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DYNAMICS OF STOCHASTIC NONCLASSICAL DIFFUSION EQUATIONS ON UNBOUNDED DOMAINS

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ABSTRACT. This article concerns the dynamics of stochastic nonclassical diffusion equation on $\mathbb{R}^N$ perturbed by a ϵ-random term, where $\epsilon \in (0,1]$ is the intension of noise. By using an energy approach, we prove the asymptotic compactness of the associated random dynamical system, and then the existence of random attractors in $H^1(\mathbb{R}^N)$. Finally, we show the upper semi-continuity of random attractors at $\epsilon = 0$ in the sense of Hausdorff semi-metric in $H^1(\mathbb{R}^N)$, which implies that the obtained family of random attractors indexed by ϵ converge to a deterministic attractor as ϵ vanishes.

1. INTRODUCTION

In this article, we consider the dynamics of solutions to the following stochastic nonclassical diffusion equation driven by an additive noise with intension ϵ:

$$\begin{gathered} u_t - \Delta u_t - \Delta u + u + f(x,u) = \mathrm{g}(x) + \epsilon h\dot{W}, \quad x \in \mathbb{R}^N, \\ u(x,\tau) = u_0(x), \quad x \in \mathbb{R}^N, \end{gathered} \tag{1.1}$$

where the initial data $u_0 \in H^1(\mathbb{R}^N)$; $\epsilon \in (0,1]$; $u = u(x,t)$ is a real valued function of $x \in \mathbb{R}^N$ and $t > \tau$; $\dot{W}(t)$ is the generalized time derivative of an infinite dimensional Wiener process $W(t)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{\omega \in C(\mathbb{R},\mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ is the σ-algebra of Borel sets induced by the compact-open topology of Ω, $\mathbb{P}$ is the corresponding Wiener measure on $\mathcal{F}$ for which the canonical Wiener process $W(t)$ satisfies that both $W(t)_{t\geq 0}$ and $W(t)_{t\leq 0}$ are usual one dimensional Brownian motions. We may identify $W(t)$ with $\omega(t)$, that is, $W(t) = W(t,\omega) = \omega(t)$ for all $t \in \mathbb{R}$.

To study system (1.1), we assume that $g \in L^2(\mathbb{R}^N)$ and $f(x,u) = f_1(x,u) + a(x)f_2(u)$ such that

$$a(.) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \tag{1.2}$$

and for every fixed $x \in \mathbb{R}^N$, $f_1(x,\cdot) \in C(\mathbb{R},\mathbb{R})$ satisfying

$$f_1(x,s)s \geq \alpha_1|s|^p - \psi_1(x), \quad \psi_1 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \tag{1.3}$$

$$|f_1(x,s)| \leq \beta_1|s|^{p-1} + \psi_2(x), \quad \psi_2 \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N), \tag{1.4}$$

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$$(f_1(x,s)-f_1(x,r))(s-r) \geq -l(s-r)^2, \tag{1.5}$$

and $f_2(\cdot) \in C(\mathbb{R},\mathbb{R})$ satisfying

$$f_2(s)s \geq \alpha_2|s|^p - \gamma, \tag{1.6}$$

$$|f_2(s)| \leq \beta_2|s|^{p-1} + \delta, \tag{1.7}$$

$$(f_2(s)-f_2(r))(s-r) \geq -l(s-r)^2, \tag{1.8}$$

where $\alpha_i, \beta_i (i=1,2)$, γ, δ and l are positive constants. The function h in (1.1) satisfies

$$h \in H^1(\mathbb{R}^N). \tag{1.9}$$

The nonclassical diffusion equation is an important mathematical model which depicts such physical phenomena as non-Newtonian flows, solid mechanics, and heat conduction, where the viscidity, the elasticity and the pressure of medium are taken into account, see e.g.[1, 2, 20]. In the deterministic case; that is, $\epsilon = 0$ in (1.1), the dynamics of nonclassical diffusion equation on bounded domains have been extensively studied by several authors in [3, 28, 29, 37]. The same model with fading memory is considered in [38, 39]. By means of the omega-limit-compactness argument, [18] obtained the pullback attractors for the nonclassical diffusion equations with variable delay on any bounded domain, where the nonlinearity is at most two orders growth.

As far as the unbounded case for the system (1.1) is concerned, most recently, by the tail estimate technique and some omega-limit-compactness argument, [23] proved the existence of global attractors in the entire space $H^1(\mathbb{R}^N)$, where the nonlinearity satisfies a similar growth as (1.2)-(1.8) but possesses certain differentiability assumptions. By a similar technique, Zhang et al [43] obtained the pullback attractors for the non-autonomous case in $H^1(\mathbb{R}^N)$, where the growth order of the nonlinearity is assumed to be controlled by the space dimension N, such that the Sobolev embedding $H^1 \hookrightarrow L^{2p-2}$ is continuous. However, it is regretted that some terms in the proof of [43, Lemma 3.4] are lost, besides the inequality (3.45) in that paper is not correct. Some similar errors can also be found in [23]. Recently, Anh *et al.* [4] established the existence of pullback attractor in the space $H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, where the nonlinearity satisfies an arbitrary polynomial growth, but some additional assumptions on the primitive function of the nonlinearity are required. To the best of our knowledge, the dynamics of system (1.1) involving random white noises has not been attacked by predecessors, even for the bounded case.

The analysis of the dynamics of stochastic partial differential equations (SPDEs) is one important topic in modern mathematical and physical fields. The notion of random attractor, developed in [14, 15, 16, 17, 27], is a suitable tool to attack this problem. The existences of random attractors for some concrete SPDEs have been extensively studied by many authors, see [11, 14, 21, 22] and references cited there. These have been involved in different spaces with different approaches, such as L^2 space [11, 14, 47] by the compact embedding, $L^{\varpi}(\varpi > 2)$ space [21, 22, 40, 46] by asymptotic a priori estimate, H_0^1 space [44, 45] by omega-limit-compactness argument. We may also find a large volume of literature on this topic for other SPDEs on bounded or unbounded domains.

However, it is a very interesting and challenging work to consider the existence of random attractors for the SPDEs defined on unbounded domains. This is because the asymptotic compactness of solutions cannot be obtained by a standard priori estimate technique as the bounded case. For the deterministic equations, this difficulty can be overcome by Ball' energy equation approach [7, 8], a tail estimate method [35, 42] and using other Banach spaces, such as the weighted space [24, 41] and etc.

Recently, Bates and his coworkers [9] generalized the tail estimates method to the random case, where the asymptotic compactness in $L^2(\mathbb{R}^N)$ for solutions of stochastic reaction-diffusion equations with additive noises is successfully proved. For the applications of this related method we may refer to [31, 32, 33, 36] and references therein. It is also worth pointing out that most recently, by using energy equation approach, Brzeźniak et al [10] obtained the asymptotic compactness of solution of stochastic 2D-Navier-Stokes equations on some unbounded domains, then the existence of random attractor for this equation is established.

In this article, the first purpose is to prove the existence of random attractor $\mathcal{A}_\epsilon$ of the initial problem (1.1) defined on $\mathbb{R}^N$. There are some problems encountered. On the one hand, it is worth noticing that for this equation, because of the term Δu_t, if the initial value u_0 belongs to $H^1(\mathbb{R}^N)$, then the solution is always in $H^1(\mathbb{R}^N)$ and has no higher regularity, which is similar to the hyperbolic case. On the other hand, the scheme in [4], which heavily relied on the assumption on the primitive function of the nonlinearity, can not be generalized to the random cases. This is because the Wiener process $W(t)$ is only continuous but not differentiable in time t and thus it is difficult to obtain the estimate of the time derivative u_t in randomly perturbed case. Thirdly, although the articles [23, 43] considered the same equations as (1.1), on account of the errors mentioned above we do not know whether or not the method developed there is applicable.

To overcome these obstacles, in this article we turn to the energy equation approach. We first prove that the weak solution of the transformed nonclassical diffusion equation is weakly continuous in $H^1(\mathbb{R}^N)$. Then the existence of a random bounded absorbing set is sufficient to show that the random dynamical system related to equations (1.1) is asymptotically compact in $H^1(\mathbb{R}^N)$. Furthermore, this asymptotic compactness is uniform in $\epsilon \in (0, 1]$, see Lemma 5.2. Some technical problems about this method in random cases are surmounted. Then the existence of random attractor in $H^1(\mathbb{R}^N)$ is proved, see Theorem 5.3.

The second goal of this article is to attack the upper semi-continuity of the random attractors $\mathcal{A}_\epsilon$ at $\epsilon = 0$ in the topology of $H^1(\mathbb{R}^N)$. Note that in the case $\epsilon = 0$, the system (1.1) is a deterministic equation and admits a global attractor $\mathcal{A}_0$ in $H^1(\mathbb{R}^N)$. It is therefore of great interest to understand both the dynamics of the stochastic equations itself and the influence of the small white noises as ϵ varies in $(0, 1]$, in particular, as $\epsilon \searrow 0$. The result on this aspect is Theorem 6.2.

The framework of this article is as follows. In section 2, we present some associated theory and notions on random dynamical systems (RDSs). In section 3, we show the existence and uniqueness of weak solution for the transformed equation with random coefficients. In section 4, we prove that the weak solutions is weakly continuous in $H^1(\mathbb{R}^N)$. In section 5, the asymptotic compactness is proved by using energy equation approach and then we establish the existence of random attractors for system (1.1) in $H^1(\mathbb{R}^N)$. In the final section, we study the convergence of the random attractors $\mathcal{A}_\epsilon$ as $\epsilon \searrow 0$.

In this article, we will use some usual notations. Denote by $(\cdot,\cdot)$ the inner product in L^2 and by $\|.\|_p$ the norm in L^p, $1 \le p \le \infty$. In particular, if $p = 2$, we omit the subscript $\|\cdot\|_2 = \|\cdot\|$. H^1 is the usual Sobolev space with norm $\|\cdot\|_{H^1}$ and H^{-1} its dual space with norm $\|\cdot\|_{H^{-1}}$. $L^p(\mathbb{R}^N, a)$ is the space with norm $\|\cdot\|_{a,p} = (\int_{\mathbb{R}^N} a(x)|\cdot|^p dx)^{1/p}$. $L^p(\tau,T;X)$ is the space of L^p functions from (τ,T) to X with norm $\|\cdot\|_{L^p(\tau,T;X)} = (\int_\tau^T \|\cdot\|_X^p dt)^{1/p}$.

2. Preliminaries on random dynamical systems

In this section, we recall some basic concepts and results related to existence and upper semi-continuity of random attractors of the RDSs. For a comprehensive exposition on this topic, there are a large volume of literature, see [5, 13, 14, 15, 16, 17, 19, 34, 12].

The basic notion in random dynamical systems is a metric dynamical system (MDS) $\vartheta \equiv (\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta_t\}_{t\in\mathbb{R}})$, which is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a group $\vartheta_t, t \in \mathbb{R}$, of measure preserving transformations of $(\Omega, \mathcal{F}, \mathbb{P})$.

A MDS ϑ is said to be ergodic under $\mathbb{P}$ if for any ϑ-invariant set $F \in \mathcal{F}$, we have either $\mathbb{P}(F) = 0$ or $\mathbb{P}(F) = 1$, where the ϑ-invariant set is in the sense that $\mathbb{P}(\vartheta_t F) = \mathbb{P}(F)$ for $F \in \mathcal{F}$ and all $t \in \mathbb{R}$.

Let X be a separable Banach space with norm $\|.\|_X$ and Borel sigma-algebra $\mathcal{B}(X)$; i.e., the smallest σ-algebra on X which contains all open subsets. Let $\mathbb{R}^+ = \{x \in \mathbb{R}; x \ge 0\}$ and 2^X be the collection of all subsets of X.

Definition 2.1. A RDS on X over a MDS ϑ is a family of $(\mathcal{B}(\mathbb{R}^+)\times\mathcal{F}\times\mathcal{B}(X)), X)$-measurable mappings

$$\varphi : \mathbb{R}^+ \times \Omega \times X \to X, \quad (t,\omega,x) \mapsto \varphi(t,\omega)x$$

such that for $\mathbb{P}$-a.e.$\omega \in \Omega$, the mappings $\varphi(t,\omega)$ satisfy the cocycle property:

$$\varphi(0,\omega) = id, \quad \varphi(t+s,\omega) = \varphi(t,\vartheta_s\omega)\circ\varphi(s,\omega)$$

for all $s,t \in \mathbb{R}^+$. A RDS over a MDS ϑ is briefly denoted by (φ,ϑ).

A RDS φ is said to be continuous if the mappings $\varphi(t,\omega) : x \mapsto \varphi(t,\omega)x$ are continuous in X for all $t \in \mathbb{R}^+$ and $\omega \in \Omega$, that is, norm-to-norm continuity.

For the nonempty sets $A, B \in 2^X$, we define the Hausdorff semi-metric by

$$d(A,B) = \sup_{x\in A}\inf_{y\in B}\|x-y\|_X.$$

In particular, $d(x,B) = d(\{x\},B)$. Note that $d(A,B) = 0$ if and only if $A \subseteq B$.

Let $\mathcal{D} \subseteq 2^X$ be given. $\mathcal{D}$ is called a sets universe if $\mathcal{D}$ satisfies the inclusion closed properties: if $D \in \mathcal{D}$ and $\hat{D} \subseteq D$, then $\hat{D} \in \mathcal{D}$.

Definition 2.2. (i) A random set $D = \{D(\omega); \omega \in \Omega\}$ is a family of nonempty subsets of X indexed by ω such that for every $x \in X$, the mapping $\omega \mapsto d(x, D(\omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable.

(ii) A random variable $r(\omega)$ is tempered with respect to ϑ if

$$\lim_{|t|\to\infty} e^{-\lambda|t|} r(\vartheta_t\omega) = 0, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ and any } \lambda > 0.$$

In the following, we give related concepts, where for convenience of our discussions in the sequel, the time variable is stated in the negative direction.

Definition 2.3. Let $\mathcal{D}$ be a universe of sets. A set $K = \{K(\omega); \omega \in \Omega\} \in \mathcal{D}$ is said to be $\mathcal{D}$-pullback absorbing for RDS (φ, ϑ) in X if for $\mathbb{P}$-a.e.$\omega \in \Omega$ and every $D \in \mathcal{D}$, there exists an absorbing time $T = T(D, \omega) < 0$ such that for all $\tau \le T$,

$$\varphi(-\tau, \vartheta_\tau\omega)D(\vartheta_\tau\omega) \subset K(\omega),$$

where $\varphi(-\tau, \vartheta_\tau\omega)D(\vartheta_\tau\omega) = \cup_{x\in D(\vartheta_\tau\omega)}\{\varphi(-\tau, \vartheta_\tau\omega)x\}$.

Note that K in Definition 2.3 is merely a subset of X (possessing the absorbing property), on which the random property in the sense of Definition 2.2(i) has not been imposed there. We also should point out that for a continuous RDS, the existence of a compact random absorbing set ensures completely the existence of a random attractor, see [14, 15, 11]. However, for our problem, we need the following generalized version of existence criterion, see [9, 19] and etc. For the random attractors of non-autonomous RDSs, we see [31] and the references therein.

Definition 2.4. Let $\mathcal{D}$ be a universe of sets. The RDS (φ, ϑ) is said to be $\mathcal{D}$-pullback asymptotically compact in X if for $\mathbb{P}$-a.e.$\omega \in \Omega$ and every $D \in \mathcal{D}$, the sequence $\{\varphi(-\tau_n, \vartheta_{-\tau_n}\omega, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $\tau_n \to -\infty$ and $x_n \in D(\vartheta_{\tau_n}\omega)$.

Theorem 2.5. *Let $\mathcal{D}$ be a universe of sets, and (φ, ϑ) a continuous RDS on X. Suppose that there exists a closed and $\mathcal{D}$-pullback random bounded absorbing set $K = \{K(\omega); \omega \in \Omega\}$ for (φ, ϑ) in X and (φ, ϑ) is $\mathcal{D}$-pullback asymptotically compact in X. Then the omega-limit set of K, $\mathcal{A} = \{\mathcal{A}(\omega); \omega \in \Omega\}$ defined by*

$$\mathcal{A}(\omega) = \cap_{s\le 0}\overline{\cup_{\tau\le s}\varphi(-\tau, \vartheta_\tau\omega)K(\vartheta_\tau\omega)} \subset K(\omega), \quad \omega \in \Omega,$$

is a $\mathcal{D}$-random attractor for (φ, ϑ) in X, in the sense that $\mathcal{A} \in \mathcal{D}$, and further for $\mathbb{P}$-a.e. $\omega \in \Omega$, there hold:

(i) *$\mathcal{A}(\omega)$ is compact random set in X;*

(ii) *the invariance property*

$$\varphi(-\tau, \omega)\mathcal{A}(\omega) = \mathcal{A}(\vartheta_{-\tau}\omega)$$

is satisfied for all $\tau \le 0$;

(iii) *in addition, the pullback convergence*

$$\lim_{\tau\to-\infty} d(\varphi(-\tau, \vartheta_\tau\omega, D(\vartheta_\tau\omega)), K(\omega)) = 0$$

holds for every $D \in \mathcal{D}$.

In the following, we recall some notions on the upper semi-continuity of the RDS. Given $\epsilon > 0$, let $(\varphi_\epsilon, \vartheta)$ be an RDS generated by an SPDE depending on the coefficient ϵ, and φ_0 the corresponding deterministic dynamical system, i.e. φ_0 is independent of the random parameter ω. Then we reformulate the result on the upper semi-continuity of random attractors in X, which can be found in [12, 31, 34].

Theorem 2.6. *Suppose that $(\varphi_\epsilon, \vartheta)$ has a random attractor $\mathcal{A}_\epsilon = \{\mathcal{A}_\epsilon(\omega); \omega \in \Omega\}$ and φ_0 has a global attractor $\mathcal{A}_0$ in X, respectively. Assume that for all $\tau \le t \le 0$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, there hold*

(i) *for every $\epsilon_n \to 0^+$, and $x_n, x \in X$ with $x_n \to x$, we have*

$$\lim_{n\to\infty} \varphi_{\epsilon_n}(t - \tau, \vartheta_\tau\omega)x_n = \varphi_0(t, \tau)x;$$

(ii) $(\varphi_\epsilon, \vartheta)$ *admits a random absorbing set* $E_\epsilon = \{E_\epsilon(\omega); \omega \in \Omega\} \in \mathcal{D}$ *such that for some deterministic positive constant* M

$$\lim \sup_{\epsilon \to 0^+} \|E_\epsilon\|_X \le M,$$

where $\|E_\epsilon\|_X = \sup_{x \in E_\epsilon} \|x\|_X$*;*

(iii) *there exists* $\epsilon_0 > 0$ *such that*

$$\cup_{0<\epsilon\le\epsilon_0} \{\mathcal{A}_\epsilon\} \quad \text{is precompact in } X. \tag{2.1}$$

Then for $\mathbb{P}$*-a.e.* $\omega \in \Omega$*, we have* $d(\mathcal{A}_\epsilon(\omega), \mathcal{A}_0) \to 0$*, as* $\epsilon \searrow 0$*.*

3. Existence and uniqueness of weak solutions

To model the white noise in the equations (1.1), based on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ defined in the introduction, we need to define a time shift on Ω by

$$\vartheta_t \omega(s) = \omega(s+t) - \omega(t), \quad \omega \in \Omega,\ t,\ s \in \mathbb{R}. \tag{3.1}$$

This shift ϑ is a group on Ω which leaves the Wiener measure $\mathbb{P}$ invariant. Specifically, $\mathbb{P}$ is ergodic with respect to ϑ. Then $\vartheta = \{\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t\in\mathbb{R}}\}$ forms an ergodic MDS, see [13].

We now convert system (1.1) with a random perturbation term into a deterministic one with a random parameter ω. For this purpose, we introduce the notation $z(t) = z(\vartheta_t\omega) = (I-\Delta)^{-1} h y(\vartheta_t \omega)$, where Δ is the Laplacian and $y(t)$ the Ornstein-Uhlenbeck(O-U) process taking the form

$$y(t) = y(\vartheta_t\omega) = -\int_{-\infty}^{0} e^s (\vartheta_t\omega)(s) ds, \quad t \in \mathbb{R},$$

where $\omega(t) = W(t)$ is one dimensional Wiener process defined in the introduction. Furthermore, $y(t)$ satisfies the stochastic differential equations

$$dy + ydt = d\omega(t) \quad \text{for all } t \in \mathbb{R}.$$

Remark 3.1. Since $y(\omega)$ is tempered, in view of [9] or [5], there exists a tempered variable $r(\omega) > 0$ such that

$$|y(\omega)|^2 + |y(\omega)|^p \le r(\omega), \tag{3.2}$$

with

$$r(\vartheta_t\omega) \le e^{\frac{\mu}{2}|t|} r(\omega), \quad t \in \mathbb{R}, \tag{3.3}$$

where we choose $0 < \mu < 2$. Note that since the inverse of $I - \Delta$ is a bounded linear operator on $H^1(\mathbb{R}^N)$, then by the Hölder inequality and using (3.2)-(3.3) and the assumption (1.9), we can deduce that

$$\|z(\vartheta_t\omega)\|_{H^1}^2 + \|z(\vartheta_t\omega)\|_p^p \le \|z(\vartheta_t\omega)\|_{H^1}^2 + c_1^p \|z(\vartheta_t\omega)\|_{H_1}^p \le c_2 e^{\frac{\mu}{2}|t|} r(\omega), \tag{3.4}$$

for $t \in \mathbb{R}$, where $c_1 > 0$ is the embedding constant of $H^1 \hookrightarrow L^p$ and c_2 a deterministic positive constant depending only on $\|h\|_{H^1}, p, c_1$.

It is easy to show that

$$(I-\Delta) z_t dt + (I-\Delta) z dt = h dW(t).$$

Let $u(t)$ satisfy (1.1). Using the change of variable $v(t) = u(t) - \epsilon z(\vartheta_t\omega)$ (where $\epsilon \in (0,1]$), $v(t)$ satisfies the equation (which depends on the random parameter ω)

$$v_t - \Delta v_t - \Delta v + v + f(x, v + \epsilon z(\vartheta_t\omega)) = \mathrm{g}, \tag{3.5}$$

with initial value condition

$$v(x,\tau) = v_0(x) = u_0(x) - \epsilon z(\vartheta_\tau \omega). \tag{3.6}$$

In addition, we assume that $p \geq 2$ for $N \leq 2$ and $2 \leq p \leq \frac{N}{N-2} + 1$ for $N \geq 3$, where the condition on growth exponent p ensures that some Sobolev embeddings hold.

Concerning the existence and uniqueness of solutions of (3.5)-(3.6), we can prove them by using the Faedo-Galerkin method and some approximation arguments, see a similar argument as [25, 6, 4]. Here, we only formulate this result and omit the proof. Before giving this, we state the definition of weak solutions.

Definition 3.2. For any $\tau \in \mathbb{R}$, a stochastic process $v(x,t), t \in [\tau, T], x \in \mathbb{R}^N$ is called a weak solution of (3.5) if and only if

$$\begin{gathered} v \in C(\tau, T; H^1(\mathbb{R}^N)) \cap L^\infty(\tau, T; H^1(\mathbb{R}^N)) \cap L^p(\tau, T; L^p(\mathbb{R}^N)), \\ \frac{dv}{dt} \in L^2(\tau, T; H^1(\mathbb{R}^N)), \quad v|_{t=\tau} = v_0, \text{ a.e. in } \mathbb{R}^N, \end{gathered}$$

and

$$\begin{aligned} &\int_\tau^T \Big((v_t, \phi) + (\nabla v_t, \nabla \phi) + (\nabla v, \nabla \phi) + (v, \phi) + (f(x, v + \epsilon z(\vartheta_t \omega)), \phi)\Big) dt \\ &= \int_\tau^T (\mathrm{g}, \phi) dt \end{aligned} \tag{3.7}$$

for all test functions $\phi \in C_0^\infty([\tau, T] \times \mathbb{R}^N)$ and $\mathbb{P}$-a.e.$\omega \in \Omega$.

Lemma 3.3. *Assume that* (1.2)–(1.9) *hold,* $g \in L^2(\mathbb{R}^N)$ *and* $v_0 \in H^1(\mathbb{R}^N)$. *Then for any* $\tau \in \mathbb{R}$, $\tau < T$,

- (i) *the initial problem* (3.5)-(3.6) *possesses a unique weak* $v(t, \omega; \tau, v_0)$ *with the initial value* $v_0 = v(\tau, \omega; \tau, v_0)$, *and*
- (ii) *the mapping* $v_0 \mapsto v(t, \omega; \tau, v_0)$ *is continuous and* $\omega \mapsto v(t, \omega; \tau, v_0)$ *is* $(\mathcal{F}, \mathcal{B}(H^1(\mathbb{R}^N) \times \mathbb{R})$*-measurable in* $H^1(\mathbb{R}^N)$ *for all* $t > \tau$.

Remark 3.4. By (3.5), it follows that the weak solution v satisfies the energy equation: for any $\tau \in \mathbb{R}$ with $\tau \leq t$,

$$\begin{aligned} \|v(t)\|_{H^1}^2 = e^{-\mu(t-\tau)} \|v(\tau)\|_{H^1}^2 - (2 - \mu) \int_\tau^t e^{-\mu(t-s)} \|v(s)\|_{H^1}^2 ds \\ - 2 \int_\tau^t e^{-\mu(t-s)} (f(x, v(s) + \epsilon z(\vartheta_s \omega)), v(s)) ds \\ + 2 \int_\tau^t e^{-\mu(t-s)} (\mathrm{g}, v(s)) ds \end{aligned} \tag{3.8}$$

where $\mu \in (0, 2)$.

Note that by Lemma 3.3 we have the measurability of solutions as mappings from $\mathbb{R}^+ \times \Omega \times H^1(\mathbb{R}^N)$ into $H^1(\mathbb{R}^N)$. Now, we are in the position to define a RDS (φ, ϑ) corresponding to the stochastic nonclassical diffusion equation (1.1). Put

$$\varphi(t - \tau, \vartheta_\tau \omega) u_0 = u(t, \omega; \tau, u_0) = v(t, \omega; \tau, u_0 - \epsilon z(\vartheta_\tau \omega)) + \epsilon z(\vartheta_t \omega), \tag{3.9}$$

for $\omega \in \Omega$, where $u_0 = u(\tau, \omega; \tau, u_0)$. Then from Lemma 3.3, (φ, ϑ) is a continuous RDS on $H^1(\mathbb{R}^N)$, where the MDS ϑ is defined in (3.1).

4. Weak-to-weak continuity of solutions in $H^1(\mathbb{R}^N)$

Although Lemma 3.3 implies that the weak solutions to (3.5)-(3.6) is norm-to-norm continuous in $H^1(\mathbb{R}^N)$, it will not be helpful for us to show the asymptotic compactness which is indispensable to the existence of a random attractor for RDS (φ, ϑ) defined by (3.9). Here, we will prove the weak continuous dependence of the solutions with respect to the initial value conditions in $H^1(\mathbb{R}^N)$. This result will be one crucial condition for us to prove the asymptotic compactness of the associated RDS (φ, ϑ).

Lemma 4.1. *Assume that* (1.2)-(1.9) *are satisfied and* $g \in L^2(\mathbb{R}^N)$. *Let the sequence* $\{v_0^{(n)}\}_{n\geq 1} \subset H^1(\mathbb{R}^N)$ *such that*

$$v_0^{(n)} \rightharpoonup v_0 \quad \text{weakly in } H^1(\mathbb{R}^N), \tag{4.1}$$

and $v^{(n)}(t)$, $v(t)$ *the corresponding weak solutions. Then there exists a subsequence (we will label again* $\{v^{(n)}(t)\}_{n\geq 1}$*) such that*

$$v^{(n)}(t) \rightharpoonup v(t) \quad \text{weakly in } H^1(\mathbb{R}^N) \text{ for all } t > \tau, \tag{4.2}$$

Furthermore, the convergence in (4.2) *is uniform in* $\epsilon \in (0,1]$ *and on the time interval* $[\tau, T]$.

Proof. We first give some estimates to show that the weak solutions $v^{(n)}(t)$ are bounded in time $t \in [\tau, T]$ and uniformly bounded in both n and ϵ in some proper spaces.

Note that as the weak convergent sequence is bounded. Then there exists a positive constant C_1 such that

$$\|v_0^{(n)}\|_{H^1}^2 \leq C_1 \quad \text{for all } n \in \mathbb{Z}^+, \tag{4.3}$$

where and in the following $C_i, i = 1, \dots, 6$ are deterministic constants independent of ϵ and n.

Now, in (3.5), we replace $v(t)$ with $v^{(n)}(t)$, take the inner products with $v^{(n)}(t)$ in $L^2(\mathbb{R}^N)$ and use the assumptions (1.3)-(1.4), (1.6)-(1.7) and (1.9) to deduce that

$$\begin{aligned} &\frac{d}{dt}\|v^{(n)}(t)\|_{H^1}^2 + \|v^{(n)}(t)\|_{H^1}^2 + \alpha_1\|u^{(n)}(t)\|_p^p + \alpha_2\|u^{(n)}(t)\|_{a,p}^p \\ &\leq C_2(\|z(\vartheta_t\omega)\|_{H^1}^2 + \|z(\vartheta_t\omega)\|_{H^1}^p) + C_3 \end{aligned} \tag{4.4}$$

is valid a.e. $t \geq \tau$. By integrating (4.4) from τ to t and using (4.3), we readily prove the following bounds, uniformly in both n and $\epsilon \in (0,1]$:

$$v^{(n)}(t) \quad \text{is uniformly bounded in } L^\infty(\tau, T; H^1(\mathbb{R}^N)), \tag{4.5}$$

$$v^{(n)}(t) \quad \text{is uniformly bounded in } L^2(\tau, T; H^1(\mathbb{R}^N)), \tag{4.6}$$

$$u^{(n)}(t) \quad \text{is uniformly bounded in } L^p(\tau, T; L^p(\mathbb{R}^N)), \tag{4.7}$$

$$u^{(n)}(t) \quad \text{is uniformly bounded in } L^p(\tau, T; L^p(\mathbb{R}^N, a)),$$

where $u^{(n)}(t) = v^{(n)}(t) + \epsilon z(\vartheta_t\omega)$. At the same time, by using (1.3) and (1.7), along with (4.7), we deduce that

$$f_1(\cdot, u^{(n)}(t)) \text{ is uniformly bounded in } L^q(\tau, T; L^q(\mathbb{R}^N)), \tag{4.8}$$

$$a(\cdot)f_2(u^{(n)}(t)) \text{ is uniformly bounded in } L^q(\tau, T; L^q(\mathbb{R}^N)), \tag{4.9}$$

where q is the conjugate of p. On the other hand, we have

$$-\Delta v^{(n)}(t) \text{ is uniformly bounded in } L^2(\tau,T;H^{-1}(\mathbb{R}^N)). \tag{4.10}$$

Hence, from (4.6) and (4.8)-(4.10) we infer that

$$\begin{aligned} &v_t^{(n)}(t) - \Delta v_t^{(n)}(t) \text{ is uniformly bounded in} \\ &L^2(\tau,T;H^{-1}(\mathbb{R}^N)) + L^q(\tau,T;L^q(\mathbb{R}^N)). \end{aligned} \tag{4.11}$$

Furthermore, in (3.5), replacing $v(t)$ by $v^{(n)}(t)$, then multiplying with $v_t^{(n)}(t)$, we find that

$$\|v_t^{(n)}(t)\|_{H^1}^2 + \frac{d}{dt}\|v^{(n)}(t)\|_{H^1}^2 \le C_4\|u^{(n)}(t)\|_{H^1}^{2p-2} + \|\psi_2\|^2 + \|\mathrm{g}\|^2) + C_5, \tag{4.12}$$

where we have used the embedding $H^1 \hookrightarrow L^{2p-2}$ under the assumptions on p and N. Then integrating (4.12) from τ to T, connection with (4.3) and (4.5), we obtain

$$v_t^{(n)}(t) \text{ is uniformly bounded in } L^2(\tau,T;H^1(\mathbb{R}^N)), \tag{4.13}$$

and therefore along with (4.6) it implies that $v^{(n)}(t) \in C(\tau,T;H^1(\mathbb{R}^N))$, see [26, Corollary 7.3].

Hence, by the compactness theorem (see, e.g. [30]) we can extract a subsequence from $\{v^{(n)}(t)\}_n$ (which we will repeatedly and wickedly label $\{v^{(n)}(t)\}_n$) such that

$$v^{(n)}(t) \rightharpoonup \hat{v}(t) \quad \text{weakly* in } L^\infty(\tau,T;H^1(\mathbb{R}^N)),$$

$$v^{(n)}(t) \rightharpoonup \hat{v}(t) \quad \text{weakly in } L^2(\tau,T;H^1(\mathbb{R}^N)), \tag{4.14}$$

$$v^{(n)}(t) \to \hat{v}(t) \quad \text{strongly in } L^2(\tau,T;L^2(B_R)), \tag{4.15}$$

where $B_R = \{x \in \mathbb{R}^N; |x| \le R\}$ for all $R > 0$. By using a similar method as [25], it is not difficult to verify that $\hat{v}(t)$ satisfies (3.5)-(3.6) in the sense of Definition 3.2. The uniqueness of solutions implies that $\hat{v}(t) = v(t)$.

For any $\tau \in \mathbb{R}$, by (4.14), we see that

$$v^{(n)}(t) \rightharpoonup v(t) \quad \text{weakly in } H^1(\mathbb{R}^N), \tag{4.16}$$

for almost every $t \ge \tau$. We then show that (4.16) holds for any $t \ge \tau$. Indeed, in terms of (4.16), for any $t \ge \tau$, we can choose a enough small number $h > 0$ such that

$$\lim_{n\to\infty} \langle v^{(n)}(t+h) - v(t+h), \phi\rangle = 0, \quad \forall \phi \in H^{-1}(\mathbb{R}^N), \tag{4.17}$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing between H^1 and its duality H^{-1}. Hence by using first (4.17) and then (4.13) we can infer that for any $t \ge \tau$,

$$\begin{aligned} &\lim_{n\to\infty} |\langle v^{(n)}(t) - v(t), \phi\rangle| \\ &\le \lim_{n\to\infty} \Big(|\langle v(t+h) - v(t), \phi > | + | < v^{(n)}(t+h) - v^{(n)}(t), \phi\rangle|\Big) \\ &\le \lim_{n\to\infty} \Big(|\langle \int_t^{t+h} v_s(s)ds, \phi\rangle| + |\langle \int_t^{t+h} v_s^{(n)}(s)ds, \phi\rangle|\Big) \\ &\le \lim_{n\to\infty} \Big(\|(v'\|_{L^2(t,t+h;H^1)} + \|(v'^{(n)}\|_{L^2(t,t+h;H^1)}\Big)h^{1/2}\|\phi\|_{H^{-1}}. \end{aligned} \tag{4.18}$$

Hence by (4.18) we know that (4.2) is proved as claimed. □

Remark 4.2. The strong convergence in (4.15) can be achieved by the compactness theorem [26, Theorem 8.1].

5. Existence of random attractors in $H^1(\mathbb{R}^N)$

We first show that the RDS (φ,ϑ) generated by the stochastic nonclassical diffusion equations (1.1) admits a closed and $\mathcal{D}_\mu$-pullback random bounded absorbing set in $H^1(\mathbb{R}^N)$, where $\mu\in(0,2)$ is based on the following consideration. In this section, our proofs are closely related to the energy equality (3.8). Throughout this paper, the number c is a generic constant independent of ϵ, t, $z(\vartheta_t\omega)$ and $v(t)$.

Lemma 5.1. Assume that (1.2)-(1.9) are satisfied and $g\in L^2(\mathbb{R}^N)$ with $\epsilon\in(0,1]$. Then there exists a closed and $\mathcal{D}_\mu$-pullback random bounded absorbing set $K_\mu=\{K_\mu(\omega);\omega\in\Omega\}$ for the RDS (φ,ϑ) in $H^1(\mathbb{R}^N)$; that is, for any $D\in\mathcal{D}_\mu$ and $\mathbb{P}$-a.e. $\omega\in\Omega$, there exists $T=T(D,\omega)<0$ such that

$$\varphi(-\tau,\vartheta_\tau\omega)D(\vartheta_\tau\omega)\subseteq K_\mu(\omega),\quad \text{for all } \tau\le T,$$

where the universe $\mathcal{D}_\mu$ is the collection of nonempty subsets $D=\{D(\omega);\omega\in\Omega\}$ of $H^1(\mathbb{R}^N)$ such that

$$\lim_{\tau\to-\infty}\Big(e^{\mu\tau}\sup_{u\in D(\vartheta_\tau\omega)}\{\|u\|_{H^1}^2\}\Big)=0, \tag{5.1}$$

where $\mu\in(0,2)$ and for every fixed μ the universe $\mathcal{D}_\mu$ is inclusion closed and $K_\mu\in\mathcal{D}_\mu$.

Proof. We first estimate each term on the right hand side of (3.8). By (1.3)-(1.4) and using a similar arguments as (4.2) in [34], we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} f_1(x,v+\epsilon z(\vartheta_t\omega))v\,dx\\ &\ge \frac{\alpha_1}{2}\|u\|_p^p-\epsilon c(\|z(\vartheta_t\omega)\|_p^p+\|z(\vartheta_t\omega)\|^2)-c(\|\psi_1\|_1+\|\psi_2\|^2).\end{aligned} \tag{5.2}$$

By using (1.6)-(1.7), we have

$$\begin{aligned}&\int_{\mathbb{R}^N} a(x)f_2(v+\epsilon z(\vartheta_t\omega))v\,dx\\ &=\int_{\mathbb{R}^N} a(x)f_2(u)u dx-\epsilon\int_{\mathbb{R}^N} a(x)f_2(u)z(\vartheta_t\omega)dx\\ &\ge \alpha_2\int_{\mathbb{R}^N} a(x)|u|^p dx-\gamma\int_{\mathbb{R}^N} a(x)dx-\epsilon\beta_2\int_{\mathbb{R}^N} a(x)|u|^{p-1}|z(\vartheta_t\omega)|dx\\ &\quad -\epsilon\delta\int_{\mathbb{R}^N} a(x)|z(\vartheta_t\omega)|dx.\end{aligned} \tag{5.3}$$

By the Young inequality, and using assumption (1.2), we obtain

$$\epsilon\beta_2\int_{\mathbb{R}^N} a(x)|u|^{p-1}|z(\vartheta_t\omega)|dx\le\frac{\alpha_2}{2}\int_{\mathbb{R}^N} a(x)|u|^p dx+\epsilon c\int_{\mathbb{R}^N}|z(\vartheta_t\omega)|^p dx, \tag{5.4}$$

$$\epsilon\delta\int_{\mathbb{R}^N} a(x)|z(\vartheta_t\omega)|dx\le\epsilon\|a\|_\infty\int_{\mathbb{R}^N}|z(\vartheta_t\omega)|^2dx+\frac{\delta^2}{4}\|a\|_1. \tag{5.5}$$

where $c=c(\alpha_2,\beta_2,p,\|a\|_\infty)$. Then, it follows from (5.3)-(5.5) that

$$\begin{aligned}&\int_{\mathbb{R}^N} a(x)f_2(v+\epsilon z(\vartheta_t\omega))v\,dx\\ &\ge\frac{\alpha_2}{2}\int_{\mathbb{R}^N} a(x)|u|^p dx-\epsilon c(\|z(\vartheta_t\omega)\|_p^p+\|z(\vartheta_t\omega)\|^2)-c\|a\|_1.\end{aligned} \tag{5.6}$$

On the other hand, we have

$$2\Big|\int_{\mathbb{R}^N} \mathrm{g} v\,dx\Big| \le (2-\mu)\|v(t)\|^2 + \frac{1}{2-\mu}\|\mathrm{g}\|^2 \le (2-\mu)\|v(t)\|_{H^1}^2 + \frac{1}{2-\mu}\|\mathrm{g}\|^2. \tag{5.7}$$

Then, we incorporate (5.2),(5.6) and (5.7) into (3.8) to yield

$$\begin{aligned} &\|v(t)\|_{H^1}^2 + \int_\tau^t e^{-\mu(t-s)}(\alpha_1\|u(s)\|_p^p + \alpha_2\|u(s)\|_{a,p}^p)ds \\ &\le e^{-\mu(t-\tau)}\|v_0\|_{H^1}^2 + \epsilon e^{-\mu t}\int_\tau^t e^{\mu s}\varsigma(\vartheta_s\omega)ds + c, \end{aligned} \tag{5.8}$$

where

$$\varsigma(\vartheta_t\omega) = c(\|z(\vartheta_t\omega)\|^2 + \|z(\vartheta_t\omega)\|_p^p).$$

We now fix $t \le 0$. From (3.4), we have

$$\begin{aligned} &\|v(t,\omega;\tau,u_0)\|_{H^1}^2 \\ &\le e^{-\mu(t-\tau)}\|v_0\|_{H^1}^2 + \epsilon e^{-\mu t}\int_\tau^t e^{\mu s}\varsigma(\vartheta_s\omega)ds + c \\ &\le e^{-\mu t}\Big(2e^{\mu\tau}\|u_0\|_{H^1}^2 + 2\epsilon e^{\mu\tau}\|z(\vartheta_\tau\omega)\|_{H^1}^2 + \epsilon\int_\tau^t e^{\mu s}\varsigma(\vartheta_s\omega)ds + c\Big) \\ &\le e^{-\mu t}\Big(2e^{\mu\tau}\|u_0\|_{H^1}^2 + 2\epsilon c e^{\mu\tau}e^{-\frac{\mu}{2}\tau}r(\omega) + \epsilon c\int_\tau^t e^{\frac{\mu}{2}s}r(\omega)ds + c\Big) \\ &\le e^{-\mu t}\Big(2e^{\mu\tau}\|u_0\|_{H^1}^2 + 2\epsilon c e^{\frac{\mu}{2}\tau}r(\omega) + \epsilon c\frac{2}{\mu}r(\omega) + c\Big). \end{aligned} \tag{5.9}$$

Therefore, it follows from (5.9) that for every $D \in \mathcal{D}_\mu$, there exists $T = T(D,\omega) < t \le 0$ such that

$$\|v(t,\omega;\tau,u_0)\|_{H^1}^2 \le ce^{-\mu t}(1+\epsilon r(\omega)), \quad \text{for all } \tau \le T. \tag{5.10}$$

By noticing that $u(0) = v(0) + \epsilon z(\omega)$, letting $t = 0$ in (5.10), we obtain

$$\begin{aligned} \|u(0,\omega;\tau,u_0)\|_{H^1}^2 &\le 2\|v(0,\omega;\tau,u_0)\|_{H^1}^2 + 2\epsilon\|z(\omega)\|_{H^1}^2 \\ &\le 2c(1+\epsilon r(\omega)) + 2\epsilon\|z(\omega)\|_{H^1}^2 \\ &\le R(\omega) =: c(1+\epsilon r(\omega)), \end{aligned} \tag{5.11}$$

for all $\tau \le T$. Observing that

$$e^{\mu\tau}R(\vartheta_\tau\omega) = e^{\mu\tau} + \epsilon e^{\mu\tau}r(\vartheta_\tau\omega) \le e^{\mu\tau} + \epsilon e^{\frac{\mu}{2}\tau}r(\omega) \to 0,$$

as $\tau \to -\infty$. Hence $K_\mu = \{\|u\|_{H^1}; \|u\|_{H^1}^2 \le c(1+\epsilon r(\omega)), \omega \in \Omega\} \in \mathcal{D}_\mu$. On the other hand, from (5.11) we find that $R(\omega)$ is measurable, and so is the set-valued mapping K_μ. Therefore K_μ is a closed and $\mathcal{D}_\mu$-pullback random bounded absorbing set for the RDS (φ,ϑ) defined in (3.9). This completes the proof. □

Now, we will use the weak-to-weak continuity of solutions in Lemma 4.1 to demonstrate the $\mathcal{D}_\mu$-pullback asymptotically compactness for the RDS (φ,ϑ) in $H^1(\mathbb{R}^N)$. In fact, we obtain the pre-compactness for the RDS (φ,ϑ) uniformly in $\epsilon \in (0,1]$, which is one of the crucial conditions for us to discuss the upper semi-continuity in section 6.

Lemma 5.2. *Assume that* (1.2)-(1.9) *are satisfied,* $g \in L^2(\mathbb{R}^N)$, $\epsilon \in (0,1]$ *and the universe* $\mathcal{D}_\mu$ *defined by* (5.1). *Then for every fixed* $\mu \in (0,2)$, *the RDS* (φ, ϑ) *corresponding to the stochastic nonclassical diffusion equations* (1.1) *is* $\mathcal{D}_\mu$*-pullback asymptotically compact in* $H^1(\mathbb{R}^N)$.

Proof. Let $\tau_n \to -\infty$, and $x_n \in D(\vartheta_{\tau_n}\omega)$ with $D \in \mathcal{D}_\mu$. It suffices to show that the sequence $\{\varphi(-\tau_n, \vartheta_{\tau_n}\omega)x_n\}_n$ is pre-compact in $H^1(\mathbb{R}^N)$.

Put $K = \{K(\omega); \omega \in \Omega\}$ omitting the subscript μ, where

$$K = \{\|u\|_{H^1}; \|u\|_{H^1}^2 \le c(1 + r(\omega))\}.$$

Then, K is also a closed and $\mathcal{D}_\mu$-pullback random bounded absorbing set (see Lemma 5.1), so $\varphi(-\tau_n, \vartheta_{\tau_n}\omega)x_n \in K(\omega)$ for $\tau_n \to -\infty$. By the boundedness of K and weak compactness theorem, there exists some $y_0 \in H^1(\mathbb{R}^N)$ such that, up to a subsequence,

$$\varphi(-\tau_n, \vartheta_{\tau_n}\omega)x_n \rightharpoonup y_0 \quad \text{weakly in } H^1(\mathbb{R}^N) \tag{5.12}$$

uniformly in $\epsilon \in (0,1]$. We need to show that the convergence in (5.12) is equivalent to the norm convergence. That is, there exists a subsequence $\{n'\} \subset \{n\}$ such that

$$\varphi(-\tau_{n'}, \vartheta_{\tau_{n'}}\omega)x_{n'} \to y_0 \quad \text{strongly in } H^1(\mathbb{R}^N). \tag{5.13}$$

To this end, it suffices to show

$$\limsup_{\tau_n \to -\infty} \|\varphi(-\tau_n, \vartheta_{\tau_n}\omega)x_n - \epsilon z(\omega)\|_{H^1} \le \|y_0 - \epsilon z(\omega)\|_{H^1}. \tag{5.14}$$

To prove the inequality (5.14), first, we give an equivalent form of the element y_0 by the RDS φ. Fix $k > \tau_n$. By the cocycle property of the RDS φ, we have

$$\varphi(-\tau_n, \vartheta_{\tau_n}\omega)x_n = \varphi(-k, \vartheta_k\omega)\varphi(-\tau_n + k, \vartheta_{\tau_n}\omega)x_n, \tag{5.15}$$

and by using again Lemma 5.1 it gives

$$\varphi(-\tau_n + k, \vartheta_{\tau_n}\omega)x_n = \varphi(-\tau_n + k, \vartheta_{\tau_n - k}\vartheta_k\omega)x_n \in K(\vartheta_k\omega), \tag{5.16}$$

if τ_n converges to $-\infty$. Then without loss generality, we may assume that for every $n \in \mathbb{Z}^+$ there exists $y_k \in K(\vartheta_k\omega)$ such that

$$\varphi(-\tau_n + k, \vartheta_{\tau_n}\omega)x_n \rightharpoonup y_k \quad \text{weakly in } H^1(\mathbb{R}^N) \tag{5.17}$$

uniformly in $\epsilon \in (0,1]$. From the definition of the RDS (φ, ϑ), the equality (5.15) can be rewrote as the form of weak solutions. As by (3.9) we have

$$\begin{aligned} \varphi(-\tau_n, \vartheta_{\tau_n}\omega)x_n &= \varphi(-k, \vartheta_k\omega)\varphi(-\tau_n + k, \vartheta_{\tau_n}\omega)x_n \\ &= u(0, \omega; k, \varphi(-\tau_n + k, \vartheta_{\tau_n}\omega)x_n) \\ &= v(0, \omega; k, \varphi(-\tau_n + k, \vartheta_{\tau_n}\omega)x_n - \epsilon z(\vartheta_k\omega)) + \epsilon z(\omega), \end{aligned} \tag{5.18}$$

so that along with (5.17) and (5.18), we infer that for every $k \in \mathbb{Z}^-$,

$$\begin{aligned} y_0 &= \text{w-}\lim_{\tau_n \to -\infty} \varphi(-\tau_n, \vartheta_{\tau_n}\omega)x_n \\ &= \text{w-}\lim_{\tau_n \to -\infty} v(0, \omega; k, \varphi(-\tau_n + k, \vartheta_{\tau_n}\omega)x_n - \epsilon z(\vartheta_k\omega)) + \epsilon z(\omega), \\ &= v(0, \omega; k, y_k - \epsilon z(\vartheta_k\omega)) + \epsilon z(\omega) = u(0, \omega; k, y_k) = \varphi(-k, \vartheta_k\omega)y_k \end{aligned} \tag{5.19}$$

uniformly in $\epsilon \in (0,1]$. On the other hand, because of

$$\varphi(-\tau_n + k, \vartheta_{\tau_n}\omega)x_n = u(k, \omega; \tau_n, x_n) = v(k, \omega; \tau_n, x_n - \epsilon z(\vartheta_{\tau_n}\omega)) + \epsilon z(\vartheta_k\omega), \tag{5.20}$$

by (5.18) and (5.20) we have

$$\varphi(-\tau_n, \vartheta_{\tau_n}\omega)x_n = v(0, \omega; k, v(k, \omega; \tau_n, x_n - \epsilon z(\vartheta_{\tau_n}\omega)) + \epsilon z(\omega). \tag{5.21}$$

Put $y^{(n)}(k) = v(k, \omega; \tau_n, x_n - \epsilon z(\vartheta_{\tau_n}\omega))$ and

$$v^{(n)}(t) = v(t, \omega; k, y_k^{(n)}), \quad v(t) = v(t, \omega; k, y_k - \epsilon z(\vartheta_k \omega)).$$

Then, from (5.17) it follows that $y^{(n)}(k)$ converges weakly to $y_k - \epsilon z(\vartheta_k \omega)$ in $H^1(\mathbb{R}^N)$.

We now consider the energy equation (3.8) on the intervals $[k, 0]$. First in terms of (5.21) and using (3.8) with $t = 0$ and $\tau = k$, we find that

$$\begin{aligned}
&\|\varphi(-\tau_n, \vartheta_{\tau_n}\omega)x_n - \epsilon z(\omega)\|_{H^1}^2 \\
&= \|v(0, \omega; k, v(k, \omega; \tau_n, x_n - \varepsilon z(\vartheta_{\tau_n}\omega)))\|_{H^1}^2 \\
&= e^{\mu k}\|y^{(n)}(k)\|_{H^1}^2 - 2\int_k^0 e^{\mu s}(f_1(x, v^{(n)}(s) + \epsilon z(\vartheta_s \omega)), v^{(n)}(s))ds \\
&\quad - 2\int_k^0 e^{\mu s}(af_2(v^{(n)}(s) + \epsilon z(\vartheta_s \omega)), v^{(n)}(s)ds \\
&\quad + 2\int_k^0 e^{\mu s}(\mathrm{g}, v^{(n)}(s))ds - (2-\mu)\int_k^0 e^{\mu s}\|v^{(n)}(s)\|_{H^1}^2 ds \\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{5.22}$$

First, we estimate I_1. In (5.8), giving $t = k$ and $\tau = \tau_n$, by utilizing (3.4), we deduce that

$$\begin{aligned}
I_1 &= e^{\mu k}\|y^{(n)}(k)\|_{H^1}^2 \\
&= e^{\mu k}\|v(k, \omega; \tau_n, x_n - \epsilon z(\vartheta_{\tau_n}\omega))\|_{H^1}^2 \\
&\le 2e^{\mu \tau_n}\|x_n\|_{H^1}^2 + 2e^{\mu \tau_n}\|z(\vartheta_{\tau_n}\omega)\|_{H^1}^2 + \int_{\tau_n}^k e^{\mu s}(\varsigma(\vartheta_s \omega) + c)ds \\
&\le 2e^{\mu \tau_n}\|x_n\|_{H^1}^2 + 2ce^{\frac{\mu}{2}\tau_n}r(\omega) + ce^{\frac{\mu}{2}k}(1 + r(\omega)),
\end{aligned}$$

and hence by $x_n \in D(\vartheta_{\tau_n}\omega)$, we obtain that

$$\lim_{\tau_n \to -\infty} I_1 \le ce^{\frac{\mu}{2}k}(1 + r(\omega)), \tag{5.23}$$

where c is a deterministic positive constant independent of k. To compute I_2, we rewrite it as

$$\begin{aligned}
I_2 &= -2\int_k^0 e^{\mu s}(f_1(x, v^{(n)}(s) + \epsilon z(\vartheta_s \omega)), v^{(n)}(s))ds \\
&= -2\int_k^0 e^{\mu s}\int_{\mathbb{R}^N(|x| \le R)} f_1(x, v^{(n)}(s) + \epsilon z(\vartheta_s \omega))v^{(n)}(s)\,dx\,ds \\
&\quad - 2\int_k^0 e^{\mu s}\int_{\mathbb{R}^N(|x| \ge R)} f_1(x, v^{(n)}(s) + \epsilon z(\vartheta_s \omega))v^{(n)}(s)\,dx\,ds \\
&= I_2' + I_2'',
\end{aligned} \tag{5.24}$$

where the radius R is large enough. To estimate I_2', we rewrite it as

$$\begin{aligned} I_2' &= -2\int_k^0 e^{\mu s}\int_{\mathbb{R}^N(|x|\le R)} f_1(x, v^{(n)}(s)+\epsilon z(\vartheta_s\omega))v^{(n)}(s)\,dx\,ds \\ &= -2\int_k^0 e^{\mu s}\int_{\mathbb{R}^N(|x|\le R)} f_1(x, u^{(n)}(s))u^{(n)}(s)\,dx\,ds \\ &\quad + 2\int_k^0 e^{\mu s}\int_{\mathbb{R}^N(|x|\le R)} f_1(x, u^{(n)}(s))z(\vartheta_s\omega)\,dx\,ds. \end{aligned} \tag{5.25}$$

Note that from (4.15), $u^{(n)}(s) \to u(s)$ for almost every $(t,x) \in [k,0]\times B_R$, where $B_R = \{x \in \mathbb{R}^N; |x| \le R\}$. Then by the continuity of f_1, we have

$$f_1(x, u^{(n)}(s))u^{(n)}(s) \to f_1(x, u(s))u(s), \quad \text{a.e. } (t,x) \in [k,0]\times B_R. \tag{5.26}$$

On the other hand, from (1.3) we see that $f_1(x,u^{(n)}(s))u^{(n)}(s) \ge -\psi_1(x), \psi_1 \in L^1(\mathbb{R}^N)$, and by the Hölder inequality,

$$\begin{aligned} &\Big|\int_k^0 e^{\mu s}\int_{\mathbb{R}^N(|x|\le R)} f_1(x, u^{(n)}(s))u^{(n)}(s)\,dx\,ds\Big| \\ &\le \Big(\int_k^0 e^{\mu s}\|f_1(x,u^{(n)}(s))\|_q^q ds\Big)^{1/q}\Big(\int_k^0 e^{\mu s}\|u^{(n)}(s)\|_p^p ds\Big)^{1/p} \le M < +\infty, \end{aligned} \tag{5.27}$$

where we have used (4.7) and (4.8), the positive constant M independent of ϵ, n. Then (5.26) and (5.27) together imply that we can utilize the Fatou-Lebesgue lemma to the nonnegative sequence $f_1(x,u^{(n)}(s))u^{(n)}(s) + \psi_1(x)$ to get that

$$\begin{aligned} &\liminf_{\tau_n\to-\infty}\int_k^0 e^{\mu s}\int_{\mathbb{R}^N(|x|\le R)} f_1(x, u^{(n)}(s))u^{(n)}(s)\,dx\,ds \\ &\ge \int_k^0 e^{\mu s}\int_{\mathbb{R}^N(|x|\le R)} \liminf_{\tau_n\to-\infty} f_1(x, u^{(n)}(s))u^{(n)}(s)\,dx\,ds \\ &= \int_k^0 e^{\mu s}\int_{\mathbb{R}^N(|x|\le R)} f_1(x, u(s))u(s)\,dx\,ds. \end{aligned} \tag{5.28}$$

Here we note that by (5.27), the left side of (5.28) is finite. On the other hand, since $f_1(x,u^{(n)}(s)) \to f_1(x,u(s))$ is weakly convergent in $L^q(k,0;L^q(B_R))$ by (4.8), and connection with our assumption (1.9), $z(\vartheta_s\omega) \in H^1 \hookrightarrow L^p$, then we have

$$\begin{aligned} &\lim_{\tau_n\to-\infty}\int_k^0 e^{\mu s}\int_{\mathbb{R}^N(|x|\le R)} f_1(x, u^{(n)}(s))z(\vartheta_s\omega)\,dx\,ds \\ &= \int_k^0 e^{\mu s}\int_{\mathbb{R}^N(|x|\le R)} f_1(x, u(s))z(\vartheta_s\omega)\,dx\,ds. \end{aligned} \tag{5.29}$$

Hence taking $\tau_n \to -\infty$ in (5.25) and then using (5.28) and (5.29) we find that

$$\limsup_{\tau_n\to-\infty} I_2' \le -2\int_k^0 e^{\mu s}\int_{\mathbb{R}^N(|x|\le R)} f_1(x, u(s))v(s)\,dx\,ds. \tag{5.30}$$

We next estimate I_2''. By using (1.3) we have

$$
\begin{aligned}
I_2'' &= -2\int_k^0 e^{\mu s}\int_{\mathbb{R}^N(|x|\geq R)} f_1(x, v^{(n)}(s)+\epsilon z(\vartheta_s\omega))v^{(n)}(s)\,dx\,ds\\
&= -2\int_k^0 e^{\mu s}\int_{\mathbb{R}^N(|x|\geq R)} f_1(x, u^{(n)}(s))u^{(n)}(s)\,dx\,ds\\
&\quad + 2\int_k^0 e^{\mu s}\int_{\mathbb{R}^N(|x|\geq R)} f_1(x, u^{(n)}(s))z(\vartheta_s\omega)\,dx\,ds\\
&\leq 2\int_k^0 e^{\mu s}\int_{\mathbb{R}^N(|x|\geq R)} \psi_1(x)\,dx\,ds\\
&\quad + 2\int_k^0 \int_{\mathbb{R}^N(|x|\geq R)} \Big(e^{\frac{1}{q}\mu s}|f_1(x, u^{(n)}(s))|\Big)\Big(e^{\frac{1}{p}\mu s}|z(\vartheta_s\omega)|\Big)\,dx\,ds\\
&\leq 2\int_k^0 e^{\mu s}\int_{\mathbb{R}^N(|x|\geq R)} \psi_1(x)\,dx\,ds + 2\Big(\int_k^0 e^{\mu s}\int_{\mathbb{R}^N(|x|\geq R)} |z(\vartheta_s\omega)|^p\,dx\,ds\Big)^{1/p}\\
&\quad \times \Big(\int_k^0 e^{\mu s}\int_{\mathbb{R}^N} |f_1(x, u^{(n)}(s))|^q\,dx\,ds\Big)^{1/q}.
\end{aligned}
\tag{5.31}
$$

Note that $\psi_1 \in L^1(\mathbb{R}^N)$, and $z(\vartheta_s\omega) \in H^1 \hookrightarrow L^p$. Then we may choose the radius R large enough such that for any $\varepsilon > 0$,

$$
\begin{gathered}
\int_k^0 e^{\mu s}\int_{\mathbb{R}^N(|x|\geq R)} \psi_1(x)\,dx\,ds \leq c\varepsilon,\\
\int_k^0 e^{\mu s}\int_{\mathbb{R}^N(|x|\geq R)} |z(\vartheta_s\omega)|^p\,dx\,ds \leq c\varepsilon.
\end{gathered}
\tag{5.32}
$$

On the other hand, by (4.8), there is constant $M > 0$ independent of n and ϵ such that

$$
\int_k^0 e^{\mu s}\int_{\mathbb{R}^N} |f_1(x, u^{(n)}(s))|^q\,dx\,ds \leq M. \tag{5.33}
$$

Then by (5.31)-(5.33) we have

$$
\limsup_{\tau_n\to-\infty} I_2'' \leq c\varepsilon, \tag{5.34}
$$

where the constant c is independent of ε.

By combining (5.30) and (5.34) into (5.24) we find that for R large enough,

$$
\limsup_{\tau_n\to-\infty} I_2 \leq c\varepsilon - 2\int_k^0 e^{\mu s}\int_{\mathbb{R}^N(|x|\leq R)} f_1(x, u(s))v(s)\,dx\,ds. \tag{5.35}
$$

Similarly we can show that

$$
\limsup_{\tau_n\to-\infty} I_3 \leq c\varepsilon - 2\int_k^0 e^{\mu s}\int_{\mathbb{R}^N(|x|\leq R)} a(x)f_2(u(s))v(s)\,dx\,ds. \tag{5.36}
$$

By the weak convergence of $\{v^{(n)}(s)\}_n$ in $L^2(k, 0; H^1(\mathbb{R}^N))$ in (4.14), we immediately get that

$$
\lim_{\tau_n\to-\infty} I_4 = \lim_{\tau_n\to-\infty}\int_k^0 e^{\mu s}(\mathrm{g}, v^{(n)}(s))\,dx\,ds = \int_k^0 e^{\mu s}(\mathrm{g}, v(s))ds, \tag{5.37}
$$

$$\liminf_{\tau_n\to-\infty} I_5 = \liminf_{\tau_n\to-\infty}\int_k^0 e^{\mu s}\|v^{(n)}(s)\|_{H^1}^2 ds \ge \int_k^0 e^{\mu s}\|v(s)\|_{H^1}^2 ds. \tag{5.38}$$

Then, we include (5.23) and (5.35)-(5.38) into (5.22), by letting $R\to+\infty$, to yield

$$\begin{aligned}
&\limsup_{\tau_n\to-\infty}\|\varphi(-\tau_n,\vartheta_{\tau_n}\omega)x_n-\epsilon z(\omega)\|_{H^1}^2\\
&\le ce^{\frac{\mu}{2}k}(1+r(\omega))-(2-\mu)\int_k^0 e^{\mu s}\|v(s)\|_{H^1}^2 ds\\
&\quad -2\int_k^0 e^{\mu s}\int_{\mathbb{R}^N} f_1(x,u(s))v(s)\,dx\,ds\\
&\quad -2\int_k^0 e^{\mu s}\int_{\mathbb{R}^N} a(x)f_2(u(s))v(s)\,dx\,ds+2\int_k^0 e^{\mu s}\int_{\mathbb{R}^N} \mathrm{g}v(s)\,dx\,ds.
\end{aligned} \tag{5.39}$$

On the other hand, from the energy equality (3.8), we have

$$\begin{aligned}
&-(2-\mu)\int_k^0 e^{\mu s}\|v(s)\|_{H^1}^2 ds-2\int_k^0 e^{\mu s}\int_{\mathbb{R}^N} f_1(x,u(s))v(s)\,dx\,ds\\
&-2\int_k^0 e^{\mu s}\int_{\mathbb{R}^N} a(x)f_2(u(s))v(s)\,dx\,ds+2\int_k^0 e^{\mu s}\int_{\mathbb{R}^N} \mathrm{g}v(s)\,dx\,ds\\
&=\|v(0,\omega;k,y_k-\epsilon z(\vartheta_k\omega))\|_{H^1}^2=\|\varphi(-k,\vartheta_k\omega)y_k-\epsilon z(\omega)\|_{H^1}^2.
\end{aligned} \tag{5.40}$$

Then it follows from (5.39)-(5.40) that

$$\begin{aligned}
&\limsup_{\tau_n\to-\infty}\|\varphi(-\tau_n,\vartheta_{\tau_n}\omega)x_n-\epsilon z(\omega)\|_{H^1}^2\\
&\le ce^{\frac{\mu}{2}k}(1+r(\omega))+\|\varphi(-k,\vartheta_k\omega)y_k-\epsilon z(\omega)\|_{H^1}^2\\
&=ce^{\frac{\mu}{2}k}(1+r(\omega))+\|y_0-\epsilon z(\omega)\|_{H^1}^2
\end{aligned} \tag{5.41}$$

by (5.19). Letting $k\to-\infty$ in (5.41), we have showed that

$$\limsup_{\tau_n\to-\infty}\|\varphi(-\tau_n,\vartheta_{\tau_n}\omega)x_n-\epsilon z(\omega)\|_{H^1}^2 \le \|y_0-\epsilon z(\omega)\|_{H^1}^2. \tag{5.42}$$

This concludes the proof. □

We now state our main result in this section.

Theorem 5.3. *Assume that* (1.2)-(1.9) *are satisfied,* $g\in L^2(\mathbb{R}^N)$. *Then the RDS* (φ,ϑ) *corresponding to the stochastic nonclassical diffusion equations* (1.1) *admits a unique* $\mathcal{D}_\mu$*-random attractor* $\mathcal{A}_\mu$ *in* $H^1(\mathbb{R}^N)$*, where the universe* $\mathcal{D}_\mu$ *is defined in* (5.1). *Furthermore, if* $0<\nu\le\mu<2$*, then* $\mathcal{A}_\nu\subseteq\mathcal{A}_\mu$.

Proof. By Lemmas 5.1 and 5.2 and by Theorem 2.5, we obtain the existence of unique $\mathcal{D}_\mu$-random attractor for the RDS (φ,ϑ) in $\mathcal{D}_\mu$ for every $\mu\in(0,2)$. On the other hand, in view of the definition of $\mathcal{D}_\mu$, if $\nu\le\mu$, then $\mathcal{D}_\nu\subseteq\mathcal{D}_\mu$. Note that $\mathcal{A}_\mu$ as a random attractor attracts every set of the universe $\mathcal{D}_\mu$, and hence attracts every set of the universe $\mathcal{D}_\nu$. Since $\mathcal{A}_\nu\in\mathcal{D}_\nu$, then $\mathcal{A}_\mu$ attracts $\mathcal{A}_\nu$ as an element of $\mathcal{D}_\nu$, i.e. for $\mathbb{P}$-a.e. $\omega\in\Omega$,

$$\lim_{\tau\to-\infty} d(\varphi(-\tau,\vartheta_\tau\omega)\mathcal{A}_\nu(\vartheta_\tau\omega),\mathcal{A}_\mu(\omega))=0,$$

where d is the Hausdorff semi-metric. By the invariant property of random attractor, $\varphi(-\tau,\vartheta_\nu\omega)\mathcal{A}_\nu(\vartheta_\tau\omega)=\mathcal{A}_\nu(\omega)$ for all $\tau<0$ and $\mathbb{P}$-a.e. $\omega\in\Omega$. Then we have

$d(\mathcal{A}_\nu(\omega), \mathcal{A}_\mu(\omega)) = 0$, which implies that $\mathcal{A}_\nu(\omega) \subseteq \mathcal{A}_\mu(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, as required. $\square$

Remark 5.4. From Theorem 5.3, it follows that the uniqueness for the RDS (φ, ϑ) related to (1.1) relies on the choice of the universe $\mathcal{D}_\mu$. If the universe $\mathcal{D}_\mu$ increases with respect to the parameter μ in the meaning of sets inclusion relation, so does the corresponding random attractors $\mathcal{A}_\mu$.

Our method can also be used for studying the non autonomous nonclassical diffusion equation

$$\begin{gathered} u_t - \Delta u_t - \Delta u + u + f(x,u) = g(x,t) + \epsilon h\dot{W}, \quad x \in \mathbb{R}^N, \\ u(x,\tau) = u_0(x), \quad x \in \mathbb{R}^N. \end{gathered}$$

6. Upper semi-continuity of random attractors at $\epsilon = 0$

In this section, to indicate the dependence of solutions of (1.1) on ϵ, we write the solutions as u_ϵ, and the corresponding RDS as $(\varphi_\epsilon, \vartheta)$. Without loss generality, we fix $\mu = 1$ in this section.

In the last section, we showed that $(\varphi_\epsilon, \vartheta)$ possesses a $\mathcal{D}$-random attractor $\mathcal{A}_\epsilon$, where $\mathcal{D}$ is defined in (5.1) with $\mu = 1$. When $\epsilon = 0$, the system (1.1) reduces into the deterministic equation

$$\begin{gathered} u_t - \Delta u_t - \Delta u + u + f(x,u) = \mathrm{g}(x), \quad x \in \mathbb{R}^N, \\ u(x,\tau) = u_0(x), \quad x \in \mathbb{R}^N, \ t > \tau, \end{gathered} \tag{6.1}$$

while the nonlinearity $f(x,u) = f_1(x,u) + a(x)f_2(u)$ satisfies that (1.2)-(1.4), (1.6)-(1.7) and additionally,

$$\frac{\partial}{\partial s} f_1(x,s) \geq -l, \quad \big|\frac{\partial}{\partial s} f_1(x,s)\big| \leq \alpha_3 |s|^{p-2} + \psi_3(x), \quad \psi_3 \in L^\infty(\mathbb{R}^N), \tag{6.2}$$

$$\frac{\partial}{\partial s} f_2(s) \geq -l, \quad \big|\frac{\partial}{\partial s} f_2(s)\big| \leq \alpha_3 |s|^{p-2} + \kappa, \tag{6.3}$$

where $\alpha_3, l, \kappa \geq 0$. Note that we have replaced the assumptions (1.5) and (1.8) by the above (6.2) and (6.3), respectively.

It is easy to check that the solution of (6.1) defines a continuous deterministic dynamical system on $H^1(\mathbb{R}^N)$, denoted by φ_0. Note that all the results in the previous section hold for $\epsilon = 0$. In particular, φ_0 admits a unique global attractor in $H^1(\mathbb{R}^N)$, denoted by $\mathcal{A}_0$.

The purpose of this section is to establish the relationships of the random attractors $\mathcal{A}_\epsilon = \{\mathcal{A}_\epsilon(\omega); \omega \in \Omega\}$ and the global attractor $\mathcal{A}_0$ when $\epsilon \to 0^+$.

We first show that, as $\epsilon \to 0^+$, the solutions of the stochastic nonclassical diffusion equations (1.1) converge to the limiting deterministic equations (6.1).

Lemma 6.1. *Suppose that $g \in L^2(\mathbb{R}^N)$, (1.2)-(1.4), (1.6)-(1.7), (1.9), and (6.2)-(6.3) hold. Given $0 < \epsilon \leq 1$, let u^ϵ and u be the solutions of equations (1.1) and (6.1) with initial conditions u_0^ϵ and u_0, respectively. Then for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $\tau \leq t \leq 0$, we have*

$$\begin{aligned} &\|u^\epsilon(t,\omega;\tau,u_0^\epsilon) - u(t;\tau,u_0)\|_{H^1}^2 \\ &\leq ce^{-c\tau}\|u_0^\epsilon - u_0\|_{H^1}^2 + c\epsilon e^{-c\tau}(\|u_0^\epsilon\|_{H^1}^2 + \|u_0\|_{H^1}^2) + c\epsilon e^{-c\tau}(1 + r(\omega)), \end{aligned}$$

where c is a deterministic positive constant independent of ϵ, and $r(\omega)$ is as in (3.4).

Proof. Put $v^\epsilon = u^\epsilon(t,\omega;\tau,u_0^\epsilon) - \epsilon z(\theta_t\omega)$ and $U = v^\epsilon - u$, where v^ϵ and u satisfy (3.5) and (6.1), respectively. Then we obtain that U is a solution of the equation

$$U_t - \Delta u_t - \Delta U + U + f(x,u^\epsilon) - f(x,u) = 0. \tag{6.4}$$

Multiplying by U and integrating over $\mathbb{R}^N$, we find that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|U\|^2 + \|\nabla U\|^2) + \|\nabla U\|^2 + \|U\|^2 \\ &= -\int_{\mathbb{R}^N} f(x,u^\epsilon)U\,dx + \int_{\mathbb{R}^N} f(x,u)U\,dx. \end{aligned} \tag{6.5}$$

Note that

$$\begin{aligned} &-\int_{\mathbb{R}^N} f(x,u^\epsilon)U\,dx + \int_{\mathbb{R}^N} f(x,u)U\,dx \\ &= -\int_{\mathbb{R}^N}(f_1(x,u^\epsilon) - f_1(x,u))U\,dx - \int_{\mathbb{R}^N} a(x)((f_2(u^\epsilon) - f_2(u))U\,dx \\ &= \int_{\mathbb{R}^N}\frac{\partial}{\partial s}f_1(x,s)(u-u^\epsilon)U\,dx + \int_{\mathbb{R}^N} a(x)\frac{\partial}{\partial s}f_2(s)(u-u^\epsilon)U\,dx \\ &=: I_1 + I_2, \end{aligned} \tag{6.6}$$

For the term I_1, it follows from (6.2) and (6.3) that

$$\begin{aligned} &I_1 \\ &= -\int_{\mathbb{R}^N}\frac{\partial}{\partial s}f_1(x,s)U^2 dx - \epsilon\int_{\mathbb{R}^N}\frac{\partial}{\partial s}f_1(x,s)z(\vartheta_t\omega)U\,dx \\ &\le l\|U\|^2 + \epsilon\alpha_3\int_{\mathbb{R}^N}(|u^\epsilon| + |u|)^{p-2}|z(\vartheta_t\omega)||U|dx + \epsilon\int_{\mathbb{R}^N}\psi_3(x)|z(\vartheta_t\omega)||U|dx \\ &\le l\|U\|^2 + c\epsilon(\|u^\epsilon\|_p^p + \|u\|_p^p + \|z(\vartheta_t\omega)\|_p^p + \|U\|_p^p) \\ &\quad + (\|U\|^2 + \epsilon\|\psi_3\|_\infty^2\|z(\vartheta_t\omega)\|^2) \\ &\le (l+1)\|U\|^2 + c\epsilon(\|u^\epsilon\|_p^p + \|u\|_p^p + \|z(\vartheta_t\omega)\|^2 + \|z(\vartheta_t\omega)\|_p^p)\,, \end{aligned} \tag{6.7}$$

where we have used that $\|U\|_p^p = \|u^\epsilon - u - z(\vartheta_t\omega)\|_p^p \le c\|u^\epsilon\|_p^p + \|u\|_p^p + \|z(\vartheta_t\omega)\|_p^p)$. Similarly, we have

$$I_2 \le (l+1)\|a\|_\infty\|U\|^2 + c\epsilon(\|u^\epsilon\|_p^p + \|u\|_p^p + \|z(\vartheta_t\omega)\|^2 + \|z(\vartheta_t\omega)\|_p^p). \tag{6.8}$$

Hence combinations (6.6)-(6.8) give

$$\begin{aligned} &-\int_{\mathbb{R}^N} f(x,u^\epsilon)U\,dx + \int_{\mathbb{R}^N} f(x,u)U\,dx \\ &\le c\|U\|^2 + c\epsilon(\|u^\epsilon\|_p^p + \|u\|_p^p + \|z(\vartheta_t\omega)\|^2 + \|z(\vartheta_t\omega)\|_p^p), \end{aligned}$$

from which and (6.5) we obtain that, using (3.4),

$$\frac{d}{dt}\|U\|_{H^1}^2 \le c\|U\|_{H^1}^2 + c\epsilon(\|u^\epsilon\|_p^p + \|u\|_p^p) + c\epsilon e^{\frac{\mu}{2}|t|}r(\omega), \tag{6.9}$$

where c is deterministic constant. We now integrate (6.9) over $[\tau, t] (t \le 0)$ to obtain

$$
\begin{aligned}
&\|U(t)\|_{H^1}^2 \\
&\le e^{c(t-\tau)}\|U(\tau)\|_{H^1}^2 + c\epsilon e^{ct}\int_\tau^t e^{-cs}(\|u^\epsilon\|_p^p + \|u\|_p^p)ds \\
&\quad + c\epsilon e^{ct} r(\omega)\int_\tau^t e^{-(\frac{\mu}{2}+c)s}ds \\
&\le e^{c(t-\tau)}\|U(\tau)\|_{H^1}^2 + c\epsilon e^{c(t-\tau)}\int_\tau^t (\|u^\epsilon\|_p^p + \|u\|_p^p)ds + c\epsilon e^{c(t-\tau)-\frac{\mu}{2}\tau} r(\omega) \\
&\le e^{-c\tau}\|U(\tau)\|_{H^1}^2 + c\epsilon e^{-c\tau}\int_\tau^t (\|u^\epsilon\|_p^p + \|u\|_p^p)ds + c\epsilon e^{-c\tau} r(\omega),
\end{aligned}
\tag{6.10}
$$

where we have used $e^{\mu t} \le 1$ for $t \le 0$. By (5.8), using (3.4) we obtain that

$$
\begin{aligned}
\int_\tau^t e^{-\mu(t-s)}\|u^\epsilon(s)\|_p^p ds &\le e^{-\mu(t-\tau)}\|v_0^\epsilon\|_{H^1}^2 + \epsilon e^{-\mu t}\int_\tau^t e^{\mu s}\varsigma(\vartheta_s\omega)ds + c \\
&\le e^{-\mu(t-\tau)}\|u_0^\epsilon - \epsilon z(\vartheta_t\omega)\|_{H^1}^2 + c\epsilon e^{-\frac{\mu}{2}t} r(\omega) + c\,.
\end{aligned}
\tag{6.11}
$$

Because $e^{-\mu(t-s)} \ge e^{-\mu(t-\tau)}$ for $\tau \le t \le 0$, by (6.11) we obtain

$$
\begin{aligned}
\int_\tau^t \|u^\epsilon(s)\|_p^p ds &\le \|u_0^\epsilon - \epsilon z(\vartheta_t\omega)\|_{H^1}^2 + c\epsilon e^{\frac{\mu}{2}t-\mu\tau} r(\omega) + c e^{\mu(t-\tau)} \\
&\le \|u_0^\epsilon - \epsilon z(\vartheta_t\omega)\|_{H^1}^2 + c\epsilon e^{-\mu\tau} r(\omega) + c e^{-\mu\tau}.
\end{aligned}
\tag{6.12}
$$

Similarly, by (6.1) we can deduce that

$$
\int_\tau^t \|u(s)\|_p^p ds \le \|u_0\|_{H^1}^2 + c. \tag{6.13}
$$

Hence, from (6.10), (6.12)-(6.13) it follows that

$$
\|U(t)\|_{H^1}^2 \le e^{-c\tau}\|U(\tau)\|_{H^1}^2 + c\epsilon e^{-c\tau}(\|u_0^\epsilon\|_{H^1}^2 + \|u_0\|_{H^1}^2) + c\epsilon e^{-c\tau}(1 + r(\omega)), \tag{6.14}
$$

then we naturally obtain

$$
\begin{aligned}
&\|u^\epsilon(t,\omega;\tau,u_0^\epsilon) - u(t,\tau,u_0)\|_{H^1}^2 \\
&= \|U(t) + \epsilon z(\vartheta_t\omega)\|_{H^1}^2 \\
&\le 2\|U(t)\|_{H^1}^2 + 2\epsilon\|z(\vartheta_t\omega)\|_{H^1}^2 \le 2\|U(t)\|_{H^1}^2 + c\epsilon e^{-\frac{\mu}{2}t} r(\omega) \\
&\le 2e^{-c\tau}\|U(\tau)\|_{H^1}^2 + c\epsilon e^{-c\tau}(\|u_0^\epsilon\|_{H^1}^2 + \|u_0\|_{H^1}^2) + c\epsilon e^{-c\tau}(1 + r(\omega)) \\
&= 2e^{-c\tau}\|u_0^\epsilon - u_0 - z(\vartheta_\tau\omega)\|_{H^1}^2 + c\epsilon e^{-c\tau}(\|u_0^\epsilon\|_{H^1}^2 + \|u_0\|_{H^1}^2) + c\epsilon e^{-c\tau}(1 + r(\omega)) \\
&\le 4e^{-c\tau}\|u_0^\epsilon - u_0\|_{H^1}^2 + c\epsilon e^{-c\tau}(\|u_0^\epsilon\|_{H^1}^2 + \|u_0\|_{H^1}^2) + c\epsilon e^{-c\tau}(1 + r(\omega)),
\end{aligned}
$$

where we have used $e^{-\mu t} \le e^{-\mu\tau}$ for $\tau \le t \le 0$. □

Theorem 6.2. *Suppose that $g \in L^2(\mathbb{R}^N)$, (1.2)-(1.4), (1.6)-(1.7), (1.9), and (6.2)-(6.3) hold. Then the random attractors $\mathcal{A}_\epsilon$ is upper semi-continuous at $\epsilon = 0$, i.e., for $\mathbb{P}$-a.e.$\omega \in \Omega$,*

$$
\lim_{\epsilon \searrow 0} d(\mathcal{A}_\epsilon(\omega), \mathcal{A}_0) = 0,
$$

where d is the Haustorff semi-metric in $H^1(\mathbb{R}^N)$.

Proof. From Lemma 6.1, we know that the RDS $(\varphi_\epsilon, \vartheta)$ converges to the DS φ_0 in $H^1(\mathbb{R}^N)$ when $\epsilon \searrow 0$ and $\|u_0^\epsilon - u_0\|_{H^1} \to 0$. On the other hand, it follows from Lemma 5.1 that for every $0 < \epsilon \le 1$, the RDS $(\varphi_\epsilon, \vartheta)$ possesses a $\mathcal{D}$-random absorbing set E_ϵ (here for brevity we do not consider the dependence of E_ϵ on μ), that is to say, for every $D \in \mathcal{D}$ and $\mathbb{P}$-a.e.$\omega \in \Omega$, there exists $T = T(D, \omega) \le 0$ such that for all $\tau \le T$

$$\|\varphi(-\tau, \vartheta_\tau \omega) D(\vartheta_\tau \omega)\|_{H^1}^2 \le M(1 + \epsilon r(\omega)),$$

where M is a deterministic constant independent of ϵ, $r(\omega)$ in (3.4) and $\mathcal{D}$ (where $\mathcal{D}_\mu = \mathcal{D}$) in (5.1). Denote

$$E_\epsilon = \{u \in H^1(\mathbb{R}^N); \|u\|_{H^1}^2 \le M(1 + \epsilon r(\omega))\}. \tag{6.15}$$

Then $E_\epsilon \in \mathcal{D}$, and for $\mathbb{P}$-a.e.$\omega \in \Omega$, it produces that

$$\limsup_{\epsilon \downarrow 0} \|E_\epsilon\|_{H^1} \le M.$$

Finally, we shall show that for $\mathbb{P}$-a.e.$\omega \in \Omega$, the union

$$A(\omega) = \cup_{0<\epsilon\le 1}\{\mathcal{A}_\epsilon(\omega)\} \text{ is precompact in } H^1(\mathbb{R}^N). \tag{6.16}$$

Indeed, for any sequence $\{u_n\}_n \subset A(\omega)$, there exists some $\epsilon_n \in (0, 1]$ such that $u_n \in \mathcal{A}_{\epsilon_n}(\omega)$ for all $n \in \mathbb{Z}^+$. According to the invariance of the random attractor $\mathcal{A}_{\epsilon_n}$, there is a sequence $v_n \in \mathcal{A}_{\epsilon_n}(\vartheta_{\tau_n}\omega)$ such that

$$u_n = \varphi_{\epsilon_n}(-\tau_n, \vartheta_{\tau_n}\omega) v_n, \quad n \in \mathbb{Z}^+.$$

By the proof of Lemma 5.2, it has showed that φ_ϵ is pre-compact in $H^1(\mathbb{R}^N)$ uniform in $\epsilon \in (0, 1]$ and thus $\varphi_{\epsilon_n}(-\tau_n, \vartheta_{\tau_n}\omega) v_n$ has a convergent subsequence in $H^1(\mathbb{R}^N)$ with respect to all ϵ_n. Therefore, we obtain that (6.16) holds true. The conditions of Theorem 2.6 are satisfied. The proof is complete. □

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