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SOLUTIONS TO NONLINEAR SCHRÖDINGER EQUATIONS FOR SPECIAL INITIAL DATA

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Abstract. This article concerns the solvability of the nonlinear Schrödinger equation with gauge invariant power nonlinear term in one space dimension. The well-posedness of this equation is known only for H^s with $s \geq 0$. Under some assumptions on the nonlinearity, this paper shows that this equation is uniquely solvable for special but typical initial data, namely the linear combinations of $\delta(x)$ and $\mathrm{p.v.}(1/x)$, which belong to $H^{-1/2-0}$. The proof in this article allows L^2-perturbations on the initial data.

1. Introduction

In this article we consider nonlinear Schrödinger equations with a gauge invariant nonlinear term

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = f(u), \tag{1.1}$$

where $u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{C}$, $f(u) = |u|^{p-1}u$, $p > 1$. The discussion in this paper is irrelevant to the sign of the nonlinear term, so we only treat the defocusing case. For $1 < p < 5$, the well-posedness in L^2 of the Cauchy problem for (1.1) was proved in [2]. However there are few results on the solvability of (1.1) for initial data in negative order Sobolev spaces. Although Kenig-Ponce-Vega [1] treated the case where $f(u) = u^2, \bar{u}^2$ or $|u|^2$ and proved the well-posedness in negative order Sobolev spaces, their result does not cover the nonlinear term that we treat in this paper. On the other hand Kita [3, 4] solved (1.1) with $u(0) = \delta(x) + L^2$-perturbation for $1 < p < 3$ by using the fact that there exists an exact solution if $u(0) = \delta(x)$. Namely $u(t) = (2\pi it)^{-1/2}\exp\{ix^2/2t - (2\pi t)^\alpha/(1-\alpha)\}$ satisfies (1.1), where $\alpha = (p-1)/2$.

In this article we consider the case where

$$u(0) = \sqrt{2\pi i}\,\lambda\delta(x) - \sqrt{\frac{2}{\pi i}}\mu\,\mathrm{p.v.}\,\frac{1}{x} + v_0. \tag{1.2}$$

Here $\lambda, \mu \in \mathbb{C}$, $v_0 \in L^2$, and p.v. means Cauchy's principle value. Unlike the case $u(0) = \delta(x)$, exact solutions for (1.1)-(1.2) are not known. This makes the problem more difficult. We introduce a first approximation of the solution to (1.1) and determine the difference of the solution itself and the first approximation by

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the contraction mapping principle. (Theorems 3.1 and 4.1) We have to assume $1 < p < 5/2$ if $\lambda = 0$ and assume $1 < p < 7/3$ for general case. Finally we remark on uniqueness of solutions. (Theorem 5.1)

Notation. Throughout this paper we put $\alpha = (p-1)/2$. $\mathcal{L} = i\partial_t + \partial_x^2/2$. $U(t) = \exp(it\partial_x^2/2)$ is the free propagator. L^r denote usual Lebesgue spaces for the space variable. For $T > 0$, $L^q_T L^r$ is the abbreviation of $L^q(0,T;L^r)$. $X_T = L^\infty_T L^2 \cap L^4_T L^\infty$ and $Y_T = L^1_T L^2 + L^{4/3}_T L^1$.

2. Preliminaries

Lemma 2.1 (Strichartz). *For any $\phi \in L^2$ and for any $F \in Y_T$, the following inequalities hold valid:*

$$\|U(t)\phi\|_{X_T} \le C\|\phi\|_2,$$
$$\|\int_0^t U(t-\tau)F(\tau)d\tau\|_{X_T} \le C\|F\|_{Y_T}.$$

The constants C are independent of T, ϕ and F.

For a proof of the above lemma, see [5, 6]. Before proceeding to the nonlinear problem, we consider the linear Cauchy problem.

Lemma 2.2. *Let $\lambda, \mu \in \mathbb{C}$ and let u_{L} be the solution of*

$$i\partial_t u_{\mathrm{L}} + \frac{1}{2}\partial_x^2 u_{\mathrm{L}} = 0 \tag{2.1}$$

with

$$u_{\mathrm{L}}(0) = \sqrt{2\pi i}\lambda\delta(x) - \sqrt{\frac{2}{\pi i}}\mu\,\mathrm{p.v.}\,\frac{1}{x}. \tag{2.2}$$

Then

$$u_{\mathrm{L}} = U(t)u_{\mathrm{L}}(0) = \frac{e^{ix^2/2t}}{\sqrt{t}}g(x/\sqrt{t}).$$

Here

$$g(a) = \lambda + \sqrt{\frac{i}{2\pi}}\mu\Big[\int_{-\infty}^a e^{\frac{-i\eta^2}{2}}d\eta - \int_a^\infty e^{\frac{-i\eta^2}{2}}d\eta\Big].$$

We remark that using integration by parts, we can easily show that

$$g(a) = \lambda \pm \mu + O(1/a) \quad \text{as } a \to \pm\infty.$$

Proof of Lemma 2.2. This is done by a direct calculation. It is well-known that

$$\mathcal{F}u_{\mathrm{L}}(0) = \sqrt{2\pi i}(\lambda + \mu\,\mathrm{sign}\,\xi),$$

where $\mathcal{F}$ is the Fourier transform. Therefore

$$u_{\mathrm{L}}(t) = \sqrt{\frac{i}{2\pi}}\int_{-\infty}^\infty e^{ix\xi - it\xi^2/2}(\lambda + \mu\,\mathrm{sign}\,\xi)d\xi \tag{2.3}$$

$$= \sqrt{\frac{i}{2\pi}}e^{ix^2/2t}\int_{-\infty}^\infty e^{-it(\xi - x/t)^2/2}(\lambda + \mu\,\mathrm{sign}\,\xi)d\xi. \tag{2.4}$$

Changing the variable as $-\sqrt{t}(\xi - x/t) = \eta$, we obtain the result. $\square$

3. Case $\lambda = 0$

In this section we consider the case

$$u(0) = -\sqrt{\frac{2}{\pi i}}\mu \,\mathrm{p.\,v.}\, \frac{1}{x} + v_0 \tag{3.1}$$

with $v_0 \in L^2$. In this case we put $A(t) = \exp(-i|\mu|^{p-1}t^{1-\alpha}/(1-\alpha))$ and put $v = u - Au_{\mathrm{L}}$ where u satisfies (1.1) and u_{L} is defined as in Lemma 2.2 with $\lambda = 0$. Then v satisfies

$$\mathcal{L}v = R + N \tag{3.2}$$

with $v(0) = v_0$, where

$$R = t^{-\alpha}(-|\mu|^{p-1} + |g(x/\sqrt{t})|^{p-1})Au_{\mathrm{L}}, \quad N = f(Au_{\mathrm{L}} + v) - f(Au_{\mathrm{L}}).$$

By Duhamel's principle, we convert this equation to the integral form

$$v(t) = U(t)v_0 - i\int_0^t U(t-\tau)\{R(\tau) + N(\tau)\}d\tau. \tag{3.3}$$

Theorem 3.1. *Let $1 < p < 5/2$ and let $v_0 \in L^2$. Then there exists $0 < T \le 1$ such that (3.3) has a unique solution in X_T.*

Proof. For simplicity we only prove a priori estimates; precise proof is done by the contraction mapping principle. We apply Lemma 2.1 to the right-hand side of (3.3) and obtain

$$\|v\|_{X_T} \le C\|v_0\|_2 + C\|R\|_{Y_T} + C\|N\|_{Y_T}. \tag{3.4}$$

By the remark to Lemma 2.2 we have

$$\|R\|_2 \le Ct^{-\alpha-1/2}\|\mu - g(x/\sqrt{t})\|_2 \le Ct^{-\alpha-1/2}\|\langle x/\sqrt{t}\rangle^{-1}\|_2 \le Ct^{-\alpha-1/4}. \tag{3.5}$$

Therefore $\|R\|_{Y_T} \le \|R; L^1_T L^2\| \le CT^{-\alpha+3/4}$. We proceed to the estimate of the second term in the right-hand side of (3.4). We can easily show that

$$|N| \le Ct^{-\alpha}|v| + C|v|^p.$$

Therefore

$$\|N\|_{Y_T} \le C\|t^{-\alpha}|v|; L^1_T L^2\| + \||v|^p; L^{q_0'}_T L^{r_0'}\|$$

with $r_0 = p+1$ and $q_0 = 4(p-1)/(p+1)$ since $L^{q_0'}_T L^{r_0'} \hookrightarrow Y_T$. Applying Hölder's inequality for the time variable we obtain

$$\begin{aligned}\|N\|_{Y_T} &\le CT^{1-\alpha}\|v; L^\infty_T L^2\| + CT^{1-\alpha/2}\|v; L^{q_0}_T L^{r_0}\|^p \\ &\le CT^{1-\alpha}\|v\|_{X_T} + CT^{1-\alpha/2}\|v\|^p_{X_T}.\end{aligned}$$

Therefore, we have proved that

$$\|v\|_{X_T} \le C\|v_0\|_2 + CT^{1-\alpha}\|v\|_{X_T} + CT^{1-\alpha/2}\|v\|^p_{X_T}.$$

If v_1 and v_2 are solutions to (3.3), we similarly obtain

$$\|v_1 - v_2\|_{X_T} \le C(T^{1-\alpha} + T^{1-\alpha/2}\sum_{j=1}^{2}\|v_j\|^{p-1}_{X_T})\|v_1 - v_2\|_{X_T}.$$

By these estimates, we can apply the contraction mapping principle. □

4. General case

We first remark that if $|\lambda+\mu| = |\lambda-\mu|$, $\lambda+\mu = 0$ or $\lambda-\mu = 0$, then we can prove analogous results to Theorem 3.1 by the same method. However in general case the method in the previous section does not work. Therefore we consider another first approximation Au_{L} of the solution as follows. Let $\rho \in C^2(\mathbb{R})$ be a real-valued function satisfying $\rho(a) = |\lambda \pm \mu|^{p-1}$ for $\pm a \geq 1$. We put

$$A(t,x) = \exp\Big[-i\int_0^t \tau^{-\alpha}\rho(x/\tau^\beta)d\tau\Big],$$
$$v = u - Au_{\mathrm{L}},$$

where $\beta > 0$ is suitably chosen, u is a solution to (1.1) with (1.2), and u_{L} is defined as in Lemma 2.2. Then v satisfies

$$\mathcal{L}v = \sum_{j=1}^4 R_j + N,$$

where

$$R_1 = \frac{1}{2}\Big[\int_0^t \tau^{-\alpha-\beta}\rho'(x/\tau^\beta)d\tau\Big]^2 Au_{\mathrm{L}}, \quad R_2 = \frac{i}{2}\int_0^t \tau^{-\alpha-2\beta}\rho''(x/\tau^\beta)d\tau Au_{\mathrm{L}},$$
$$R_3 = i\int_0^t \tau^{-\alpha-\beta}\rho'(x/\tau^\beta)d\tau A\partial_x u_{\mathrm{L}}, \quad R_4 = t^{-\alpha}[|g(x/\sqrt{t})|^{p-1} - \rho(x/t^\beta)]Au_{\mathrm{L}}$$

Similarly as in the previous section, we convert this equation to the integral form

$$v(t) = U(t)v_0 - i\int_0^t U(t-\tau)\{\sum_{j=1}^4 R_j(\tau) + N(\tau)\}d\tau. \tag{4.1}$$

We look for the solution to (4.1) by the contraction mapping principle.

Theorem 4.1. *Let $1 < p < 7/3$ and let $(p-2)_+ < \beta < (3-p)/2$. Then there exists $0 < T \leq 1$ such that (4.1) has a unique solution in X_T.*

Proof. We only show a priori estimates. By Lemma 2.1,

$$\begin{aligned}\|v\|_{X_T} &\leq C\|v_0\|_2 + C\sum_{j=1}^4 \|R_j\|_{Y_T} + C\|N\|_{Y_T}\\ &\leq C\|v_0\|_2 + C\|R_1; L_T^{4/3}L^1\| + C\|R_2; L_T^{4/3}L^1\|\\ &\quad + C\|R_3; L_T^1 L^2\| + C\|R_4; L_T^1 L^2\| + C\|N\|_{Y_T}.\end{aligned}$$

We estimate the right-hand side term by term.

$$\begin{aligned}\|R_1\|_1 &\leq Ct^{-1/2}\Big[\int_0^t \tau^{-\alpha-\beta}\|\rho'(x/\tau^\beta)\|_2 d\tau\Big]^2\\ &= Ct^{-1/2}\Big(\int_0^t \tau^{-\alpha-\beta/2}d\tau\Big)^2 = Ct^{3/2-2\alpha-\beta}\end{aligned}$$

if $\alpha + \beta/2 < 1$, and hence $\|R_1; L_T^{4/3}L^1\| \leq CT^{9/4-2\alpha-\beta}$ under this condition. Similarly $\|R_2; L^{4/3}L^1\| \leq CT^{5/4-\alpha-\beta}$ if $\alpha + \beta < 1$. By the estimate

$$|R_3| \leq C\int_0^t \tau^{-\alpha-\beta}|\rho'(x/\tau^\beta)|d\tau(|x|/t^{3/2} + 1/t),$$

we obtain

$$\|R_3\|_2 \le Ct^{-1/2-\alpha+\beta/2} + Ct^{-\alpha-\beta/2}$$

if $\alpha - \beta/2 < 1$ and $\alpha + \beta/2 < 1$. Therefore

$$\|R_3; L^1_T L^2\| \le CT^{1/2-\alpha+\beta/2} + CT^{1-\alpha-\beta/2}$$

if $\alpha - \beta/2 < 1/2$ and $\alpha + \beta/2 < 1$.

The estimate of R_4 is done as follows. We can easily show that

$$|R_4| \le \begin{cases} Ct^{-\alpha-1/2}, & |x/\sqrt{t}| \le 1 \text{ or } |x/t^\beta| \le 1, \\ Ct^{-\alpha}/|x|, & |x/\sqrt{t}| \ge 1 \text{ and } |x/t^\beta| \ge 1. \end{cases}$$

We first consider the case where $\beta \ge 1/2$. Then $|x/t^\beta| \ge 1$ follows from $|x/\sqrt{t}| \ge 1$ since we may assume $0 < t < T < 1$. So we divide the spatial real-axis into the parts $|x/\sqrt{t}| \le 1$ and $|x/\sqrt{t}| \ge 1$ and we denote the corresponding parts of R_4 by $R_{4,<}$ and $R_{4,>}$. By the estimate above we have $\|R_{4,<}\|_2 \le Ct^{-\alpha-1/4}$ and hence $\|R_{4,<}; L^1_T L^2\| \le CT^{-\alpha+3/4}$ if $\alpha < 3/4$. We can similarly estimate $R_{4,>}$ and we obtain $\|R_4; L^1_T L^2\| \le CT^{-\alpha+3/4}$ if $\alpha < 3/4$. If $\beta < 1/2$ we divide the spatial real-axis into the parts $|x/t^\beta| \le 1$ and $|x/t^\beta| \ge 1$ and estimate R_4 similarly. Then we obtain $\|R_{4,<}; L^1_T L^2\| \le CT^{-\alpha-\beta/2+1/2}$ if $\alpha - \beta/2 < 1/2$ and $\|R_{4,>}; L^1_T L^2\| \le CT^{-\alpha-\beta/2+1}$ if $\alpha + \beta/2 < 1$. On the other hand, the estimate of N is same as in the previous section. Collecting all the estimates above, we can conclude that

$$\|v\|_{X_T} \le C\|v_0\|_2 + CT^\epsilon + CT^{1-\alpha}\|v\|_{X_T} + CT^{1-\alpha/2}\|v\|^p_{X_T} \tag{4.2}$$

with some $\epsilon > 0$, under the conditions that

$$\alpha + \beta < 1 \quad \text{and} \quad \alpha - \beta/2 < 1/2,$$

which is possible if $0 < \alpha < 2/3$ and $(2\alpha - 1)_+ < \beta < 1 - \alpha$, or equivalently the assumption for p and β. The estimate for the difference of two solutions is the same as in the previous section. □

5. A remark on the uniqueness

In the previous two sections we discuss the unique existence of the integral equations (3.3) or (4.1). However the uniqueness of the solution to (1.1) with (3.1) or (1.2) may fail because different first approximations derive another solutions. In this section we consider this problem. Let $\tilde{\rho} \in C^2(\mathbb{R})$ be a different real-valued function from ρ in the previous section but let $\tilde{\rho}$ satisfy $\tilde{\rho}(a) = |\lambda \pm \mu|^{p-1}$ for $\pm a \ge 1$. Let $\tilde{\beta} > 0$, and we put

$$\tilde{A}(t,x) = \exp\Big[-i\int_0^t \tau^{-\alpha}\tilde{\rho}(x/\tau^{\tilde{\beta}})d\tau\Big],$$
$$\tilde{v} = u - \tilde{A}u_{\mathrm{L}}.$$

As in the previous section, we convert this equation with $\tilde{v}(0) = v_0$ into integral form and solve this integral equation by the contraction mapping principle. We want to prove that $v + Au_{\mathrm{L}} = \tilde{v} + \tilde{A}u_{\mathrm{L}}$, where v and A are the ones in the previous section. To this end it is sufficient to show the following.

Theorem 5.1. *Let $1 < p < 5/2$, $\beta > (p-2)_+$ and let $\tilde{\beta}$ satisfy the same condition. Then $Au_{\mathrm{L}} - \tilde{A}u_{\mathrm{L}} \in X_T$.*

Proof. By the estimate

$$|Au_{\mathrm{L}} - \tilde{A}u_{\mathrm{L}}| \le Ct^{-1/2} \int_0^t \tau^{-\alpha} |\rho(x/\tau^{\beta}) - \tilde{\rho}(x/\tau^{\tilde{\beta}})| d\tau, \tag{5.1}$$

we can easily show that $\|Au_{\mathrm{L}} - \tilde{A}u_{\mathrm{L}}; L_T^4 L^\infty\| \le CT^{3/4-\alpha}$ if $\alpha < 3/4$. On the other hand, the right-hand side of (5.1) does not exceed

$$Ct^{-1/2} \int_0^t \tau^{-\alpha} |\rho(x/\tau^{\beta}) - |g(x/\sqrt{\tau})|^{p-1}| d\tau + Ct^{-1/2} \int_0^t \tau^{-\alpha} |\tilde{\rho}(x/\tau^{\tilde{\beta}}) - |g(x/\sqrt{\tau})| d\tau,$$

where g is defined in Lemma 2.2, it suffices to show that the first integral in the above quantity belongs to X_T. Since

$$|\rho(x/\tau^{\beta}) - |g(x/\sqrt{\tau})|^{p-1}| \le \begin{cases} C, & |x/\sqrt{\tau}| \le 1 \text{ or } |x/\tau^{\beta}| \le 1, \\ C\sqrt{\tau}/|x|, & |x/\sqrt{\tau}| \ge 1 \text{ and } |x/\tau^{\beta}| \ge 1, \end{cases}$$

we obtain

$$\|t^{-1/2} \int_0^t \tau^{-\alpha} |\rho(x/\tau^{\beta}) - |g(x/\sqrt{\tau})|^{p-1}| d\tau; L_T^\infty L^2\| \le CT^{-\alpha+\beta/2+1/2} + CT^{-\alpha-\beta/2+1}$$

if $\alpha < 3/4$ and $\beta > (2\alpha - 1)_+$. Thus we have proved the theorem. □

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