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WELL-POSEDNESS OF AN ELLIPTIC EQUATION WITH INVOLUTION

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Abstract. In this article, we study a mixed problem for an elliptic equation with involution. This problem is reduced to boundary value problem for the abstract elliptic equation in a Hilbert space with a self-adjoint positive definite operator. Operator tools permits us to obtain stability and coercive stability estimates in Hölder norms, in t, for the solution.

1. Introduction

Elliptic equations have important applications in a wide range of applications such as physics, chemistry, biology and ecology and other fields. In mathematical modeling, elliptic equations are used together with boundary conditions specifying the solution on the boundary of the domain. Dirichlet and Neumann conditions are examples of classical boundary conditions. The role played by coercive inequalities (well-posedness) in the study of local boundary-value problems for elliptic and parabolic differential equations is well known (see, e.g., [20, 29] and the references therein). Mathematical models of various physical, chemical, biological or environmental processes often involve nonclassical conditions. Such conditions are usually identified as nonclassical boundary conditions and reflect situations when a data on the domain boundary can not be measured directly, or when the data on the boundary depends on the data inside the domain. Well-posedness of various classical and nonclassical boundary value problems for partial differential and difference equations has been studied extensively by many researchers with the operator method tool (see [1, 2, 3, 4, 5, 6, 8, 11, 14, 15, 16, 19, 26, 27, 28]).

The theory of functional-differential equations with the involution has received less attention than functional-differential equations. Except for a few works [1, 21, 30] parabolic differential and difference equations with the involution are not studied enough in the literature.

For example, in [30], the mixed problem for a parabolic partial differential equation with the involution with respect to t

$$u_t(t,x) = au_{xx}(t,x) + bu_{xx}(-t,x), \quad 0 < x < l,\ -\infty < t < \infty$$

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with the Dirichlet condition in x was studied. The Fourier method is common used method to get existence of unbounded solutions and non existence of solution dependent on coefficients a and b.

Papers [22, 12], the mixed problem for a first-order partial differential equation with the involution was investigated. The Fourier method was used to find a classical solution of the mixed problem for a first-order differential equation with involution. Application of the Fourier method was substantiated using refined asymptotic formulas obtained for eigenvalues and eigenfunctions of the corresponding spectral problem. The Fourier series representing the formal solution was transformed using certain techniques, and the possibility of its term-by-term differentiation was proved.

The paper [13] was devoted to the study of first order linear problems with involution and periodic boundary value conditions. First, it was proved a correspondence between a large set of such problems with different involutions to later focus attention to the case of the reflection. Then, different cases for which a Green's function can be obtained explicitly, it was derived several results in order to obtain information about its sign. More general existence and uniqueness of solution results were established.

In [17, 18], the basic properties of systems of eigenfunctions and associated functions for one kind of generalized spectral problems for a second-order and a first-order ordinary differential operators.

In [25], the notion of regularity of boundary conditions for the simplest second-order differential equation with a deviating argument was introduced. The Riesz basis property for a system of root vectors of the corresponding generalized spectral problem with regular boundary conditions (in the sense of the introduced definition) was established. Examples of irregular boundary conditions to which the theory of Il'in basis property can be applied were given.

In [24], a nonclassical operator L in $L_2(-1,1)$, generated by the differential expression with shifted argument

$$Lu := -u''(-x), \quad -1 < x < 1 \tag{1.1}$$

and the boundary conditions

$$\alpha_j u'(-1) + \beta_j u'(1) + \alpha_{j1} u(-1) + \beta_{j1} u(1) = 0, \quad j = 1, 2 \tag{1.2}$$

was considered. For the spectral problem corresponding to (1.1) and (1.2), the author introduces a concept of regular boundary conditions (1.2). In some sense, the definition is similar to that of strong (Birkhoff) regular boundary conditions (1.2) for second-order ordinary differential equations. The main result of the paper states that a system of eigenfunctions and associated functions of the operator L forms an unconditional basis of the space $L_2(-1,1)$.

In the paper [23], spectral problem for a model second-order differential operator with an involution was considered. An operator was given by the differential expression $Lu = -u''(-x)$ and boundary conditions of general form. A criterion for the basis property of the systems of eigenfunctions of this operator in terms of the coefficients in the boundary conditions was obtained.

In this article, we study the mixed problem for an elliptic equation with the involution

$$
\begin{gathered}
-\frac{\partial^2 u(t,x)}{\partial t^2} = (a(x)u_x(t,x))_x + \beta(a(-x)u_x(t,-x))_x - \sigma u(t,x) + f(t,x), \\
-l < x < l,\ 0 < t < T, \\
u(t,-l) = u(t,l), \quad u_x(t,-l) = u_x(t,l), \quad 0 \le t \le T, \\
u(0,x) = \varphi(x), \quad u(T,x) = \psi(x), \quad -l \le x \le l, \\
\varphi(-l) = \varphi(l), \quad \psi(-l) = \psi(l), \quad \varphi'(-l) = \varphi'(l), \quad \psi'(-l) = \psi'(l),
\end{gathered}
\tag{1.3}
$$

where $u(t,x)$ is unknown function, $\varphi(x), \psi(x), a(x)$, and $f(t,x)$ are sufficiently smooth functions, $a \ge a(x) = a(-x) \ge \delta > 0$ and $\sigma > 0$ is a sufficiently large number.

Here, we study problem (1.3) for an elliptic equation with the involution by using the operator tool in monograph [10]. We establish stability estimates in the $C([0,T], L_2[-l,l])$ norm, and coercive stability estimates in the $C^\alpha([0,T], L_2[-l,l])$ and $C^\alpha_{0T}([0,T], L_2[-l,l])$ norms for the solution of this problem.

2. Preliminaries and statement of main results

To formulate our results, we introduce the Hilbert $L_2[-l,l]$ of all integrable functions f defined on $[-l,l]$, equipped with the norm

$$
\|f\|_{L_2[-l,l]} = \Big(\int_{-l}^{l} |f(x)|^2 dx \Big)^{1/2}.
$$

We introduce the inner product in $L_2[-l,l]$ by

$$
\langle u, v \rangle = \int_{-l}^{l} u(x)v(x)dx.
$$

In this article, $C^\alpha([0,T],E)$ and $C^\alpha_{0T}([0,T],E)$ $(0 < \alpha < 1)$ stand for Banach spaces of all abstract continuous functions $\varphi(t)$ defined on $[0,T]$ with values in E satisfying a Hölder condition for which the following norms are finite

$$
\|\varphi\|_{C^\alpha([0,T],E)} = \|\varphi\|_{C([0,T],E)} + \sup_{0 \le t < t+\tau \le T} \frac{\|\varphi(t+\tau) - \varphi(t)\|_E}{\tau^\alpha},
$$

$$
\|\varphi\|_{C^\alpha_{0T}([0,T],E)} = \|\varphi\|_{C([0,T],E)} + \sup_{0 \le t < t+\tau \le T} \frac{(t+\tau)^\alpha (T-t)^\alpha \|\varphi(t+\tau) - \varphi(t)\|_E}{\tau^\alpha},
$$

respectively. Here, $C([0,T],E)$ stands for the Banach space of all abstract continuous functions $\varphi(t)$ defined on $[0,T]$ with values in E equipped with the norm

$$
\|\varphi\|_{C([0,T],E)} = \max_{0 \le t \le T} \|\varphi(t)\|_E.
$$

Definition 2.1. An operator A densely defined in a Banach space E with domain $D(A)$ is called positive in E, if its spectrum σ_A lies in the interior of the sector of angle φ, $0 < \varphi < \pi$, symmetric with respect to the real axis, and moreover on the edges of this sector $S_1(\varphi) = \{\rho e^{i\varphi} : 0 \le \rho \le \infty\}$ and $S_2(\varphi) = \{\rho e^{-i\varphi} : 0 \le \rho \le \infty\}$, and outside of the sector the resolvent $(\lambda - A)^{-1}$ is subject to the bound (see [9])

$$
\|(A - \lambda)^{-1}\|_{E \to E} \le \frac{M}{1 + |\lambda|}.
$$

The infimum of all such angles φ is called the spectral angle of the positive operator A and is denoted by $\varphi(A) = \varphi(A, E)$. The operator A is said to be strongly positive in a Banach space E if $\varphi(A, E) < \frac{\pi}{2}$.

Throughout this article, we will indicate with M positive constants which can be different from time to time and we are not interested in precise. We will write $M(\alpha, \beta, \cdots)$ to stress the fact that the constant depends only on $\alpha, \beta, \cdots$.

With the help of the positive operator A, we introduce the fractional spaces $E_\alpha = E_\alpha(E, A), 0 < \alpha < 1$, consisting of all $v \in E$ for which the following norm is finite [9]:

$$\|v\|_{E_\alpha} = \|v\|_E + \sup_{\lambda > 0} \lambda^{1-\alpha} \|A \exp\{-\lambda A\} v\|_E. \tag{2.1}$$

Finally, we introduce a differential operator A^x defined by the formula

$$A^x v(x) = -(a(x)v_x(x)_x - \beta(a(-x)v_x(-x))_x + \sigma v(x) \tag{2.2}$$

with the domain $D(A^x) = \{u, u_{xx} \in L_2[-l, l] : u(-l) = u(l), u'(-l) = u'(l)\}$.

We can rewrite problem (1.3) in the following abstract form

$$-u_{tt}(t) + Au(t) = f(t), \quad 0 < t < T,\ u(0) = \varphi,\ u(T) = \psi \tag{2.3}$$

in a Hilbert space $H = L_2[-l, l]$ with the unbounded operator $A = A^x$ defined by formula (2.2). Here, $f(t) = f(t, x)$ and $u(t) = u(t, x)$ are known and unknown abstract functions respectively and they are defined on $(0, T)$ with values in $H = L_2[-l, l]$, $\varphi = \varphi(x)$, $\psi = \psi(x)$, and $a = a(x)$ are given smooth elements of $H = L_2[-l, l]$.

The main result of present paper is the following theorem on stability estimates of (1.3) in spaces $C([0, T)], L_2[-l, l])$ and coercive stability estimates in $C^\alpha([0, T], L_2[-l, l])$ and $C_{0T}^\alpha([0, T], L_2[-l, l])$ norms for the solution of problem (1.3).

Theorem 2.2. *Assume that* $\delta - a|\beta| \geq 0, \varphi(x), \varphi_{xx}(x), \psi(x), \psi_{xx}(x) \in L_2[-l, l]$ *and* $f(t, x) \in C_{0T}^\alpha([0, T], L_2[-l, l])$. *Then the solution of* (1.3) *satisfies stability estimates*

$$\|u\|_{C([0,T],L_2[-l,l])} \leq M(\delta, \sigma, \beta, l)[\|\varphi\|_{L_2[-l,l]} + \|\psi\|_{L_2[-l,l]} + \|f\|_{C([0,T],L_2[-l,l])}],$$

and the coercive stability estimates

$$\begin{aligned} &\|u_{tt}\|_{C_{0T}^\alpha([0,T],L_2[-l,l])} + \|u_{xx}\|_{C_{0T}^\alpha([0,T],L_2[-l,l])} \\ &\leq M(\delta, \sigma, \alpha, \beta, l)\left[\|\varphi_{xx}\|_{L_2[-l,l]} + \|\psi_{xx}\|_{L_2[-l,l]} + \|f\|_{C_{0T}^\alpha([0,T],L_2[-l,l])}\right]. \end{aligned}$$

Theorem 2.3. *Assume* $\delta - a|\beta| \geq 0,\ \varphi(x), \varphi_{xx}(x), \psi(x), \psi_{xx}(x) \in L_2[-l, l]$ *and*

$$\begin{gathered} (a(x)\psi_x(x))_x + \alpha(a(-x)\psi_x(-x))_x - \sigma\psi(x) + f(T, x) = 0, \\ (a(x)\varphi_x(x))_x + \alpha(a(-x)\varphi_x(-x))_x - \sigma\varphi(x) + f(0, x) = 0 \end{gathered}$$

and $f(t, x) \in C^\alpha([0, T], L_2[-l, l])$. *Then the solution of* (1.3) *satisfies coercive stability estimate*

$$\|u_{tt}\|_{C^\alpha([0,T],L_2[-l,l])} + \|u_{xx}\|_{C^\alpha([0,T],L_2[-l,l])} \leq M(\delta, \sigma, \alpha, \beta, l)\|f\|_{C^\alpha([0,T],L_2[-l,l])}.$$

The proofs of Theorem 2.2 and 2.3 are based on the following abstract Theorem on stability of problem (2.3) in $C([0, T], E)$ space and coercive stability in $C^\alpha([0, T], E)$ and $C_{0T}^\alpha([0, T], E)$ spaces and on self-adjointness and positive definite of the unbounded operator $A = A^x$ defined by formula (2.2) in $L_2[-l, l]$ space.

Theorem 2.4 ([10]). *Let A be positive operator in a Banach space E and $f \in C_{0T}^{\alpha}([0,T],E)$ $(0<\alpha<1)$. Then, for the solution of the boundary value problem* (2.3)*, stability and coercive stability inequalities*

$$\begin{aligned}
&\|u\|_{C([0,T],E)} \le M[\|\varphi\|_E + \|\psi\|_E + \|f\|_{C([0,T],E)}],\\
&\|u''\|_{C_{0T}^{\alpha}([0,T],E)} + \|Au\|_{C_{0T}^{\alpha}([0,T],E)}\\
&\le M\big[\|A\varphi\|_E + \|A\psi\|_E + \frac{1}{\alpha(1-\alpha)}\|f\|_{C_{0T}^{\alpha}([0,T],E)}\big]
\end{aligned}$$

hold. Moreover, assume that $A\varphi - f(0) = 0$, $A\psi - f(T) = 0$ and $f \in C^{\alpha}([0,T],E)$ $(0<\alpha<1)$. Then, for the solution of the boundary value problem (2.3)*, the coercive stability inequality*

$$\|u''\|_{C^{\alpha}([0,T],E)} + \|Au\|_{C^{\alpha}([0,T],E)} \le \frac{M}{\alpha(1-\alpha)}\|f\|_{C^{\alpha}([0,T],E)}$$

holds.

In the next Section, the self-adjointness and positive definiteness of the operator $A = A^x$ defined by formula (2.2) in $L_2[-l,l]$ space will be studied.

3. Self-adjointness and positive definiteness

Theorem 3.1. *Assume that $\delta - a|\beta| \ge 0$. Then, the operator $A = A^x$ defined by formula* (2.2) *is a self-adjoint and positive definite operator in $L_2[-l,l]$ space with the spectral angle $\varphi(A,H) = 0$.*

Proof. We will prove the following identity

$$\langle A^x u, v\rangle = \langle u, A^x v\rangle, u, v \in D(A^x), \tag{3.1}$$

and estimate

$$\langle A^x u, u\rangle \ge \sigma\langle u, \quad u\rangle, u \in D(A^x). \tag{3.2}$$

Applying definition of the inner product and integrating by part, we obtain

$$\begin{aligned}
\langle A^x u, v\rangle &= -\int_{-l}^{l}(a(x)u_x(x))_x v(x)dx - \beta\int_{-l}^{l}(a(-x)u_x(-x))_x v(x)dx\\
&\quad + \sigma\int_{-l}^{l}u(x)v(x)dx\\
&= -a(l)u_x(l)v(l) + a(-l)u_x(-l)v(-l) + \int_{-l}^{l}a(x)u_x(x)v_x(x)dx\\
&\quad + \beta[-a(-l)u_x(-l)v(-l) + a(l)u_x(l)v(l)]\\
&\quad + \beta\int_{-l}^{l}a(-x)u_x(-x)v_x(x)dx + \sigma\int_{-l}^{l}u(x)v(x)dx.
\end{aligned}$$

From conditions $a(x) = a(-x)$ and $u, v \in D(A^x)$, it follows that

$$-a(l)u_x(l)v(l) + a(-l)u_x(-l)v(-l) = 0.$$

Then,

$$\begin{aligned}
\langle A^x u, v\rangle &= \int_{-l}^{l}a(x)u_x(x)v_x(x)dx + \beta\int_{-l}^{l}a(x)u_x(x)v_x(-x)dx\\
&\quad + \sigma\int_{-l}^{l}u(x)v(x)dx.
\end{aligned} \tag{3.3}$$

In a similar manner, one establishes formula

$$\langle u, A^x v\rangle = \int_{-l}^{l} a(x)u_x(x)v_x(x)dx + \beta\int_{-l}^{l} u_x(x)a(-x)v_x(-x)dx + \sigma\int_{-l}^{l} u(x)v(x)dx.$$

Therefore, from these formulas and condition $a(x) = a(-x)$ it follows identity (3.1). Now, we will prove the estimate (3.2). Applying the identity (3.3), we obtain

$$\begin{aligned}
&\langle A^x u, u\rangle \\
&= \int_{-l}^{l} a(x)u_x(x)u_x(x)dx + \beta\int_{-l}^{l} u_x(x)a(-x)u_x(-x)dx + \sigma\int_{-l}^{l} u(x)u(x)dx \\
&\geq \sigma\langle u, u\rangle + \delta\int_{-l}^{l} u_x(x)u_x(x)dx + \beta\delta\int_{-l}^{l} a(-x)u_x(x)u_x(-x)dx.
\end{aligned}$$

Using the Cauchy inequality, we obtain

$$\begin{aligned}
\int_{-l}^{l} a(-x)u_x(x)u_x(-x)dx &\leq a\Big(\int_{-l}^{l} |u_x(x)|^2dx\Big)^{1/2}\Big(\int_{-l}^{l} |u_x(-x)|^2dx\Big)^{1/2} \\
&= a\langle u_x, u_x\rangle.
\end{aligned}$$

Since $\beta \geq -|\beta|$, we have

$$\beta\int_{-l}^{l} a(-x)u_x(x)u_x(-x)dx \geq -|\beta|a\langle u_x, u_x\rangle.$$

Then

$$\langle A^x u, u\rangle \geq \sigma\langle u, u\rangle + (\delta - |\beta|a)\langle u_x, u_x\rangle \geq \sigma\langle u, u\rangle.$$

The proof is complete. □

4. Conclusion

In the present study, mixed problem (1.3) for an elliptic equation with the involution is investigated. The stability estimates in $C([0,T], L_2[-l,l])$ norm and coercive stability estimates in $C^\alpha([0,T], L_2[-l,l])$ and $C^\alpha_{0T}([0,T], L_2[-l,l])$ norms for the solution of this problem are established.

Moreover, applying results of paper [5] and the present paper, the nonlocal problem for an elliptic equation with the involution

$$\begin{gathered}
-\frac{\partial^2 u(t,x)}{\partial t^2} = (a(x)u_x(t,x))_x + \beta(a(-x)u_x(t,-x))_x - \sigma u(t,x) + f(t,x), \\
-l < x < l,\ 0 < t < T, \\
u(t,-l) = u(t,l), \quad u_x(t,-l) = u_x(t,l), \quad 0 \leq t \leq T, \\
u(0,x) = u(T,x) + \varphi(x), \quad u_t(0,x) = u_t(T,x) + \psi(x), \quad -l \leq x \leq l, \\
\varphi(-l) = \varphi(l), \quad \psi(-l) = \psi(l), \quad \varphi'(-l) = \varphi'(l), \psi'(-l) = \psi'(l)
\end{gathered} \tag{4.1}$$

can be studied. Here, $u(t,x)$ is an unknown function, $\varphi(x), \psi(x), a(x)$, and $f(t,x)$ are sufficiently smooth functions, $a \geq a(x) = a(-x) \geq \delta > 0$ and $\sigma > 0$ is a sufficiently large number. The stability estimates in $C([0,T], L_2[-l,l])$ norm and coercive stability estimates in $C^\alpha([0,T], L_2[-l,l])$ and $C^\alpha_{0T}([0,T], L_2[-l,l])$ norms for the solution of problem (4.1) can be established. Finally, applying the result of the monograph [10], the high order of accuracy two-step difference schemes for the numerical solution of mixed problems (1.3) and (4.1) can be presented. Of course, the stability estimates for the solution of these difference schemes have been established without any assumptions about the grid steps.

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References

[1] Agarwal, R. P.; Bohner, M., Li, W. T.; *Nonoscillation and Oscillation Theory for Functional Differential Equations.* Monographs and Textbooks in Pure and Applied Mathematics. Vol. 267. New York: Marcel Dekker, 2004.

[2] Agmon, S.; *Lectures on Elliptic Boundary Value Problems.* New Jersey, D. Van Nostrand, Princeton, 1965.

[3] Agmon, S.; Douglis, S. A.; Nirenberg, L.; *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II.* Communications on Pure and Applied Mathematics. Vol. 17, 35-92, 1964.

[4] Ashyralyev, A; *Well-posedeness of the difference schemes for elliptic equations in $C_\tau^{\beta,\gamma}(E)$ spaces.* Applied Mathematics Letter., Vol. 22, 390-395, 2009.

[5] Ashyralyev, A.; *On well-posedness of the nonlocal boundary value problem for elliptic equations.* Numerical Functional Analysis and Optimization. Vol. 24. No. 1-2, 1-15, 2003.

[6] Ashyralyev, A.; Ashyralyyev, C.; *On the problem of determining the parameter of an elliptic equation in a Banach space.* Nonlinear Analysis-Modelling and Control. Vol. 19. No. 3. SI, 350-366, 2014.

[7] Ashyralyev, A.; Claudio, C.; Piskarev, S.; *On well-posedness of difference schemes for abstract elliptic problems in $L_p(0,1,E)$ spaces.* Numerical Functional Analysis and Optimization. Vol. 29. No. 1-2, 43-65, 2008.

[8] Ashyralyev, A.; Sarsenbi, A. M.; *Well-posedness of a parabolic equation with nonlocal boundary condition.* Boundary value problems. Article Number 38. DOI: 10.1186/s13661-015-0297-5, 2015.

[9] Ashyralyev, A.; Sobolevskii, P. E.; *Well-Posedness of Parabolic Difference Equations.* Operator Theory: Advances and Applications. Basel, Boston, Berlin: BirkhauserVerlag, 1994.

[10] Ashyralyev, A.; Sobolevskii, P. E.; *New Difference Schemes for Partial Differential Equations.* Operator Theory: Advances and Applications. Basel, Boston, Berlin: BirkhauserVerlag, 2004.

[11] Ashyralyyev, C.; *High order approximation of the inverse elliptic problem with Dirichlet-Neumann conditions.* Filomat. Vol. 28. No. 5, 947-962. DOI: 10.2298/FIL1405947A, 2014.

[12] Burlutskaya, M. Sh.; *Mixed problem for a first-order partial differential equation with involution and periodic boundary conditions.* Computational Mathematics and Mathematical Physics. Vol. 54. No. 1, 1-10, 2014.

[13] Cabada, A.; Tojo, F.; *Existence results for a linear equation with reflection, non-constant coefficient and periodic boundary conditions.* Journal of Mathematical Analysis and Applications. Vol. 412. No. 1, 529-546, 2014.

[14] Golubeva, N. D.; Pulkina, L. S.; *A nonlocal problem with integral conditions.* Mathematical Notes. Vol. 59. No. 3-4, 326-328. DOI: 10.1007/BF02308548, 1996.

[15] Gorbachuk, V. L.; Gorbachuk, M. L.; *Boundary Value Problems for Differential - Operator Equations.* Kiev: Naukova Dumka, 1984 (in Russian).

[16] Grisvard, P.; *Elliptic Problems in Nonsmooth Domains.* London, Pitman Advanced Publishing Program, 1985.

[17] Kopzhassarova, A. A.; Lukashov, A. L.; Sarsenbi, A. M.; *Spectral properties of non-self-adjoint perturbations for a spectral problem with involution.* Abstract and Applied Analysis. Vol. 2012. Article ID 590781, 2012.

[18] Kopzhassarova, A. A.; Sarsenbi, A. M.; *Basis properties of eigenfunctions of second- order differential operators with involution.* Abstract and Applied Analysis. Vol. 2012. Article ID 576843, 2012.

[19] Krein, S. G.; *Linear Differential Equations in Banach space.* Moscow: Nauka, 1996 (in Russian).

[20] Ladyzhenskaya, O. A.; Ural'tseva, N. N.; *Linear and Quasilinear Equations of Elliptic Type.* Moscow: Nauka, 1973.
[21] Przeworska-Rolewicz, D.; *Equations with Transformed Argument. An Algebraic Approach.* Amsterdam, Warszawa: Elsevier Scientific Publishing Comp. PWN - Polish Scientific Publishers, 1973.
[22] Sadybekov, M. A.; Sarsenbi, A. M.; *Mixed problem for a differential equation with involution under boundary conditions of general form.* AIP Conference Proceedings. Ed. Ashyralyev, and A; Lukashov, A. Vol. 1470, 225-227, 2012.
[23] Sadybekov, M. A.; Sarsenbi, A. M.; *Criterion for the basis property of the eigenfunction system of a multiple differentiation operator with an involution.* Differential Equations. Vol. 48. Issue 8, 1112-1118, 2012.
[24] Sarsenbi, A. M.; *Unconditional bases related to a nonclassical second-order differential operator.* Differential Equations. Vol. 46. Issue 4, 509-514, 2010.
[25] Sarsenbi, A. M.; Tengaeva, A. A.; *On the basis properties of root functions of two generalized eigenvalue problems.* Differential Equations. Vol. 48. Issue 2, 306-308, 2012.
[26] Skubachevskii, A. L.; *Elliptic Functional Differential Equations and Applications.* Operator Theory: Advances and Applications. Basel, Boston, Berlin: Birkhauser Verlag, 1997.
[27] Sobolevskii, P. E.; *On elliptic equations in a Banach space.* Differentsial'nye Uravneniya. Vol. 4. No. 7, 1346-1348, 1969 (in Russian).
[28] Sobolevskii, P. E.; *The theory of semigroups and stability of difference schemes in operator theory in function spaces.* (Proc. Sch, Novosibirsk, 1975), 304-307. Novosibirsk: Nauka (Sib.Otdel), 1977 (in Russian).
[29] Vishsik, M. L.; Myshkis, A. D.; Oleinik, O. A.; *Partial Differential Equations.* Vol. 1, 563-599. Moscow: Fizmatgiz, 1959.
[30] Wiener, J.; *Generalized Solutions of Functional-Differential Equations.* New Jersey, River Edge: World Scientific Pub., 1993.

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