r)} \|\Delta_k \bar{u}_j^{m+1}\|_{L^r_t L^p} &\lesssim 2^{(j-k)\beta/r} (1 - e^{-ctr2^{k\beta}})^{1/r} 2^{js} \|\Delta_k u_{0,j}\|_{L^p} \\ &\quad + 2^{j(s+\beta/r)} 2^{(j-k)\beta} e^{cV(t)} V^{1-\beta/2}(t) \|u_j^{m+1}\|_{L^r_t L^p} \\ &\quad + 2^{(j-k)(1+\beta/r)} e^{cV(t)} \int_0^t 2^{js} \big(\|f_j\|_{L^p} + \|R_j\|_{L^p}\big) d\tau. \end{aligned} \tag{4.4}$$

Let $M_0 \in \mathbb{Z}$ to be fixed later. Decomposing

$$u_j^{m+1} = S_{j-M_0} \bar{u}_j^{m+1} \circ \psi_j^{-1} + \sum_{k \ge j-M_0} \Delta_k \bar{u}_j^{m+1} \circ \psi_j^{-1}.$$

For all $t \in [0,T]$, there holds

$$\|u_j^{m+1}\|_{L^r_t L^p} \lesssim e^{cV(t)} \big(\|S_{j-M_0} \bar{u}_j^{m+1}\|_{L^r_t L^p} + \sum_{k \ge j-M_0} \|\Delta_k \bar{u}_j^{m+1}\|_{L^r_t L^p}\big). \tag{4.5}$$

By Lemma 4.1 and Lemma 4.2,

$$\|S_{j-M_0}\bar{u}_j^{m+1}\|_{L_t^rL^p} \lesssim e^{cV(t)}\big(e^{cV(t)}-1+2^{-M_0}\big)\|u_j^{m+1}\|_{L_t^rL^p}. \tag{4.6}$$

Since $\Delta_k u_{0,j}=0$ for $|k-j|\geq 2$ and $e^{cV(t)}-1+2^{-M_0}\lesssim e^{-c'V(t)}2^{-M_0}$, multiplying (4.5) by $2^{j(s+\beta/r)}$ and using (4.4) and (4.6), we obtain

$$\begin{aligned}
2^{j(s+\beta/r)}\|u_j^{m+1}\|_{L_t^rL^p} &\lesssim e^{cV(t)}2^{M_0\beta/r}\big(1-e^{-ctr2^{j\beta}}\big)^{1/r}2^{js}\|u_{0,j}\|_{L^p}\\
&\quad + e^{cV(t)}2^{j(s+\beta/r)}\big(2^{-M_0}+2^{M_0\beta}V^{1-\beta/2}(t)\big)\|u_j^{m+1}\|_{L_t^rL^p}\\
&\quad + e^{cV(t)}2^{M_0(1+\beta/r)}\int_0^t 2^{js}\big(\|f_j\|_{L^p}+\|R_j\|_{L^p}\big)d\tau.
\end{aligned}$$

Now we choose M_0 to be the unique integer such that $2c2^{-M_0}\in(1/8,1/4]$ and $T_1\leq T$ be the largest real number such that

$$cV(T_1)\leq \min\big(\ln 2, (\frac{2^{-M_0\beta}}{8c^{\beta/2}})^{\frac{2}{2-\beta}}\big).$$

Thus for $t\in[0,T_1]$,

$$2^{j(s+\beta/r)}\|u_j^{m+1}\|_{L_t^rL^p} \lesssim \big(1-e^{-ctr2^{j\beta}}\big)^{1/r}2^{js}\|u_{0,j}\|_{L^p} + \int_0^t 2^{js}\big(\|f_j\|_{L^p}+\|R_j\|_{L^p}\big)d\tau.$$

Taking the l^1-norm we conclude that

$$\begin{aligned}
&\|u^{m+1}\|_{\mathcal{L}_t^r\dot{B}_{p,1}^{s+\beta/r}}\\
&\lesssim \sum_{j\in\mathbb{Z}}\big(1-e^{-ctr2^{j\beta}}\big)^{1/r}2^{js}\|u_{0,j}\|_{L^p} + \sum_{j\in\mathbb{Z}}\int_0^t 2^{js}\big(\|f_j\|_{L^p}+\|R_j\|_{L^p}\big)d\tau.
\end{aligned}$$

□

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