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BASICITY IN L_p OF ROOT FUNCTIONS FOR DIFFERENTIAL EQUATIONS WITH INVOLUTION

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Abstract. We consider the differential equation

$$\alpha u''(-x) - u''(x) = \lambda u(x), \quad -1 < x < 1,$$

with the nonlocal boundary conditions $u(-1) = 0$, $u'(-1) = u'(1)$ where $\alpha \in (-1, 1)$. We prove that if $r = \sqrt{(1-\alpha)/(1+\alpha)}$ is irrational then the system of its eigenfunctions is complete and minimal in $L_p(-1, 1)$ for any $p > 1$, but does not constitute a basis. In the case of a rational value of r we specify the way of choosing the associated functions which provides the system of all root functions of the problem forms a basis in $L_p(-1, 1)$.

1. Introduction and statement of results

This article continues the research started in [19] where a full spectral analysis in $L_2(-1, 1)$ is given to the problem

$$\begin{gathered} \alpha u''(-x) - u''(x) = \lambda u(x), \quad -1 < x < 1, \\ u(-1) = 0, \quad u'(-1) = u'(1). \end{gathered} \tag{1.1}$$

The differential expression in (1.1) contains the involution transform of the argument x while the parameter α belongs to $(-1, 1)$.

When α equals zero the problem (1.1) becomes the known Samarskii-Ionkin problem [13] which gives the classical example of a boundary-value problem with regular, but not strongly regular boundary conditions. It has an infinite number of associated functions, and these functions could be tuned to produce (together with eigenfunctions) an unconditional basis in $L_2(-1, 1)$.

Such problems have a typical instability. Both the basicity of root functions and the equiconvergence of the related spectral decomposition with the Fourier trigonometric series could disappear at either of the following situations: (a) after a small change of associated functions in their root subspaces [12]; (b) after a perturbation of the differential expression by adding subordinate terms $a_1(x)u'(x) + a_2(x)u(x)$ with sufficiently small coefficients [11, 21]; (c) after a small shift of the boundary conditions; e.g., of the form $u'(0) = u'(1) + \varepsilon u(1)$, $\varepsilon \in (0, 1)$ [12]. Vladimir A. Il'in called such instability the essential nonself-adjointness of the problem.

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The considered boundary-value problem (1.1) for the differential equation with involution encapsulates the same instability but with respect to its parameter α.

Proposition 1.1 ([19]). *Denote*

$$r = \sqrt{(1-\alpha)/(1+\alpha)}. \tag{1.2}$$

Then

(1) *for any positive r, the system of root functions of* (1.1) *is complete and minimal in $L_2(-1,1)$;*
(2) *if r is irrational then there are no associated functions while the eigenfunctions of* (1.1) *do not constitute a basis in $L_2(-1,1)$;*
(3) *if r is rational then there is an infinite number of associated functions which could be chosen to make the whole system of root functions of* (1.1) *an unconditional basis in $L_2(-1,1)$.*

In this paper we obtain an analogous result in any Lebesgue space $L_p(-1,1)$, $1<p<\infty$. We prove the following results.

Theorem 1.2. *Let r in* (1.2) *be a positive irrational number. Then the system of eigenfunctions of* (1.1) *is complete and minimal in $L_p(-1,1)$, $1<p<\infty$, but is not uniformly minimal, and therefore does not constitute a basis in $L_p(-1,1)$.*

Theorem 1.3. *Let r be rational. Then the system of root functions of* (1.1) *is complete and minimal in $L_p(-1,1)$, $1<p<\infty$, and the associated functions could be chosen in such a way that the whole system forms a basis in $L_p(-1,1)$.*

The functional-differential equations with involutions evoked interest of mathematicians in early 1940s. Since 1970s the qualitative theory of first-order differential equations with involution is cultivated rather extensively (see, e.g., books by Przeworska-Rolewicz [26], Wiener [34] and the recent research by Watkins [33]). Boundary-value problems for second and higher order equations have been studied in [9, 25, 26, 34]. Cabada and Tojo added a new element in the previous studies: the construction of the Green function [6, 7]. Spectral topics (the basicity of root functions, equiconvergence of spectral expansions) for first- and second-order operators which contain involution in their main terms are discussed in [16, 17, 18, 27, 28].

Since the pioneering paper by Ionkin [13] for the heat flow equation and the introduction of a new approach to these problems by Il'in, there have been many research papers on non-local boundary value problems (see the overview in [12]). Among the recent ones – the papers of Aleroev, Kirane and Malik [1], Ashyralyev and Sarsenbi [3], Furati, Iyiola and Kirane [10], Kerimov [15], Makin [22], Mokin [24], Sarsenbi [29], Sarsenbi and Tengaeva [30].

For non-Hilbert spaces, the spectral properties of conventional differential operators were considered in [2, 5, 8, 20, 31].

2. The case of irrational number r

As in [19], one can easily calculate the spectrum of (1.1):

$$\sigma = \{0; \pi^2(1\pm\alpha)n^2,\ n\in\mathbb{N}\} \tag{2.1}$$

and the corresponding eigenfunctions:

$$
\begin{gathered}
\lambda_0 = 0 : u_0(x) = x + 1, \quad \lambda'_l = \pi^2(1+\alpha)l^2 : u_l^{(1)}(x) = \sin(\pi l x), \\
\lambda''_k = \pi^2(1-\alpha)k^2 : u_k^{(2)}(x) = \cos(\pi k x) + \frac{\cos \pi k}{\sin(\pi r k)} \sin(\pi r k x), \quad l, k \in \mathbb{N}.
\end{gathered} \tag{2.2}
$$

The dual system is formed by eigenfunctions of the adjoint problem

$$
\begin{gathered}
\alpha v''(-x) - v''(x) = \lambda v(x), \quad -1 < x < 1, \\
v(-1) = v(1), \quad \alpha v'(-1) = -v'(1),
\end{gathered} \tag{2.3}
$$

namely,

$$
\begin{gathered}
\lambda_0 = 0 : v_0(x) = 1/2, \quad \lambda''_k = \pi^2(1-\alpha)k^2 : v_k^{(2)}(x) = \cos(\pi k x), \\
\lambda'_l = \pi^2(1+\alpha)l^2 : v_l^{(1)}(x) = \sin(\pi l x) + \frac{\cos \pi l}{r \sin(\pi l / r)} \cos\Big(\frac{\pi l x}{r}\Big), \quad l, k \in \mathbb{N}.
\end{gathered} \tag{2.4}
$$

Recall that the system $\{e_n\}$ in a Banach space $\mathcal{B}$ is called *complete* in $\mathcal{B}$ if it spans $\mathcal{B}$ and is *minimal* if neither element in this system belongs to the span of others.

It is known [14, pp. 6–8] that

- the system $\{e_n\}$ is minimal in $\mathcal{B}$ if and only if it has the dual system $\{e_n^*\}$ in $\mathcal{B}^*$;
- if $\mathcal{B}$ is reflexive then the system $\{e_n\}$ is complete in $\mathcal{B}$ if and only if it is total, i.e. the relations $e^*(e_n) = 0$ for all n with a given $e^* \in \mathcal{B}^*$ yield $e^* = 0$.

Lemma 2.1. *Both systems* (2.2) *and* (2.4) *are complete and minimal in* $L_p(-1,1)$ *for any* $p > 1$.

Proof. The minimality of the systems (2.2) and (2.4) is provided by their mutual biorthogonality. Their completeness follows from totality. For instance, consider a function $f \in L_q(-1,1)$, $q^{-1} + p^{-1} = 1$, which is orthogonal to each function in (2.2). Then, as $f(x)$ is orthogonal to the functions $u_k^{(1)}(x)$, and due to the fact that the trigonometric system forms a basis in L_q [14, p.128], the function $f(x)$ a.e. coincides with an even function. Thus, we have

$$
0 = \int_{-1}^{1} f(x) u_k^{(2)}(x) dx = \sin(\pi r k) \int_{-1}^{1} f(x) \cos(\pi k x) dx
$$

and, since $r \notin \mathbb{Q}$, $f(x)$ is orthogonal to $\cos(\pi k x)$, $k \in \mathbb{N}$, and therefore, it is a.e. a constant function on $[-1,1]$. The relation $\int_{-1}^{1} f(x) u_0(x) dx = 0$ provides $f(x)$ vanishes a.e. on $[-1,1]$. The proof is complete. $\square$

The system $\{e_n\} \subset \mathcal{B}$ is called *uniformly minimal* in $\mathcal{B}$ [23] if its dual system $\{e_n^*\} \subset \mathcal{B}^*$ satisfies the relation

$$
\sup_n \Big(\|e_n\| \cdot \|e_n^*\| \Big) < \infty. \tag{2.5}
$$

Lemma 2.2. *Neither system* (2.2) *nor* (2.4) *is uniformly minimal in* $L_p(-1,1)$, $p > 1$.

Proof. Let us consider the system (2.2) in the space $L_p(-1,1)$. Taking into account that the $L_q(-1,1)$-norms of functions $v_k^{(2)}(x)$ in (2.4) ($q^{-1}+p^{-1}=1$) satisfy the estimates
$$2^{1/q} \ge \|v_k^{(2)}\|_q \ge 2^{-1/p}\|v_k^{(2)}\|_1 \ge 2^{-1/p}, \tag{2.6}$$
we show that there exists such a sequence k_n of positive integers such that the norm $\|u_{k_n}^{(2)}\|_p$ tends to infinity. Evaluating the L_1-norm of the function $u_k^{(2)}(x)$:
$$\begin{aligned} \int_{-1}^{1} |u_k^{(2)}(x)|dx &\ge \frac{1}{|\sin(\pi rk)|}\int_{-1}^{1}|\sin(\pi rkx)|dx - 2 \\ &\ge \frac{1}{|\sin(\pi rk)|}\Big(1-\frac{\sin(2\pi kr)}{2\pi kr}\Big) - 2 \end{aligned} \tag{2.7}$$
one notes (see [32, p.25]) that the inequality $|\frac{1}{r}-\frac{k}{s}| < 1/s^2$ has infinitely many solutions $k=k_n$, $s=s_n\in\mathbb{N}$. Hence $|\pi rk_n - \pi s_n| < \pi r/s_n$ and $|\sin(\pi rk_n)| < |\sin(\pi r/s_n)|$. Therefore, the right-hand side of inequality (2.7) blows up as $k = k_n\to\infty$ which means that the norm
$$\|u_{k_n}^{(2)}\|_p \ge 2^{(1-p)/p}\|u_{k_n}^{(2)}\|_1$$
also tends to infinity.

Together with estimate (2.6), this shows that the condition of uniform minimality (2.5) is not valid for the functions $u_{k_n}^{(2)}(x)$ and $v_{k_n}^{(2)}(x)$. The proof is complete. □

A system $\{e_n\}\subset\mathcal{B}$ is called a *basis* in $\mathcal{B}$ if, for any $f\in\mathcal{B}$, there exists a unique convergent to f series: $\sum_{n=1}^{\infty}\alpha_n e_n = f$. In this case the series is called the biorthogonal series for f and $\alpha_n = e_n^*(f)$ for any n. Any basis in $\mathcal{B}$ is a uniformly minimal system [23].

It follows from Lemma 2.2 that the systems (2.2) and (2.4) do not form bases in $L_p(-1,1)$ whatever $1<p<\infty$. Then the proof of Theorem 1.2 is complete.

3. The case of rational number r

Now let r be equal to the irreducible fraction $\frac{m_1}{m_2}$ ($m_1,m_2\in\mathbb{N}$). Then the spectrum (2.1) of problem (1.1) contains two subsequences that glue to each other:
$$\lambda_n^* \equiv \lambda'_{m_1 n} = \lambda''_{m_2 n} \quad \forall n\in\mathbb{N}. \tag{3.1}$$
These eigenvalues have multiplicity 2 and there are one eigenfunction and one associated function corresponding to them in each problem (1.1) and (2.3). The straightforward calculation shows that the biorthogonal pairs are formed by the functions (we use notation from (2.2) and (2.4))
$$\begin{gathered} u_0(x),\quad u_l^{(1)}(x),\quad l\not\equiv 0 \pmod{m_1}, \\ u_k^{(2)}(x),\quad k\not\equiv 0 \pmod{m_2}, \\ u_n^*(x) = \sin(\pi m_1 nx), \\ u_{n,1}^*(x) = \frac{x\cos(\pi m_1 nx) + (-1)^{(m_1+m_2)n}\cos(\pi m_2 nx)}{2(1+\alpha)\pi m_1 n} + a_n u_n^*(x),\quad n\in\mathbb{N}, \end{gathered} \tag{3.2}$$

for problem (1.1), and

$$
\begin{gathered}
v_0(x), \quad v_l^{(1)}(x), \quad l \not\equiv 0 \pmod{m_1}, \\
v_k^{(2)}(x), \quad k \not\equiv 0 \pmod{m_2}, \\
v_n^*(x) = 2(1+\alpha)\pi m_1 n(-1)^{(m_1+m_2)n}\cos(\pi m_2 nx), \\
v_{n,1}^*(x) = -r^{-1}(-1)^{(m_1+m_2)n} x\sin(\pi m_2 nx) + \sin(\pi m_1 nx) - a_n v_n^*(x), \quad n \in \mathbb{N},
\end{gathered} \tag{3.3}
$$

for problem (2.3). The functions $u_n^*(x), u_{n,1}^*(x)$ in (3.2) and $v_n^*(x), v_{n,1}^*(x)$ in (3.3) for each $n \in \mathbb{N}$ are the eigen- and associated functions which correspond to the sequence $\{\lambda_n^*\}$ in (3.1). The constants $a_n \in \mathbb{R}$ could be taken arbitrarily.

Lemma 3.1. *Systems* (3.2) *and* (3.3) *are complete and minimal in* $L_p(-1,1)$, $p > 1$.

The proof of Lemma 3.1 mimics the proof of Lemma 2.1, with minor changes. We omit it.

Lemma 3.2. *If* $a_n = O(1/n)$, $n \to \infty$, *then the systems* (3.2) *and* (3.3) *are uniformly minimal in* $L_p(-1,1)$, $p > 1$. *If* $\lim_{n\to\infty} na_n = \infty$ *then these systems are not uniformly minimal and, therefore, do not form bases.*

Proof. We start with eigenfunctions of the biorthogonal pair $u_l^{(1)}(x)$ and $v_l^{(1)}(x)$, $l \not\equiv 0 \pmod{m_1}$. Their norms satisfy the estimates:

$$
\|u_l^{(1)}\|_p \le 2^{1/p}, \quad \|v_l^{(1)}\|_q \le 2^{1/q}\Big(1 + \Big(r|\sin(\pi l/r)|\Big)^{-1}\Big).
$$

The right-hand part of the second estimate is bounded because for $l \not\equiv 0 \pmod{m_1}$ the number $l/r = lm_2/m_1$ is not integer and hence $|\sin(\pi l/r)| \ge \sin(\pi/m_1)$.

Similarly one can prove the boundedness of $\|u_k^{(2)}\|_p \cdot \|v_k^{(2)}\|_q$ for $k \not\equiv 0 \pmod{m_2}$.

In the case $\lambda = \lambda_n^*$ the biorthogonal pairs are formed by the functions $u_n^*(x), v_{n,1}^*(x)$ and $u_{n,1}^*(x), v_n^*(x)$. For all $n \in \mathbb{N}$ the relations

$$
c_1 \le \|u_n^*\|_p \le c_2, \quad c_1 n \le \|v_n^*\|_q \le nc_2 \tag{3.4}
$$

are valid with some positive constants c_1, c_2.

If $a_n = O(1/n)$ then

$$
\|u_{n,1}^*\|_p \le \frac{c_3}{n}, \quad \|v_{n,1}^*\|_q \le c_3,
$$

and, by virtue of (3.4), the uniform minimality condition (2.5) is satisfied.

If $\lim_{n\to\infty} na_n = \infty$ then we come to the estimates

$$
\|u_{n,1}^*\|_p \ge c_4|a_n| > 0, \quad \|v_{n,1}^*\|_q \ge c_4|a_n|n > 0,
$$

which mean that $\|u_n^*\|_p \cdot \|v_{n,1}^*\|_q$ and $\|u_{n,1}^*\|_p \cdot \|v_n^*\|_q$ disagree with (2.5). The proof is complete. $\square$

Further we consider the uniformly minimal systems (3.2) and (3.3) and for simplicity suppose that $a_n \equiv 0$ for any n. Since the natural normalization of the functions $u_{n,1}^*(x)$ and $v_n^*(x)$ makes these systems uniformly bounded on $[-1,1]$, the known result of Gaposhkin [23] provides they could form only conditional bases in $L_p(-1,1)$ for $p \ne 2$. Therefore, in order to study their basis properties we should specify the order of root functions in (3.2) and (3.3). In $L_2(-1,1)$ the order of root functions is irrelevant since they form an unconditional basis [19].

The proposed order will correspond to the order of functions in the classical trigonometric system. The biorthogonal system which consists of root functions of the problem (1.1) and the related root functions of the adjoint problem (2.3) starts with the pair

$$\begin{bmatrix} u_0(x) \\ v_0(x) \end{bmatrix} = \begin{bmatrix} x+1 \\ 1/2 \end{bmatrix},$$

which is followed by the juxtaposed blocks ($k = 1, 2, \dots$) of coupled pairs

$$\begin{aligned} &\begin{bmatrix} u_k^{(1)}(x) & u_k^{(2)}(x) \\ v_k^{(1)}(x) & v_k^{(2)}(x) \end{bmatrix} \\ &= \begin{bmatrix} \sin(\pi kx) & \cos(\pi kx) + \frac{\cos \pi k}{\sin \pi rk} \sin(\pi rkx) \\ \sin(\pi kx) + \frac{\cos \pi k}{r \sin \frac{\pi k}{r}} \cos(\frac{\pi kx}{r}) & \cos(\pi kx) \end{bmatrix}. \end{aligned}$$

However if $k \equiv 0 \pmod{m_1}$ then the first column of the block should be replaced by the column

$$\begin{bmatrix} \sin(\pi kx) \\ \sin(\pi kx) - r^{-1}(-1)^{(1+r)k/r} x \sin(\frac{\pi kx}{r}) \end{bmatrix};$$

if $k \equiv 0 \pmod{m_2}$ then the second column is also replaced by the column

$$\begin{bmatrix} (2(1+\alpha)\pi kr)^{-1} \Big[(-1)^{(1+r)k} \cos(\pi kx) + x \cos(\pi krx) \Big] \\ 2(1+\alpha)\pi kr (-1)^{(1+r)k} \cos(\pi kx) \end{bmatrix}.$$

Hence the partial sums of the biorthogonal series with respect to the root functions of the problem (1.1) take the form (we use the notation $K_1 = m_1\mathbb{N}$ and $K_2 = m_2\mathbb{N}$)

$$\begin{aligned} S_N(x,f) &= (f, v_0) u_0(x) + \sum_{\substack{1 \le k \le N \\ k \notin K_1}} (f, v_k^{(1)}) u_k^{(1)}(x) + \sum_{\substack{1 \le k \le N \\ k \notin K_2}} (f, v_k^{(2)}) u_k^{(2)}(x) \\ &\quad + \sum_{\substack{1 \le k \le N \\ k \in K_1}} \Big(f(t), \sin(\pi kt) - r^{-1}(-1)^{(1+r)k/r} t \sin(\frac{\pi kt}{r}) \Big) \sin(\pi kx) \\ &\quad + \sum_{\substack{1 \le k \le N \\ k \in K_2}} \Big(f(t), \cos(\pi kt) \Big) \Big[\cos(\pi kx) + (-1)^{(1+r)k} x \cos(\pi krx) \Big]. \end{aligned} \tag{3.5}$$

This sum evidently contains the partial sum of the Fourier trigonometric series:

$$\begin{aligned} &S_N^{(0)}(x,f) \\ &= (f, 1/2) + \sum_{k=1}^{N} \Big\{ (f(t), \cos(\pi kt)) \cos(\pi kx) + (f(t), \sin(\pi kt)) \sin(\pi kx) \Big\}, \end{aligned} \tag{3.6}$$

the remaining items group into the following sums:

$$S_N^{(1)}(x,f) = \sum_{\substack{1 \le k \le N \\ k \notin K_1}} \frac{\cos \pi k}{r \sin \frac{\pi k}{r}} \Big(f(t), \cos\big(\frac{\pi kt}{r}\big) \Big) \sin(\pi kx),$$

$$S_N^{(2)}(x,f) = \sum_{\substack{1 \le k \le N \\ k \notin K_2}} \frac{\cos \pi k}{\sin(\pi kr)} \Big(f(t), \cos(\pi kt) \Big) \sin(\pi krx),$$

$$S_N^{(3)}(x,f) = -\sum_{\substack{1\le k\le N\\ k\in \overline{K}_1}} r^{-1}(-1)^{(1+r)k/r}\Big(f(t), t\sin\big(\frac{\pi kt}{r}\big)\Big)\sin(\pi kx),$$
$$S_N^{(4)}(x,f) = \sum_{\substack{1\le k\le N\\ k\in \overline{K}_2}} (-1)^{(1+r)k}\Big(f(t),\cos(\pi kt)\Big)x\cos(\pi krx). \tag{3.7}$$

To analyze these four sums, we decompose $f(x)$ into the sum of its even and odd components

$$f(x) = f_+(x) + f_-(x) \equiv \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$$

and note that for the odd component $f_-(x)$ all the sums in (3.7) vanish.

In $S_N^{(3)}(x,f_+)$ we make the substitution $k=m_1n$ and for simplicity suppose that m_1+m_2 is even. Then this sum takes the form

$$S_N^{(3)}(x,f_+) = -r^{-1}\sum_{\substack{1\le k\le N\\ k=m_1n}}\int_0^1 f_+(t)t\sin(\pi m_2nt)dt\cdot\sin(\pi m_1nx)$$

and further substitutions $\tau = m_2t, y=m_1x$ transform it into the sum

$$S_N^{(3)}(x,f_+) = -(rm_2^2)^{-1}\sum_{\substack{1\le k\le N\\ k=m_1n}}\int_0^{m_2} f_+\Big(\frac{\tau}{m_2}\Big)\tau\sin(\pi n\tau)d\tau\cdot\sin(\pi ny).$$

It could be easily interpreted as a sum of m_2 partial sums of Fourier trigonometric series for functions which L_p-norms are $O(1)\|f\|_p$. A similar conclusion could be made about $S_N^{(4)}(x,f_+)$.

The sum $S_N^{(2)}(x,f_+)$ naturally splits into m_2-1 items in accordance with the remainder $k_1 = k \pmod{m_2}$, $k_1=\overline{1,m_2-1}$. We suppose, for simplicity, that k_1 and m_1+m_2 are even. Then the corresponding parts of the sum equal

$$\begin{aligned}
S_N^{(2,k_1)}(x,f_+) = \frac{1}{\sin(\pi k_1 r)}\sum_{\substack{1\le k\le N\\ k=k_1+m_2n}}\Big\{&\int_0^1 f(t)\cos(\pi k_1t)\cos(\pi m_2nt)dt\\
&\times\Big[\cos(\pi m_1nx)\sin(\pi k_1rx)+\sin(\pi m_1nx)\cos(\pi k_1rx)\Big]\\
&-\int_0^1 f(t)\sin(\pi k_1t)\sin(\pi m_2nt)dt\\
&\times\Big[\cos(\pi m_1nx)\sin(\pi k_1rx)+\sin(\pi m_1nx)\cos(\pi k_1rx)\Big]\Big\}.
\end{aligned}$$

Similar to $S_N^{(3)}(x,f_+)$ this expression consists of four items which are combinations of the partial sums of Fourier trigonometric series for functions which L_p-norms are $O(1)\|f\|_p$, and of the partial sums of conjugate trigonometric series which converge in $L_p(0,1)$ to functions which L_p-norms are also $O(1)\|f\|_p$ by Riesz theorem [4, p. 566]. The remaining sum $S_N^{(1)}(x,f_+)$ is considered similarly.

It is known [4, pp.593–594] that if $F(x)\in L_p$ then the partial sums $\sigma_N(x,F)$ of its Fourier trigonometric series and the partial sums $\sigma_N^*(x,F)$ of its conjugate series satisfy the estimate

$$\|\sigma_N(x,F)\|_p \le c\|F\|_p, \quad \|\sigma_N^*(x,F)\|_p \le c\|F\|_p$$

uniformly with respect to N.

It follows from (3.5)–(3.7) that

$$\|S_N(x,f)\|_p \le \|(f,1/2)x\|_p + \|S_N^{(0)}(x,f)\|_p + \sum_{j=1}^{4} \|S_N^{(j)}(x,f_+)\|_p = O(1)\|f\|_p \quad (3.8)$$

uniformly with respect to N.

The system of root functions of the problem (1.1) is complete and minimal in $L_p(-1,1)$ (Lemma 3.1), therefore (see, e.g., [14, p. 11]) the estimate (3.8) is sufficient for its basicity in $L_p(-1,1)$ for $p>1$. Theorem 1.3 is proved.

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