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EXPONENTIAL P-STABILITY OF STOCHASTIC ∇-DYNAMIC EQUATIONS ON DISCONNECTED SETS

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Abstract. The aim of this article is to consider the existence of solutions, finiteness of moments, and exponential p-stability of stochastic ∇-dynamic equations on an arbitrary closed subset of $\mathbb{R}$, via Lyapunov functions. This work can be considered as a unification and generalization of works dealing with random difference and stochastic differential equations.

1. Introduction

The direct method has become the most widely used tool for studying the exponential stability of stochastic equations. For differential equations, we mention the very interesting book by Khas'minskii [12] in which author uses the Lyapunov functions to study stability. Foss and Konstantopoulos [8] presented an overview of stochastic stability methods, mostly motivated by stochastic network applications. Socha [24] considered the exponential p-stability of singularly perturbed stochastic systems for the "slow" and "fast" components of the full-order system. Govindan [9] proved the existence and uniqueness of a mild solution under two sets of hypotheses and considered the exponential second moment stability of the solution process for stochastic semilinear functional differential equations in a Hilbert space. We also refer to [15, 16] in which authors considered stochastic asymptotic stability and boundedness for stochastic differential equations with respect to semimartingale via multiple Lyapunov functions. The long-time behavior of densities of the solutions is studied in [20] by using Khas'minskii function. For random difference systems, we can refer the reader to [18, 22, 23], for stability of nonlinear systems.

Recently, a method for the unified analysis of equations of motion in continuous and discrete cases within the framework of the theory of time scales has drawn a lot attention. For deterministic cases, in [4], author used the Lyapunov function of quadratic form to study the stability of linear dynamic equations. Hoffacker and Tisdell examined the stability and instability of the equilibrium point of nonlinear dynamic equations [13]. Martynyuk presented systematically the stability theory of dynamic equations in [17].

While the stability of deterministic dynamic equations on time scales has been investigated for a long time, as far as we know, there is not much in mathematical

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literature for the stochastic case, and no work dealing with the stability of stochastic dynamic equations. Here, we mention some of the first attempts on this direction. In [11], the authors developed the theory of Brownian motion. Sanyal in his Ph. D. Dissertation [21] tries to define *stochastic integral and stochastic dynamic equations* on time scale with the positive graininess. Lungan and Lupulescu in [14] consider random dynamical systems with random Δ-integral. Gravagne and Robert deal with the bilateral Laplace transforms in [10]. The Doob-Meyer decomposition theorem and definition of stochastic ∇-integral with respect to square integrable martingale on any arbitrary time scale also Itô's formula are studied in [6, 7]. Recently, Bohner et al [3] investigate stochastic dynamic equations on time scale by considering an integral with respect to the restriction of a standard Wiener process on time scale. However, this way can not be applied to define the stochastic integral in general case since when one deals with a martingale defined on time scale and we do not know whenever it can be extended to a regular martingale on $\mathbb{R}$.

The aim of this article is to use Lyapunov functions to consider the existence, finiteness of moments, and long term behavior of solutions for ∇-stochastic dynamic equations on arbitrary closed subset of $\mathbb{R}$. We study

$$d^\nabla X(t) = f(t, X(t_-))d^\nabla t + g(t, X(t_-))d^\nabla M(t)$$
$$X(a) = x_a \in \mathbb{R}^d, \quad t \in \mathbb{T}_a,$$

where $(M_t)_{t\in\mathbb{T}_a}$ is a $\mathbb{R}$-valued square integrable martingale and $f : \mathbb{T}_a \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{T}_a \times \mathbb{R} \to \mathbb{R}$ are two Borel functions. We emphasis that martingale M is defined only on $\mathbb{T}_a$. This work can be considered as a unification and generalization of works dealing with these areas of stochastic difference and differential equations.

In working on stochastic multi-dimensional dynamic equations with respect to discontinuous martingale on time scales, it rises many difficulties, especially the complicated calculations and they require some improvements. Besides, some estimates of stochastic calculus for continuous time are not automatically valid on arbitrary time scale and we need to change them into a suitable form to obtain similar results.

The organization of this paper is as follows. We introduce some basic notion and definitions for time scale and for square integrable martingales in Section 2. Section 3 deals with the existence and the finiteness of moments of solutions for stochastic dynamic equations with respect to a square integrable martingale in case the coefficients satisfy locally Lipschitz conditions. Section 4 is concerned with necessary and sufficient conditions for the exponential p-stability of stochastic dynamic equations.

2. Preliminaries

Let $\mathbb{T}$ be a closed subset of $\mathbb{R}$, enclosed with the topology inherited from the standard topology on $\mathbb{R}$. Let $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \mu(t) = \sigma(t) - t$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \nu(t) = t - \rho(t)$ (supplemented by $\sup\emptyset = \inf\mathbb{T}, \inf\emptyset = \sup\mathbb{T}$). A point $t \in \mathbb{T}$ is said to be *right-dense* if $\sigma(t) = t$, *right-scattered* if $\sigma(t) > t$, *left-dense* if $\rho(t) = t$, *left-scattered* if $\rho(t) < t$ and *isolated* if t is simultaneously right-scattered and left-scattered. The set ${}_k\mathbb{T}$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a right-scattered minimum; otherwise it is $\mathbb{T}$ without this right-scattered minimum. Similarly, $\mathbb{T}^k$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum; otherwise it is $\mathbb{T}$ without this left-scattered maximum. A function f defined on $\mathbb{T}$ is *regulated* if there exist the left-sided limit at every left-dense point and right-sided limit at every right-dense point. A regulated function is called *ld-continuous* if it is continuous at every left-dense point. Similarly, one has the notion of rd-continuous. For every $a, b \in \mathbb{T}$, by $[a, b]$, we mean the set $\{t \in \mathbb{T} : a \le t \le b\}$. Denote $\mathbb{T}_a = \{t \in \mathbb{T} : t \ge a\}$ and by $\mathcal{R}$ (*resp.* $\mathcal{R}^+$) the set of all rd-continuous and regressive (resp. positive regressive) functions. For any function f defined on $\mathbb{T}$, we write f^ρ for the function $f \circ \rho$; i.e., $f_t^\rho = f(\rho(t))$ for all $t \in_k \mathbb{T}$ and $\lim_{\sigma(s)\uparrow t} f(s)$ by $f(t_-)$ or f_{t_-} if this limit exists. It is easy to see that if t is left-scattered then $f_{t_-} = f_t^\rho$. Let

$$\mathbb{I} = \{t : t \text{ is left-scattered}\}.$$

Clearly, the set $\mathbb{I}$ of all left-scattered points of $\mathbb{T}$ is at most countable.

Throughout of this paper, we suppose that the time scale $\mathbb{T}$ has bounded graininess, that is $\nu_* = \sup\{\nu(t) : t \in_k \mathbb{T}\} < \infty$.

Let A be an increasing right continuous function defined on $\mathbb{T}$. We denote by μ_∇^A the Lebesgue ∇-measure associated with A. For any μ_∇^A-measurable function $f : \mathbb{T} \to \mathbb{R}$ we write $\int_a^t f_\tau \nabla A_\tau$ for the integral of f with respect to the measures μ_∇^A on $(a, t]$. It is seen that the function $t \to \int_a^t f_\tau \nabla A_\tau$ is cadlag. It is continuous if A is continuous. In case $A(t) \equiv t$ we write simply $\int_a^t f_\tau \nabla\tau$ for $\int_a^t f_\tau \nabla A_\tau$. For details, we can refer to [5].

In general, there is no relation between the Δ-integral and ∇-integral. However, in case the integrand f is regulated one has

$$\int_a^b f(\tau_-)\nabla\tau = \int_a^b f(\tau)\Delta\tau \quad \forall a, b \in \mathbb{T}^k,$$

Indeed, by [5, Theorem 6.5],

$$\begin{aligned}\int_a^b f(\tau)\Delta\tau &= \int_{[a;b)} f(\tau)d\tau + \sum_{a\le s<b} f(s)\mu(s)\\ &= \int_{(a,b]} f(\tau_-)d\tau + \sum_{a<s\le b} f(s_-)\nu(s) = \int_a^b f(\tau_-)\nabla\tau.\end{aligned}$$

Therefore, if $p \in \mathcal{R}$ then the exponential function $e_p(t, t_0)$, defined by [2, Definition 2.30, pp. 59], is solution of the initial value problem

$$y^\nabla(t) = p(t_-)y(t_-), \;\; y(t_0) = 1, \;\; t > t_0. \tag{2.1}$$

Also if $p \in \mathcal{R}$, $e_{\ominus p}(t, t_0)$ is the solution of the equation

$$y^\nabla(t) = -p(t_-)y(t), \quad y(t_0) = 1, \quad t > t_0,$$

where $\ominus p(t) = \frac{-p(t)}{1+\mu(t)p(t)}$. Later, we need the following lemma.

Lemma 2.1 ([2, 7])**.** *Let $u(t)$ be a regulated function and $u_a, \alpha \in \mathbb{R}_+$. Then, the inequality*

$$u(t) \le u_a + \alpha \int_a^t u(\tau_-)\nabla\tau \quad \forall t \in \mathbb{T}_a$$

implies

$$u(t) \le u_a e_\alpha(t, a) \quad \forall t \in \mathbb{T}_a.$$

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{T}_a}, \mathbb{P})$ be a probability space with filtration $\{\mathcal{F}_t\}_{t\in\mathbb{T}_a}$ satisfying the usual conditions (i.e., $\{\mathcal{F}_t\}_{t\in\mathbb{T}_a}$ is increasing and $\cap\{\mathcal{F}_{\rho(s)} : s \in \mathbb{T}, s > t\} = \mathcal{F}_t$ for all $t \in \mathbb{T}_a$ while $\mathcal{F}_a$ contains all P-null sets). Denote by $\mathcal{M}_2$ the set of the square integrable $\mathcal{F}_t$-martingales and by $\mathcal{M}_2^r$ the subspace of the space $\mathcal{M}_2$ consisting of martingales with continuous characteristics. For any $M \in \mathcal{M}_2$, set

$$\widehat{M}_t = M_t - \sum_{s\in(a,t]} (M_s - M_{\rho(s)}).$$

It is clear that $\widehat{M}_t$ is an $\mathcal{F}_t$-martingale and $\widehat{M}_t = \widehat{M}_{\rho(t)}$ for any $t \in \mathbb{T}$. Further,

$$\langle \widehat{M} \rangle_t = \langle M \rangle_t - \sum_{s\in(a,t]} (\langle M \rangle_s - \langle M \rangle_{\rho(s)}). \tag{2.2}$$

Therefore, $M \in \mathcal{M}_2^r$ if and only if $\widehat{M} \in \mathcal{M}_2^r$. In this case, $\widehat{M}$ can be extended to a regular martingale defined on $\mathbb{R}$.

Denote by $\mathfrak{B}$ the class of Borel sets in $\mathbb{R}$ whose closure does not contain the point 0. Let $\delta(t, A)$ be the number of jumps of the M on the $(a, t]$ whose values fall into the set $A \in \mathfrak{B}$. Since the sample functions of the martingale M are cadlag, the process $\delta(t, A)$ is defined with probability 1 for all $t \in \mathbb{T}_a, A \in \mathfrak{B}$. We extend its definition over the whole Ω by setting $\delta(t, A) \equiv 0$ if the sample $t \to M_t(\omega)$ is not cadlag. Clearly the process $\delta(t, A)$ is $\mathcal{F}_t$-adapted and its sample functions are nonnegative, monotonically nondecreasing, continuous from the right and take on integer values. We also define $\widehat{\delta}(t, A)$ for $\widehat{M}_t$ by a similar way. Let $\widetilde{\delta}(t, A) = \sharp\{s \in (a, t] : M_s - M_{\rho(s)} \in A\}$. It is evident that

$$\delta(t, A) = \widehat{\delta}(t, A) + \widetilde{\delta}(t, A). \tag{2.3}$$

Further, for fixed t, $\delta(t, \cdot), \widehat{\delta}(t, \cdot)$ and $\widetilde{\delta}(t, \cdot)$ are measures.

The functions $\delta(t, A), \widehat{\delta}(t, A)$ and $\widetilde{\delta}(t, A), t \in \mathbb{T}_a$ are $\mathcal{F}_t$-regular submartingales for fixed A. By Doob-Meyer decomposition, each process has a unique representation of the form

$$\delta(t, A) = \zeta(t, A) + \pi(t, A), \quad \widehat{\delta}(t, A) = \widehat{\zeta}(t, A) + \widehat{\pi}(t, A),$$
$$\widetilde{\delta}(t, A) = \widetilde{\zeta}(t, A) + \widetilde{\pi}(t, A),$$

where $\pi(t, A), \widehat{\pi}(t, A)$ and $\widetilde{\pi}(t, A)$ are natural increasing integrable processes and $\zeta(t, A), \widehat{\zeta}(t, A)$, $\widetilde{\zeta}(t, A)$ are martingales. We find a version of these processes such that they are measures when t is fixed. By denoting

$$\widehat{M}^c_t = \widehat{M}_t - \widehat{M}^d_t,$$

where

$$\widehat{M}^d_t = \int_a^t \int_{\mathbb{R}} u\widehat{\zeta}(\nabla\tau, du),$$

we obtain

$$\langle \widehat{M} \rangle_t = \langle \widehat{M}^c \rangle_t + \langle \widehat{M}^d \rangle_t, \quad \langle \widehat{M}^d \rangle_t = \int_a^t \int_{\mathbb{R}} u^2 \widehat{\pi}(\nabla\tau, du). \tag{2.4}$$

Throughout this article, we suppose that $\langle M \rangle_t$ is absolutely continuous with respect to Lebesgue measure μ_∇, i.e., there exists $\mathcal{F}_t$-adapted progressively measurable process K_t such that

$$\langle M\rangle_t = \int_a^t K_\tau \nabla\tau. \tag{2.5}$$

Further, for any $T\in\mathbb{T}_a$,

$$\mathbb{P}\{\sup_{a\le t\le T}|K_t|\le N\}=1, \tag{2.6}$$

where N is a constant (possibly depending on T).

The relations (2.2), (2.4) imply that $\langle\widehat{M}^c\rangle_t$ and $\langle\widehat{M}^d\rangle_t$ are is absolutely continuous with respect to μ_∇ on $\mathbb{T}$. Thus, there exists $\mathcal{F}_t$-adapted, progressively measurable bounded process $\widehat{K}_t^c$ and $\widehat{K}_t^d$ satisfying

$$\langle\widehat{M}^c\rangle_t=\int_a^t \widehat{K}_\tau^c\nabla\tau,\quad \langle\widehat{M}^d\rangle_t=\int_a^t \widehat{K}_\tau^d\nabla\tau,$$

and the following relation holds

$$\mathbb{P}\{\sup_{a\le t\le T}\widehat{K}_t^c+\widehat{K}_t^d\le N\}=1.$$

Moreover, it is easy to show that $\widehat{\pi}(t,A)$ is absolutely continuous with respect to μ_∇ on $\mathbb{T}$, that is, it can be expressed as

$$\widehat{\pi}(t,A)=\int_a^t\widehat{\Upsilon}(\tau,A)\nabla\tau, \tag{2.7}$$

with an $\mathcal{F}_t$-adapted, progressively measurable process $\widehat{\Upsilon}(t,A)$. Since $\mathfrak{B}$ is generated by a countable family of Borel sets, we can find a version of $\widehat{\Upsilon}(t,A)$ such that the map $t\to\widehat{\Upsilon}(t,A)$ is measurable and for t fixed, $\widehat{\Upsilon}(t,\cdot)$ is a measure. Hence, from (2.4) we see that

$$\langle\widehat{M}^d\rangle_t=\int_a^t\int_{\mathbb{R}}u^2\widehat{\Upsilon}(\tau,du))\nabla\tau.$$

This means that

$$\widehat{K}_t^d=\int_{\mathbb{R}}u^2\widehat{\Upsilon}(t,du)).$$

For the process $\widetilde{\pi}(t,A)$ we can write

$$\widetilde{\pi}(t,A)=\sum_{s\in(a,t]}\mathbb{E}[1_A(M_s-M_{\rho(s)})\big|\mathcal{F}_{\rho(s)}].$$

Putting

$$\widetilde{\Upsilon}(t,A)=\begin{cases}\frac{\mathbb{E}[1_A(M_t-M_{\rho(t)})|\mathcal{F}_{\rho(t)}]}{\nu(t)} & \text{if } \nu(t)>0,\\ 0 & \text{if } \nu(t)=0\end{cases}$$

yields

$$\widetilde{\pi}(t,A)=\int_a^t\widetilde{\Upsilon}(\tau,A)\nabla\tau. \tag{2.8}$$

Further, by the definition if $\nu(t)>0$ we have

$$\int_{\mathbb{R}}u\widetilde{\Upsilon}(t,du)=\frac{\mathbb{E}[M_t-M_{\rho(t)}\big|\mathcal{F}_{\rho(t)}]}{\nu(t)}=0, \tag{2.9}$$

and

$$\int_{\mathbb{R}}u^2\widetilde{\Upsilon}(t,du)=\frac{\mathbb{E}[(M_t-M_{\rho(t)})^2\big|\mathcal{F}_{\rho(t)}]}{\nu(t)}=\frac{\langle M\rangle_t-\langle M\rangle_{\rho(t)}}{\nu(t)}.$$

Let $\Upsilon(t,A)=\widehat{\Upsilon}(t,A)+\widetilde{\Upsilon}(t,A)$. From (2.3) we see that

$$\pi(t,A)=\int_a^t \Upsilon(\tau,A)\nabla\tau.$$

Denote by $\mathcal{L}_1^{\rm loc}(\mathbb{T}_a,\mathbb{R})$ the family of real valued, $\mathcal{F}_t$-progressively measurable processes $f(t)$ with $\int_a^T|f(\tau)|\nabla\tau<+\infty$ a.s. for every $T>a$ and by $\mathcal{L}_2(\mathbb{T}_a;M)$ the space of all real valued, $\mathcal{F}_t$-predictable processes $\phi(t)$ satisfying $\mathbb{E}\int_a^T\phi^2(\tau)\nabla\langle M\rangle_\tau<\infty$, for any $T>a$. Consider a d-tuple of semimartingales $X(t)=(X_1(t),\dots,X_d(t))$ defined by

$$X_i(t)=X_i(a)+\int_a^t f_i(\tau)\,\nabla\tau+\int_a^t g_i(\tau)\nabla M_\tau,$$

where $f_i\in\mathcal{L}_1^{\rm loc}(\mathbb{T}_a,\mathbb{R})$ and $g_i\in\mathcal{L}_2(\mathbb{T}_a;M)$ for $i=\overline{1,d}$. For any twice differentiable function V, put

$$\begin{aligned}
&\mathcal{A}V(t,x)\\
&=\sum_{i=1}^d\frac{\partial V(t,x)}{\partial x_i}(1-1_{\mathbb{I}}(t))f_i(t)+\Big(V(t,x+f(t)\nu(t))-V(t,x)\Big)\Phi(t)\\
&\quad+\frac12\sum_{i,j}\frac{\partial^2V(t,x)}{\partial x_ix_j}g_i(t)g_j(t)\widehat{K}_t^c-\sum_{i=1}^d\frac{\partial V(t,x)}{\partial x_i}g_i(t)\int_{\mathbb{R}}u\widehat{\Upsilon}(t,du)\\
&\quad+\int_{\mathbb{R}}(V\big(t,x+f(t)\nu(t)+g(t)u\big)-V(t,x+f(t)\nu(t)))\Upsilon(t,du),
\end{aligned}\tag{2.10}$$

with $f=(f_1,f_2,\dots,f_d)$; $g=(g_1,g_2,\dots,g_d)$ and

$$\Phi(t)=\begin{cases}0 & \text{if } t \text{ is left-dense}\\ 1/\nu(t) & \text{if } t \text{ is left-scattered.}\end{cases}$$

Let $C^{1,2}(\mathbb{T}_a\times\mathbb{R}^d;\mathbb{R})$ be the set of all functions $V(t,x)$ defined on $\mathbb{T}_a\times\mathbb{R}^d$, having continuous ∇-derivative in t and continuous second derivative in x. Using the Itô's formula in [7] we see that for any $V\in C^{1,2}(\mathbb{T}_a\times\mathbb{R}^d;\mathbb{R}_+)$

$$V(t,X(t))-V(a,X(a))-\int_a^t\Big(V^{\nabla_\tau}(\tau,X(\tau_-))+\mathcal{A}V(\tau,X(\tau_-))\Big)\nabla\tau\tag{2.11}$$

is a locally integrable martingale, where V^{∇_t} is partial ∇-derivative of $V(t,x)$ in t.

3. Existence of solutions and finiteness of moments for stochastic dynamic equations

Consider a ∇-stochastic dynamic equations on $\mathbb{T}$ of the form

$$\begin{gathered}
d^\nabla X(t)=f(t,X(t_-))d^\nabla t+g(t,X(t_-))d^\nabla M(t)\\
X(a)=x_a\in\mathbb{R}^d,\quad t\in\mathbb{T}_a,
\end{gathered}\tag{3.1}$$

where $f:\mathbb{T}_a\times\mathbb{R}^d\to\mathbb{R}^d$ and $g:\mathbb{T}_a\times\mathbb{R}^d\to\mathbb{R}^d$ are two Borel functions. Under the global Lipschitz and linear growth rate conditions of the coefficients f,g, there exists uniquely a solution for Cauchy problem (3.1) (see: [7]). We now consider the case where the coefficients are locally Lipschitz.

Theorem 3.1. *Suppose that for any $k > 0$ and $T > a$, there exists a constant $L_{T,k} > 0$ such that*

$$\|f(t,x) - f(t,y)\|^2 \vee \|g(t,x) - g(t,y)\|^2 \le L_{T,k}\|x-y\|^2, \tag{3.2}$$

for all $x, y \in \mathbb{R}^d$ with $\|x\| \vee \|y\| \le k$ and $t \in [a,T]$. Further, there are positive constants $c = c(T); b = b(T)$ and a nonnegative function $V \in C^{1,2}([a,T] \times \mathbb{R}^d; \mathbb{R}_+)$ satisfying

$$V^{\nabla_t}(t,x) + \mathcal{A}V(t,x) \le cV(t_-,x) + b \quad \forall (t,x) \in [a,T] \times \mathbb{R}^d, \tag{3.3}$$

and $\lim_{x\to\infty} \inf_{t\in[a,T]} V(t,x) = \infty$. Then, (3.1) has a unique solution $X_{a,x_a}(t)$ defined on $\mathbb{T}_a$. In addition, if there exists a positive constant $c_1 = c_1(T)$ such that

$$c_1\|x\|^p \le V(t,x) \quad \forall (t,x) \in [a,T] \times \mathbb{R}^d, \tag{3.4}$$

then

$$\mathbb{E}\|X_{a,x_a}(t)\|^p \le \frac{1}{c_1}(V(a,x_a) + \frac{b}{c})e_c(t,a) \quad \forall t \in [a,T].$$

Proof. For each $k \ge k_0 = [\|x_a\|] + 1$, define the truncation function

$$f_k(t,x) = \begin{cases} f(t,x) & \text{if } \|x\| \le k \\ f(t, \frac{kx}{\|x\|}) & \text{if } \|x\| > k, \end{cases}$$

and $g_k(t,x)$ is defined by a similar way. The functions f_k and g_k satisfy the global Lipschitz condition and the linear growth rate condition. Hence, by [7, Theorem 3.2] there exists a unique solution $X_k(\cdot)$ to the equation

$$\begin{gathered} d^{\nabla}X(t) = f_k(t, X(t_-))d^{\nabla}t + g_k(t, X(t_-))d^{\nabla}M(t) \\ X(a) = x_a \in \mathbb{R}^d, \ \forall t \in [a,T]. \end{gathered} \tag{3.5}$$

Define the stopping time

$$\theta_k = \inf\{t \in [a,T] : |X_k(t)| \ge k\}, \quad \theta_{k_0} = a.$$

It is easy to see that θ_k is increasing and

$$X_k(t) = X_{k+1}(t) \quad \text{if } a \le t \le \theta_k. \tag{3.6}$$

Let $\theta_\infty = \lim_{k\to\infty} \theta_k$ and the process $X_{a,x_a}(t) = X(t)$, $a \le t \le \theta_\infty$ be given by

$$X(t) = X_k(t), \quad \theta_{k-1} \le t < \theta_k, \quad k \ge k_0.$$

Using (3.6) one gets $X(t\wedge\theta_k) = X_k(t\wedge\theta_k)$. It follows from (3.5) that

$$\begin{aligned} X(t\wedge\theta_k) &= x_a + \int_a^{t\wedge\theta_k} f_k(\tau, X(\tau_-))\nabla\tau + \int_a^{t\wedge\theta_k} g_k(\tau, X(\tau_-))\nabla M_\tau \\ &= x_a + \int_a^{t\wedge\theta_k} f(\tau, X(\tau_-))\nabla\tau + \int_a^{t\wedge\theta_k} g(\tau, X(\tau_-))\nabla M_\tau, \end{aligned}$$

for any $t \in [a,T]$ and $k \ge 1$. We show that $\lim_{k\to\infty}\theta_k = T$ a.s. Indeed, by (2.11) it yields

$$\begin{aligned} \mathbb{E}[V(\theta_k\wedge t, X(\theta_k\wedge t))] &= V(a,x_a) + \mathbb{E}\int_a^{t\wedge\theta_k} \Big(V^{\nabla_\tau}(\tau, X(\tau_-)) + \mathcal{A}V(\tau, X(\tau_-))\Big)\nabla\tau \\ &\le V(a,x_a) + \int_a^t (c\mathbb{E}V(\theta_k\wedge\tau_-, X(\theta_k\wedge\tau_-)) + b)\nabla\tau. \end{aligned}$$

Using Lemma 2.1 with the function $u(t) = \mathbb{E}[V(\theta_k \wedge t, X(\theta_k \wedge t))] + \frac{b}{c}$ gets

$$\mathbb{E}V(\theta_n \wedge t, X(\theta_n \wedge t)) \leq \big(V(a, x_a) + \frac{b}{c}\big)e_c(t, a).$$

On the other hand, on the set $\{\theta_\infty < T\}$ we have $\limsup_{t \to \theta_\infty} \|X(t)\| = \infty$. Therefore, the assumption $\lim_{x\to\infty} \inf_{t\in[a,T]} V(t, x) = \infty$ follows $\mathbb{P}\{\theta_\infty < T\} = 0$, i.e., the solution $X_{a,x_a}(t)$ is defined on $\mathbb{T}_a$.

The uniqueness follows immediately from the uniqueness of solutions of (3.5). When the condition (3.4) is satisfied we see that

$$c_1\mathbb{E}\|X_{a,x_a}(t \wedge \theta_n)\|^p \leq \mathbb{E}[V(t \wedge \theta_n, X_{a,x_a}(t \wedge \theta_n))] \leq (V(a, x_a) + \frac{b}{c})e_c(t, a).$$

Letting $n \to \infty$ yields

$$c_1\mathbb{E}\|X_{a,x_a}(t)\|^p \leq (V(a, x_a) + \frac{b}{c})e_c(t, a)$$

or

$$\mathbb{E}\|X_{a,x_a}(t)\|^p \leq \frac{1}{c_1}(V(a, x_a) + \frac{b}{c})e_c(t, a).$$

The proof is complete. □

Corollary 3.2. *Suppose that the conditions* (2.5)*;* (2.6) *and* (3.2) *hold and the linear growth condition*

$$\|f(t, x)\|^2 \vee \|g(t, x)\|^2 \leq G(1 + \|x\|^2) \quad \forall(t, x) \in [a, T] \times \mathbb{R}^d, \tag{3.7}$$

is satisfied. We suppose further that $\int_{\mathbb{R}} |u|\widehat{\Upsilon}(t, du) \leq m_1$ *a.s where* m_1 *is a constant. Then* (3.1) *has a unique solution* $X_{a,x_a}(t)$ *defined on* $\mathbb{T}_a$ *satisfying*

$$\mathbb{E}\|X_{a,x_a}(t)\|^2 \leq (1 + \|x_a\|^2)e_c(t, a),$$

where c *is a constant.*

Proof. From (2.4), (2.7), it follows that $\int_{\mathbb{R}} u^2\Upsilon(t, du) < N$ for all $t \in [a, T]$. Using the Lyapunov function $V(t, x) = 1 + \|x\|^2$ gets

$$\begin{aligned}
&\mathcal{A}V(t, x)\\
&= 2(1 - 1_{\mathbb{I}}(t))x^T f(t, x) + \|g(t, x)\|^2 \widehat{K}^c_t\\
&\quad + (\|x + f(t, x)\nu(t)\|^2 - \|x\|^2)\Phi(t) - 2x^T g(t, x) \int_{\mathbb{R}} u\widehat{\Upsilon}(t, du)\\
&\quad + \int_{\mathbb{R}} (\|x + f(t, x)\nu(t) + g(t, x)u\|^2 - \|x + f(t, x)\nu(t)\|^2)\Upsilon(t, du)\\
&= 2x^T f(t, x) + 2x^T g(t, x) \int_{\mathbb{R}} u\widetilde{\Upsilon}(t, du) + 2\nu(t)f(t, x)^T g(t, x) \int_{\mathbb{R}} u\widehat{\Upsilon}(t, du)\\
&\quad + \|g(t, x)\|^2 \widehat{K}^c_t + \|f(t, x)\|^2\nu(t) + \|g(t, x)\|^2 \int_{\mathbb{R}} u^2\Upsilon(t, du)\\
&\leq (1 + G(1 + 2N + 2m_1\nu_* + \nu_*))(1 + \|x\|^2) = cV(x),
\end{aligned}$$

where $c = 1 + G(1 + 2N + 2m_1\nu_* + \nu_*)$. Moreover, $\|x\|^2 \leq 1 + \|x\|^2 = V(x)$. Thus, (3.3) and (3.4) are satisfied. Using Theorem 3.1 we can complete the proof. □

We note that in the continuous case, if the linear growth rate condition (3.7) holds and the boundedness conditions (2.5), (2.6) of characteristic $\langle M\rangle_t$ are satisfied, then all moments of the solutions are finite. This property may no longer valid on time scale as it is shown in the following example.

Example 3.3. Consider two random variables ξ_1, ξ_2 valued in $\mathbb{Z}\setminus\{0\}$ with

$$\mathbb{P}\{\xi_1 = \pm i\} = \frac{k}{|i|^5}, \quad \mathbb{P}[\xi_2 = j \mid \xi_1 = i] = C_i |j|^{-(4+\frac{1}{|i|})}.$$

It is seen that $8/3 \geq \sum_{j\in\mathbb{Z}\setminus\{0\}} |j|^{-4} > C_i^{-1} > \sum_{j\in\mathbb{Z}\setminus\{0\}} |j|^{-5} > 1$ and $\mathbb{E}[\xi_2 \mid \xi_1] = 0$. Therefore, the sequence $M_1 = \xi_1$ and $M_2 = \xi_1 + \xi_2$ is a martingale. Further,

$$\begin{aligned}
\mathbb{E}[\xi_2^2 \mid \xi_1 = i] &= \sum_{j\in\mathbb{Z}\setminus\{0\}} j^2 \mathbb{P}[\xi_2 = j \mid \xi_1 = i] \\
&= C_i \sum_{j\in\mathbb{Z}\setminus\{0\}} \frac{j^2}{|j|^{4+\frac{1}{|i|}}} \leq C_i \sum_{j\in\mathbb{Z}\setminus\{0\}} \frac{1}{|j|^2}.
\end{aligned}$$

Thus $\langle M\rangle_t$ is bounded. On the other hand,

$$\begin{aligned}
\mathbb{E}|\xi_2|^3 &= \sum_{i,j\in\mathbb{Z}\setminus\{0\}} |j|^3 \mathbb{P}[\xi_2 = j \mid \xi_1 = i]\mathbb{P}\{\xi_1 = i\} \\
&= k \sum_{i,j\in\mathbb{Z}\setminus\{0\}} C_i \frac{1}{|j|^{(1+\frac{1}{|i|})}|i|^5} \\
&\leq 4k \sum_{i\in\mathbb{Z}\setminus\{0\}} \frac{1}{i^4} < \infty,
\end{aligned}$$

which implies

$$\mathbb{E}|M_1|^3 < \infty,\ \mathbb{E}|M_2|^3 \leq 4(\mathbb{E}\xi_1^3 + \mathbb{E}\xi_2^3) < \infty.$$

Consider the dynamic equation on the time scale $\mathbb{T} = \{1, 2\}$

$$\begin{gathered}
d^\nabla X_t = -X_{t_-} d^\nabla t + X_{t_-} d^\nabla M_t \\
X_1 = \xi_1.
\end{gathered}$$

This equation has a unique solution $X_1 = \xi_1$ and $X_2 = \xi_1\xi_2$. However,

$$\begin{aligned}
\mathbb{E}|X_2|^3 = \mathbb{E}|\xi_1\xi_2|^3 &= \sum_{i,j\in\mathbb{Z}\setminus\{0\}} |ij|^3 \mathbb{P}[\xi_2 = j \mid \xi_1 = i]\mathbb{P}\{\xi_1 = i\} \\
&\geq \frac{3k}{8} \sum_{i,j\in\mathbb{Z}\setminus\{0\}} \frac{1}{i^2|j|^{1+\frac{1}{|i|}}} \\
&\geq \frac{3k}{8} \sum_{i\in\mathbb{Z}\setminus\{0\}} |i|\frac{1}{i^2} = \infty.
\end{aligned}$$

In the following we give conditions ensuring the finiteness of p-moment of the solution of (3.1).

Theorem 3.4. *Suppose that linear growth condition* (3.7) *and the conditions* (2.5), (2.6) *hold. Further, there are two constants m_1, m_p such that*

$$\int_{\mathbb{R}} |u|\widehat{\Upsilon}(t, du) \leq m_1, \quad \int_{\mathbb{R}} |u|^p \Upsilon(t, du) \leq m_p \quad \forall t \in [a, T] \tag{3.8}$$

almost surely. Then, the solution $X_{a,x_a}(t)$ of (3.1) starting in x_a satisfies the estimate

$$\mathbb{E}\|X_{a,x_a}(t)\|^p \le (\|x_a\|^p + 1)e_H(t,a), \quad a \le t \le T \tag{3.9}$$

where H is a constant.

Proof. Since $\int_{\mathbb{R}} |u|^2 \Upsilon(t,du) = \langle M\rangle_t \le N := m_2$, we can suppose that $p \ge 2$. Applying (2.10) to the Lyapunov function $V(t,x) = \|x\|^p$ we have

$$\begin{aligned}\mathcal{A}V(t,x) &= p\|x\|^{p-2}(1-1_{\mathbb{I}}(t))x^T f(t,x) + (\|x+f(t,x)\nu(t)\|^p - \|x\|^p)\Phi(t)\\ &\quad + \frac{p}{2}\|x\|^{p-2}\|g(t,x)\|^2\widehat{K}_t^c + \frac{p(p-2)}{2}\|x\|^{p-4}|x^T g(t,x)|^2\widehat{K}_t^c\\ &\quad + \int_{\mathbb{R}}[\|x+f(t,x)\nu(t)+g(t,x)u\|^p - \|x+f(t,x)\nu(t)\|^p]\Upsilon(t,du)\\ &\quad - p\|x\|^{p-2}x^T g(t,x)\int_{\mathbb{R}} u\widehat{\Upsilon}(t,du).\end{aligned}$$

Using Taylor's expansion for the function $\|x+y\|^p$ at $y=0$ obtains

$$\begin{aligned}&\|x+f(t,x)\nu(t)\|^p - \|x\|^p\\ &= p\|x\|^{p-2}x^\top f(t,x)\nu(t) + \frac{p}{2}\|x+\theta f(t,x)\nu(t)\|^{p-2}\|f(t,x)\|^2\nu(t)^2\\ &\quad + \frac{p(p-2)}{2}\|x+\theta f(t,x)\nu(t)\|^{p-4}|(x+\theta f(t,x)\nu(t))^\top f(t,x)|^2\nu(t)^2.\end{aligned}$$

where $0 \le \theta \le 1$. It is seen that

$$\begin{aligned}\|x+\theta f(t,x)\nu(t)\|^{p-2}\|f(t,x)\|^2 &\le (\|x\| + \|f(t,x)\nu(t)\|)^{p-2}\|f(t,x)\|^2\\ &\le (\sqrt{1+\|x\|^2} + \sqrt{G(1+\|x\|^2)}\nu_*)^{p-2}G(1+\|x\|^2)\\ &= G(1+\sqrt{G}\nu_*)^{p-2}(1+\|x\|^2)^{p/2},\end{aligned}$$

and

$$\begin{aligned}&\|x+\theta f(t,x)\nu(t)\|^{p-4}|(x+\theta f(t,x)\nu(t))^\top f(t,x)|^2\\ &\le \|x+\theta f(t,x)\nu(t)\|^{p-2}\|f(t,x)\|^2\\ &\le G(1+\sqrt{G}\nu_*)^{p-2}(1+\|x\|^2)^{p/2}.\end{aligned}$$

Similarly, the Taylor's expansion of the function $\|x+f(t,x)\nu(t)+y\|^p$ at $y=0$ leads us

$$\begin{aligned}&\|x+f(t,x)\nu(t)+g(t,x)u\|^p - \|x+f(t,x)\nu(t)\|^p\\ &= p\|x+f(t,x)\nu(t)\|^{p-2}(x+f(t,x)\nu(t))^\top g(t,x)u\\ &\quad + \frac{p}{2}\|x+f(t,x)\nu(t)+\eta g(t,x)u\|^{p-2}\|g(t,x)u\|^2\\ &\quad + \frac{p(p-2)}{2}\|x+f(t,x)\nu(t)+\eta g(t,x)u\|^{p-4}\\ &\quad \times |(x+f(t,x)\nu(t)+\eta g(t,x)u)^\top g(t,x)u|^2,\end{aligned}$$

where $0 \le \eta \le 1$. By defining $c_p = 2^{p-1}$ if $p > 1$ and $c_p = 1$ if $p \le 1$ we have

$$\begin{aligned}&\|x+f(t,x)\nu(t)\|^{p-2}(x+f(t,x)\nu(t))^\top g(t,x)u\\ &\le \|x+f(t,x)\nu(t)\|^{p-1}\|g(t,x)u\|\\ &\le \sqrt{G}(1+\sqrt{G}\nu_*)^{p-1}|u|(1+\|x\|^2)^{p/2},\end{aligned}$$

and

$$\begin{aligned}
&\|x+f(t,x)\nu(t)+\eta g(t,x)u\|^{p-2}\|g(t,x)u\|^2\\
&\le c_{p-2}(\|x+f(t,x)\nu(t)\|^{p-2}+(\|g(t,x)\|u)^{p-2})\|g(t,x)u\|^2\\
&\le c_{p-2}(G(1+\sqrt{G}\nu_*)^{p-2}u^2+G^{p/2}|u|^p)(1+\|x\|^2)^{p/2}.
\end{aligned}$$

Further,

$$\begin{aligned}
&\|x+f(t,x)\nu(t)+\eta g(t,x)u\|^{p-4}|(x+f(t,x)\nu(t)+\eta g(t,x)u)^\top g(t,x)u|^2\\
&\le \|x+f(t,x)\nu(t)+\eta g(t,x)u\|^{p-2}\|g(t,x)u\|^2\\
&\le c_{p-2}(G(1+\sqrt{G}\nu_*)^{p-2}u^2+G^{p/2}|u|^p)(1+\|x\|^2)^{p/2}.
\end{aligned}$$

Therefore, by using (2.9), (3.8) we obtain

$$\begin{aligned}
&\mathcal{A}V(t,x)\\
&= p\|x\|^{p-2}(1-1_{\mathbb{I}}(t))x^Tf(t,x)\\
&\quad+(p\|x\|^{p-2}x^\top f(t,x)\nu(t)+\frac{p}{2}\|x+\theta f(t,x)\nu(t)\|^{p-2}\|f(t,x)\|^2\nu(t)^2\\
&\quad+\frac{p(p-2)}{2}\|x+\theta f(t,x)\nu(t)\|^{p-4}|(x+\theta f(t,x)\nu(t))^\top f(t,x)|^2\nu(t)^2)\Phi(t)\\
&\quad+\frac{p}{2}\|x\|^{p-2}\|g(t,x)\|^2\widehat{K}_t^c+\frac{p(p-2)}{2}\|x\|^{p-4}|x^Tg(t,x)|^2\widehat{K}_t^c\\
&\quad+p\int_{\mathbb{R}}\|x+f(t,x)\nu(t)\|^{p-2}(x+f(t,x)\nu(t))^\top g(t,x)u\widetilde{\Upsilon}(t,du)\\
&\quad+p\int_{\mathbb{R}}\|x+f(t,x)\nu(t)\|^{p-2}(x+f(t,x)\nu(t))^\top g(t,x)u\widehat{\Upsilon}(t,du)\\
&\quad+\frac{p}{2}\int_{\mathbb{R}}\|x+f(t,x)\nu(t)+\eta g(t,x)u\|^{p-2}\|g(t,x)u\|^2\Upsilon(t,du)\\
&\quad+\frac{p(p-2)}{2}\int_{\mathbb{R}}\|x+f(t,x)\nu(t)+\eta g(t,x)u\|^{p-4}|(x+f(t,x)\nu(t)\\
&\quad+\eta g(t,x)u)^\top g(t,x)u|^2\Upsilon(t,du)-p\|x\|^{p-2}x^Tg(t,x)\int_{\mathbb{R}}u\widehat{\Upsilon}(t,du)\\
&\le\Big\{p\sqrt{G}(1+m_1)+\frac{p(p-1)}{2}G(N+(1+\sqrt{G}\nu_*)^{p-2}(\nu_*+c_{p-2}N))\\
&\quad+p\sqrt{G}(1+\sqrt{G}\nu_*)^{p-1}m_1+c_{p-2}\frac{p(p-1)}{2}G^{p/2}m_p\Big\}(1+\|x\|^2)^{p/2}\\
&\le HV(x)
\end{aligned}$$

where H is defined

$$\begin{aligned}
H=c_{p/2}&\Big\{p\sqrt{G}(1+m_1)+\frac{p(p-1)}{2}G\Big(N+(1+\sqrt{G}\nu_*)^{p-2}(\nu_*+c_{p-2}N)\Big)\\
&+p\sqrt{G}(1+\sqrt{G}\nu_*)^{p-1}m_1+c_{p-2}\frac{p(p-1)}{2}G^{p/2}m_p\Big\}.
\end{aligned}\tag{3.10}$$

By Theorem 3.1, we obtain

$$\mathbb{E}\|X_{a,x_a}(t)\|^p\le(\|x_a\|^p+1)e_H(t,a),\quad a\le t\le T.$$

The proof is complete. $\square$

4. Exponential p-stability

By (2.1), the Δ-exponential function e_p is also a solution of a ∇-dynamic equations. Therefore, in the following, instead of using $\widehat{e}_p$, we use e_p to define the exponential stability although we are working with stochastic ∇-dynamic equations. Let the process K_t be bounded on $\mathbb{T}_a$, i.e., the constant N in (2.5) does not depend on $T > a$. Suppose that for any $s \ge a; x_s \in \mathbb{R}^d$, the solution $X_{s,x_s}(t)$ with initial condition $X_{s,x_s}(s) = x_s$ of (3.1) exists uniquely and it is defined on $\mathbb{T}_s$. Further,

$$f(t,0) \equiv 0; \quad g(t,0) \equiv 0. \tag{4.1}$$

This assumption implies that (3.1) has the trivial solution $X_{s,0}(t) \equiv 0$.

Definition 4.1. The trivial solution of (3.1) is said to be exponentially p-stable if there is a positive constant α such that for any $s > a$ there exists $\Gamma = \Gamma(s) > 1$, such that

$$\mathbb{E}\|X_{s,x_s}(t)\|^p \le \Gamma \|x_s\|^p e_{\ominus\alpha}(t,s) \quad \text{on } t \ge s, \tag{4.2}$$

holds for all $x_s \in \mathbb{R}^d$.

If one can choose Γ independent of s, the trivial solution of (3.1) is said to be uniformly exponentially p-stable.

Remark 4.2. Since $\ominus\alpha(t) \le -\frac{\alpha}{1+\alpha\nu_*}$ for all $t \in \mathbb{T}$, $0 < e_{\ominus\alpha}(t,s) \le e_{-\frac{\alpha}{1+\alpha\nu_*}}(t,s)$ and $e_{-\frac{\alpha}{1+\alpha\nu_*}}(t,s) \to 0$ as $t \to \infty$. Thus, if $\alpha > 0$ then $\lim_{t\to\infty} e_{\ominus\alpha}(t,s) = 0$. The advantage of using $e_{\ominus\alpha}(t,s)$ is that the requirement $-\alpha \in \mathcal{R}^+$ is not necessary.

Theorem 4.3. *Suppose that there exist a function $V(t,x) \in C^{1,2}(\mathbb{T}_a \times \mathbb{R}^d; \mathbb{R}_+)$, positive constants $\alpha_1, \alpha_2, \alpha_3$ such that*

$$\alpha_1\|x\|^p \le V(t,x) \le \alpha_2\|x\|^p, \tag{4.3}$$

$$V^{\nabla_t}(t,x) + \mathcal{A}V(t,x) \le -\alpha_3 V(t_-,x) \quad \forall (t,x) \in \mathbb{T}_a \times \mathbb{R}^d, \tag{4.4}$$

where the differential operator $\mathcal{A}$is defined with respect to (3.1)*. Then, the trivial solution $x \equiv 0$ of* (3.1) *is uniformly exponentially p-stable.*

Proof. Let α be a positive number satisfying $\frac{\alpha}{1+\alpha\nu(t)} < \alpha_3$ for all $t \in \mathbb{T}$ and let $s \ge a, x_s \in \mathbb{R}^d$. To simplify notations, we write $X(t)$ for $X_{s,x_s}(t)$. For each $n > \|x_s\|$, define the stopping time

$$\theta_n = \inf\{t \ge s : \quad \|X(t)\| \ge n\}.$$

Obviously, $\theta_n \to \infty$ as $n \to \infty$ almost surely. By (4.13), calculating expectations we obtain

$$\begin{aligned}
&\mathbb{E}[e_\alpha(t\wedge\theta_n,s)V(t\wedge\theta_n, X(t\wedge\theta_n))] \\
&= V(s,x_s) + \mathbb{E}\int_s^{t\wedge\theta_n} e_\alpha(\theta_n\wedge\tau_-,s)\Big[\alpha V(\tau_-,X(\tau_-)) \\
&\quad + (1+\alpha\nu(\tau))(V^{\nabla_\tau}(\tau,X(\tau_-)) + \mathcal{A}V(\tau,X(\tau_-)))\Big]\nabla\tau.
\end{aligned}$$

Using (4.4) and the inequality $\frac{\alpha}{1+\alpha\nu(t)} < \alpha_3$ obtains

$$\alpha V(\tau_-,X(\tau_-)) + (1+\alpha\nu(\tau))\big(V^{\nabla_\tau}(\tau,X(\tau_-)) + \mathcal{A}V(\tau,X(\tau_-))\big) \le 0.$$

Therefore,

$$\alpha_1 e_\alpha(t\wedge\theta_n,s)\mathbb{E}\|X(t\wedge\theta_n)\|^p \le \mathbb{E}[e_\alpha(t\wedge\theta_n,s)V(t\wedge\theta_n,X(t\wedge\theta_n))]$$

$$\le V(s,x_s) \le \alpha_2\|x_s\|^p.$$

Letting $n\to\infty$ yields

$$\alpha_1 e_\alpha(t,s)\mathbb{E}\|X(t)\|^p \le \alpha_2\|x_s\|^p.$$

Hence,

$$\mathbb{E}\|X_{s,x_s}(t)\|^p \le \frac{\alpha_2}{\alpha_1}\|x_s\|^p e_{\ominus\alpha}(t,s).$$

The proof is complete. □

We now consider the inverse problem by showing that if the trivial solution of (3.1) is uniformly exponentially p-stable then such a Lyapunov function exits. Firstly, we study the differentiability of solutions with respect to the initial conditions and the continuity with respect to coefficients.

Lemma 4.4 (Burkholder inequality on time scales)**.** *For any $p\ge 2$ there exist positive constants B_p such that if $\{M_t\}_{t\in\mathbb{T}_a}$ is an $\mathcal{F}_t$-martingale with $\mathbb{E}|M_t|^p<\infty$ and $M_a=0$ then*

$$\mathbb{E}\sup_{a\le s\le t}|M_s|^p \le B_p\Big(\mathbb{E}\langle M\rangle_t^{p/2} + \mathbb{E}\sum_{a\le s\le t}|\nabla^* M_s|^p\Big),$$

where $\nabla^* M_s = M_s - M_{s_-}$.

Proof. By Doob's inequality, we have

$$\mathbb{E}\sup_{a\le s\le t}|M_s|^p \le \Big(\frac{p}{p-1}\Big)^p\mathbb{E}|M_t|^p.$$

Otherwise, we see that the martingale $\widehat{M}_t$ can be extended to a regular martingale on $[a;\infty)_{\mathbb{R}}$. Therefore, by using proof of [19, Lemma 5] we obtain

$$\mathbb{E}|\widehat{M}_t|^p \le \widehat{B}_p\Big(\mathbb{E}\langle\widehat{M}\rangle_t^{p/2} + \mathbb{E}\sum_{a\le s\le t}|\nabla^*\widehat{M}_s|^p\Big),$$

for a constant $\widehat{B}_p$. Further, the martingale $\widetilde{M}_t$ is a sum of random variables. Then, applying [1, Theorem 13.2.15, pp.416] yields

$$\mathbb{E}|\widetilde{M}_t|^p \le \widetilde{B}_p\Big(\mathbb{E}\langle\widetilde{M}\rangle_t^{p/2} + \mathbb{E}\sum_{a\le s\le t}|\nabla^*\widetilde{M}_s|^p\Big).$$

Consequently,

$$\begin{aligned}
\mathbb{E}\sup_{a\le s\le t}|M_s|^p &\le 2^{p-1}\Big(\frac{p}{p-1}\Big)^p\Big(\mathbb{E}|\widehat{M}_t|^p + \mathbb{E}|\widetilde{M}_t|^p\Big)\\
&\le 2^{p-1}\Big(\frac{p}{p-1}\Big)^p\Big[\widehat{B}_p\Big(\mathbb{E}\langle\widehat{M}\rangle_t^{p/2} + \mathbb{E}\sum_{a\le s\le t}|\nabla^*\widehat{M}_s|^p\Big) + \widetilde{B}_p\Big(\mathbb{E}\langle\widetilde{M}\rangle_t^{p/2}\\
&\quad + \mathbb{E}\sum_{a\le s\le t}|\nabla^*\widetilde{M}_s|^p\Big)\Big]\\
&\le B_p\Big(\mathbb{E}(\langle\widehat{M}\rangle_t + \langle\widetilde{M}\rangle_t)^{p/2} + \mathbb{E}\sum_{a\le s\le t}(|\nabla^*\widehat{M}_s|^p + |\nabla^*\widetilde{M}_s|^p)\Big)\\
&= B_p\Big(\mathbb{E}\langle M\rangle_t^{p/2} + \mathbb{E}\sum_{a\le s\le t}|\nabla^* M_s|^p\Big)
\end{aligned}$$

where $B_p = 2^p(\frac{p}{p-1})^p\max\big\{\widehat{B}_p,\widetilde{B}_p\big\}$. The proof is complete. □

Theorem 4.5. *Let $p \geq 2, M \in \mathcal{M}_2$ such that the conditions* (2.5), (2.6) *and* (3.8) *hold and let $g \in \mathcal{L}_2((a,T];M)$ with*
$$\int_a^t \mathbb{E}|g(\tau)|^p \nabla\tau < \infty \ \ \forall t \in \mathbb{T}_a.$$
Then
$$\mathbb{E} \sup_{a \leq t \leq T} \Big| \int_a^t g(\tau)\nabla M_\tau \Big|^p \leq C_p \int_a^T \mathbb{E}|g(\tau)|^p \nabla\tau,$$
where $C_p = B_p\{(T-a)^{\frac{p}{2}-1} N^{p/2} + m_p\}$.

Proof. Set
$$x_t = \int_a^t g(\tau)\nabla M_\tau, \quad t \in [a,T].$$
The process x_t is a square martingale with the characteristic
$$\langle x\rangle_t = \int_a^t |g(\tau)|^2 \nabla\langle M\rangle_\tau.$$
Since $\langle M\rangle_t$ is continuous, so is $\langle x\rangle_t$. Applying Lemma 4.4 to the martingale (x_t) obtains
$$\begin{aligned}
&\mathbb{E} \sup_{a \leq r \leq t} |x_r|^p \\
&\leq B_p\Big\{\mathbb{E}\langle x\rangle_t^{p/2} + \mathbb{E} \sum_{a \leq s \leq t} |\nabla^* x_s|^p\Big\} \\
&= B_p\Big\{\mathbb{E}\langle x\rangle_t^{p/2} + \mathbb{E} \int_a^t \int_{\mathbb{R}} |g(\tau)u|^p \delta(\nabla\tau, du)\Big\} \\
&= B_p\Big\{\mathbb{E}\Big[\int_a^t |g(\tau)|^2 \nabla\langle M\rangle_\tau\Big]^{p/2} + \mathbb{E} \int_a^t \int_{\mathbb{R}} |g(\tau)u|^p \pi(\nabla\tau, du)\Big\} \\
&\leq B_p\Big\{(t-a)^{\frac{p}{2}-1} N^{p/2} \int_a^t \mathbb{E}|g(\tau)|^p \nabla\tau + \mathbb{E}\int_a^t |g(\tau)|^p \int_{\mathbb{R}} |u|^p \Upsilon(\tau, du)\nabla\tau\Big\} \\
&\leq B_p\big\{(t-a)^{\frac{p}{2}-1} N^{p/2} + m_p\big\} \int_a^t \mathbb{E}|g(\tau)|^p \nabla\tau.
\end{aligned}$$
By putting $C_p = B_p\big[(T-a)^{\frac{p}{2}-1} N^{p/2} + m_p\big]$ we complete the proof. □

Lemma 4.6. *Let $T, s \in \mathbb{T}_a; T > s$ and $p \geq 2$ fixed. Suppose that the condition* (3.8) *holds and process $\zeta(t)$ is the solution of the stochastic equation*
$$\zeta(t) = \varphi(t) + \int_s^t \psi(\tau)\zeta(\tau_-)\nabla\tau + \int_s^t \chi(\tau)\zeta(\tau_-)\nabla M_\tau, \ \ \forall t \in [s,T]. \tag{4.5}$$
We assume that the functions $\varphi(t), \psi(t)$ and $\chi(t)$ are $\mathcal{F}_t$-adapted and that there exists a constant $K > 0$ such that with probability 1, *$\|\psi(t)\| \leq K$ and $\|\chi(t)\| \leq K$. Then*
$$\mathbb{E} \sup_{s \leq t \leq T} \|\zeta(t)\|^p \leq 3^{p-1} \mathbb{E} \sup_{s \leq t \leq T} \|\varphi(t)\|^p e_{H_1}(T,s), \tag{4.6}$$
where $H_1 = 3^{p-1}K^p((T-s)^{p-1} + C_p)$.

Proof. For any $n > 0$ denote $\theta_n = \inf\{t > s : \|\zeta(t)\| > n\}$. From (4.5) we have

$$\begin{aligned}
&\mathbb{E} \sup_{s\le r\le t} \|\zeta(r\wedge\theta_n)\|^p \\
&\le 3^{p-1}\Big(\mathbb{E}\sup_{s\le r\le T}\|\varphi(r)\|^p + \mathbb{E}\sup_{s\le r\le t}\Big\|\int_s^{r\wedge\theta_n}\psi(\tau)\zeta(\tau_-)\nabla\tau\Big\|^p \\
&\quad + \mathbb{E}\sup_{s\le r\le t}\Big\|\int_s^{r\wedge\theta_n}\chi(\tau)\zeta(\tau_-)\nabla M_\tau\Big\|^p\Big) \\
&\le 3^{p-1}\Big(\mathbb{E}\sup_{s\le r\le T}\|\varphi(r)\|^p + K^p(T-a)^{p-1}\int_s^{t\wedge\theta_n}\mathbb{E}\|\zeta(\tau_-)\|^p\nabla\tau \\
&\quad + C_pK^p\int_s^{t\wedge\theta_n}\mathbb{E}\|\zeta(\tau_-)\|^p\nabla\tau\Big) \quad \text{(by Theorem 4.5)} \\
&= 3^{p-1}\Big(\mathbb{E}\sup_{s\le r\le T}\|\varphi(r)\|^p + K^p\Big((T-a)^{p-1}+C_p\Big)\int_s^{t\wedge\theta_n}\mathbb{E}\|\zeta(\tau_-)\|^p\nabla\tau\Big) \\
&= 3^{p-1}\mathbb{E}\sup_{s\le r\le T}\|\varphi(r)\|^p + H_1\int_s^t\sup_{s\le r\le\tau_-}\mathbb{E}\|\zeta(r\wedge\theta_n)\|^p\nabla\tau,
\end{aligned}$$

where $H_1 = 3^{p-1}K^p((T-s)^{p-1}+C_p)$. Using Lemma 2.1 one gets

$$\mathbb{E}\sup_{s\le t\le T}\|\zeta(t\wedge\theta_n)\|^p \le 3^{p-1}\mathbb{E}\sup_{s\le t\le T}\|\varphi(t)\|^p e_{H_1}(T,s).$$

Letting $n\to\infty$ yields (4.6). The proof is complete. $\square$

Lemma 4.7. *Suppose that the coefficients of* (3.1) *are continuous in s, x and they have continuous bounded first and second partial derivatives and condition* (3.8) *holds for $p \ge 4$. Then, the solution $X_{s,x}(t)$, $s \le t \le T$, with initial condition $X_{s,x}(s) = x$ of* (3.1) *is twice differentiable with respect to x. Further, the derivatives*

$$\frac{\partial}{\partial x_i}(X_{s,x}(t)), \quad \frac{\partial^2}{\partial x_i\partial x_j}(X_{s,x}(t))$$

are continuous in x in mean square.

Proof. Suppose that the derivatives $f'_x(t,x), g'_x(t,x), f''_{xx}(t,x), g''_{xx}(t,x)$ are bounded by a constant λ. To simplify notations we put $Y_{s,\Delta x}(t) = X_{s,x+\Delta x}(t) - X_{s,x}(t)$. Using Lagrange theorem we see that for any $i = 1, 2, \dots, d$, there exists $\theta_i, \xi_i \in [0;1]$ such that

$$\begin{aligned}
&f_i(t, X_{s,x}(t_-) + Y_{s,\Delta x}(t_-)) - f_i(t, X_{s,x}(t_-)) \\
&= \sum_{j=1}^d \frac{\partial f_i}{\partial x_j}(t, X_{s,x}(t_-) + \theta_i Y_{s,\Delta x}(t_-))Y_{i,s,\Delta x}(t_-), \\
&g_i(t, X_{s,x}(t_-) + Y_{s,\Delta x}(t_-)) - g_i(t, X_{s,x}(t_-)) \\
&= \sum_{j=1}^d \frac{\partial g_i}{\partial x_j}(t, X_{s,x}(t_-) + \xi_i Y_{s,\Delta x}(t_-))Y_{i,s,\Delta x}(t_-).
\end{aligned} \tag{4.7}$$

Let $A_{s,\Delta x}(t)$ be the matrix with entries $a^{ij}_{s,\Delta x}(t) = \frac{\partial f_i}{\partial x_j}(t, X_{s,x}(t_-) + \theta_i Y_{s,\Delta x}(t_-))$, and let $B_{s,\Delta x}(t)$ be the matrix with entries $b^{ij}_{s,\Delta x}(t) = \frac{\partial g_i}{\partial x_j}(t, X_{s,x}(t_-) + \xi_i Y_{s,\Delta x}(t_-))$. Then (4.7) can be rewritten

$$f(t, X_{s,x}(t_-) + Y_{s,\Delta x}(t_-)) - f(t, X_{s,x}(t_-)) = A_{s,\Delta x}(t)Y_{s,\Delta x}(t_-),$$

$$g(t, X_{s,x}(t_-) + Y_{s,\Delta x}(t_-)) - g(t, X_{s,x}(t_-)) = B_{s,\Delta x}(t) Y_{s,\Delta x}(t_-).$$

Hence,

$$Y_{s,\Delta x}(t) = \Delta x + \int_s^t A_{s,\Delta x}(\tau) Y_{s,\Delta x}(\tau_-) \nabla \tau + \int_s^t B_{s,\Delta x}(\tau) Y_{s,\Delta x}(\tau_-) \nabla M_\tau.$$

Since $A_{s,\Delta x}(t)$ and $B_{s,\Delta x}(t)$ are bounded by a constant λ, by using Lemma 4.6 one has

$$\mathbb{E} \sup_{s \le t \le T} \|Y_{s,\Delta x}(t)\|^2 \le 3\|\Delta x\|^2 e_{H_2}(T, s), \tag{4.8}$$

where $H_2 = 3\lambda^2(T - s + C_2)$. As a consequence, $\mathbb{E} \sup_{s \le t \le T} \|Y_{s,\Delta x}(t)\|^2 \to 0$ as $\|\Delta x\| \to 0$ in probability. Let $\zeta_{s,x}(t)$ be the solution of the variation dynamic equation

$$\zeta_{s,x}(t) = I + \int_s^t f'_x(\tau, X_{s,x}(\tau_-)) \zeta_{s,x}(\tau_-) \nabla \tau + \int_s^t g'_x(\tau, X_{s,x}(\tau_-) \zeta_{s,x}(\tau_-) \nabla M_\tau,$$

for all $s \le t \le T$. Since f'_x and g'_x are bounded by constant λ,

$$\mathbb{E} \sup_{s \le t \le T} \|\zeta_{s,x}(t)\|^4 \le 27 e_{H_3}(T, s), \tag{4.9}$$

where $H_3 = 27\lambda^4((T - s)^3 + C_4)$. Define

$$\zeta_{\Delta x}(t) = Y_{s,\Delta x}(t) - \zeta_{s,x}(t)\Delta x \quad \forall\, s \le t \le T.$$

The process $\zeta_{\Delta x}(t)$ satisfies Equation

$$\zeta_{\Delta x}(t) = \phi_{\Delta x}(t) + \int_s^t A_{s,\Delta x}(\tau) \zeta_{\Delta x}(\tau_-) \nabla \tau + \int_s^t B_{s,\Delta x}(\tau) \zeta_{\Delta x}(\tau_-) \nabla M_\tau,$$

where,

$$\begin{aligned} \phi_{\Delta x}(t) = &\int_s^t [(A_{s,\Delta x}(\tau) - f'_x(\tau, X_{s,x}(\tau_-))) \zeta_{s,x}(\tau_-) \Delta x] \nabla \tau \\ &+ \int_s^t [(B_{s,\Delta x}(\tau) - g'_x(\tau, X_{s,x}(\tau_-))) \zeta_{s,x}(\tau_-) \Delta x] \nabla M_\tau. \end{aligned}$$

Applying Lemma 4.6 again one gets

$$\mathbb{E} \sup_{s \le t \le T} \|\zeta_{\Delta x}(t)\|^2 \le 3\mathbb{E} \sup_{s \le t \le T} \|\phi_{\Delta x}(t)\|^2 e_{H_2}(T, s). \tag{4.10}$$

Since $f'_x(t, x), g'_x(t, x)$ are continuous and $\mathbb{E} \sup_{s \le t \le T} \|Y_{s,\Delta x}(t)\|^2 \to 0$ as $\|\Delta x\| \to 0$ in probability,

$$\lim_{\Delta x \to 0} (\|A_{s,\Delta x}(t) - f'_x(t, X_{s,x}(t_-))\| + \|B_{s,\Delta x}(t) - g'_x(t, X_{s,x}(t_-))\|) = 0$$

in probability. Hence, by the boundedness of A, B, f', g', we obtain

$$\begin{aligned} &\mathbb{E}\big[\sup_{s \le t \le T} \frac{\|\phi_{\Delta x}(t)\|^2}{\|\Delta x\|^2} \big] \\ &\le 2(T - s) \int_s^T \mathbb{E}\|A_{s,\Delta x}(\tau) - f'_x(\tau, X_{s,x}(\tau_-)) \zeta_{s,x}(\tau_-)\|^2 \nabla \tau \\ &\quad + 8 \int_s^T \mathbb{E}\|B_{s,\Delta x}(\tau) - g'_x(\tau, X_{s,x}(\tau_-)) \zeta_{s,x}(\tau_-)\|^2 \nabla \langle M \rangle_\tau \to 0 \end{aligned} \tag{4.11}$$

as $\|\Delta x\| \to 0$. Thus, (4.10) and (4.11) imply

$$\mathbb{E} \sup_{t\le s\le T} \frac{\|\zeta_{\Delta x}(s)\|}{\|\Delta x\|} = 0 \quad \text{as } \Delta x \to 0.$$

This means

$$\zeta_{s,x}(t) = \frac{\partial}{\partial x} X_{s,x}(t) \quad \forall s \le t \le T.$$

The mean square continuity of $\zeta_{s,x}(t)$ with respect to x again follows from the continuity of $f'_x(t, X_{s,x}(t))$ and $g'_x(t, X_{s,x}(t))$.

We prove the existence of $\frac{\partial^2 X_{s,x}(t)}{\partial x^2}$. To simplify notations, if F is a bilinear mapping, we write Fh^2 for $F(h,h)$. Let bilinear mapping $\eta_{s,x}(t)$ be the solution of the second variation dynamic equation

$$\begin{aligned}\eta_{s,x}(t) &= \int_s^t f''_{xx}(\tau, X_{s,x}(\tau_-))\zeta^2_{s,x}(\tau_-)\nabla\tau + \int_s^t f'_x(\tau, X_{s,x}(\tau_-))\eta_{s,x}(\tau_-)\nabla\tau\\ &\quad + \int_s^t g''_{xx}(\tau, X_{s,x}(\tau_-))\zeta^2_{s,x}(\tau_-)\nabla M_\tau + \int_s^t g'_x(\tau, X_{s,x}(\tau_-))\eta_{s,x}(\tau_-)\nabla M_\tau,\end{aligned}$$

for all $s \le t \le T$. Using Lemma 4.6 and (4.9) we see that

$$\mathbb{E} \sup_{s\le t\le T} \|\eta_{s,x}(t)\|^2 \le \infty. \tag{4.12}$$

Define

$$\eta_{\Delta x}(t) = \zeta_{s,x+\Delta x}(t)\Delta x - \zeta_{s,x}(t)\Delta x - \eta_{s,x}(t)(\Delta x)^2, \quad s \le t \le T.$$

The process $\eta_{\Delta x}(t)$ satisfies the equation

$$\begin{aligned}\eta_{\Delta x}(t) &= \psi_{\Delta x}(t) + \int_s^t f'_x(\tau, X_{s,x+\Delta x}(\tau_-))\eta_{\Delta x}(\tau_-)\nabla\tau\\ &\quad + \int_s^t g'_x(\tau, X_{s,x+\Delta x}(\tau_-))\eta_{\Delta x}(\tau_-)\nabla M_\tau,\end{aligned} \tag{4.13}$$

where,

$$\begin{aligned}\psi_{\Delta x}(t) &= \int_s^t \Big[\Big(f'_x(\tau, X_{s,x+\Delta x}(\tau_-)) - f'_x(\tau, X_{s,x}(\tau_-))\\ &\quad - f''_{xx}(\tau, X_{s,x}(\tau_-)\Big)\zeta_{s,x}(\tau_-)\Delta x\Big)\zeta_{s,x}(\tau_-)\Delta x\\ &\quad + (f'_x(\tau, X_{s,x+\Delta x}(\tau_-)) - f'_x(\tau, X_{s,x}(\tau_-)))\eta_{s,x}(\tau_-)(\Delta x)^2\Big]\nabla\tau\\ &\quad + \int_s^t \Big[\big(g'_x(\tau, X_{s,x+\Delta x}(\tau_-)) - g'_x(\tau, X_{s,x}(\tau_-))\\ &\quad - g''_{xx}(\tau, X_{s,x}(\tau_-))\zeta_{s,x}(\tau_-)\Delta x\big)\zeta_{s,x}(\tau_-)\Delta x\\ &\quad + \big(g'_x(\tau, X_{s,x+\Delta x}(\tau_-)) - g'_x(\tau, X_{s,x}(\tau_-))\big)\eta_{s,x}(\tau_-)(\Delta x)^2\Big]\nabla M_\tau.\end{aligned}$$

Using Lemma 4.6 one obtains

$$\mathbb{E}\|\eta_{\Delta x}(t)\|^2 \le \mathbb{E} \sup_{s\le t\le T} \|\psi_{\Delta x}(t)\|^2 e_{H_2}(T,s), \tag{4.14}$$

where $H_2 = 3\lambda^2(T - s + 4N)$. It is easy to see that

$$\mathbb{E} \sup_{s\le t\le T} \Big\| \int_s^t \Big[\big(f'_x(\tau, X_{s,x+\Delta x}(\tau_-)) - f'_x(\tau, X_{s,x}(\tau_-)) - f''_{xx}(\tau, X_{s,x}(\tau_-))$$

$$
\begin{aligned}
&\times \zeta_{s,x}(\tau_-)\Delta x\big)\zeta_{s,x}(\tau_-)\Delta x\Big]\nabla\tau\big\|^2\\
&\leq 2(T-s)\mathbb{E}\int_s^T \big\|\big(f'_x(\tau, X_{s,x+\Delta x}(\tau_-))\\
&- f'_x(\tau, X_{s,x}(\tau_-)) - f''_{xx}(\tau, X_{s,x}(\tau_-))Y_{s,\Delta x}(\tau_-)\big)\zeta_{s,x}(\tau_-)\Delta x\big\|^2\nabla\tau\\
&\quad + 2(T-s)\mathbb{E}\int_s^T \|f''_{xx}(\tau, X_{s,x}(\tau_-))(Y_{s,\Delta x}(\tau_-) - \zeta_{s,x}(\tau_-)\Delta x)\\
&\quad \times \zeta_{s,x}(\tau_-)\Delta x\|^2\nabla\tau = o(\|\Delta x\|^4);
\end{aligned}
$$

$$
\begin{aligned}
&\int_s^T \mathbb{E}\Big\|\big(f'_x(\tau, X_{s,x+\Delta x}(\tau_-)) - f'_x(\tau, X_{s,x}(\tau_-))\big)\eta_{s,x}(\tau_-)(\Delta x)^2\Big\|^2\nabla\tau\\
&= o(\|\Delta x\|^4);
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\sup_{s\leq t\leq T}\Big\|\int_s^t \Big[\big(g'_x(\tau, X_{s,x+\Delta x}(\tau_-)) - g'_x(\tau, X_{s,x}(\tau_-)) - g''_{xx}(\tau, X_{s,x}(\tau_-))\\
&\times \zeta_{s,x}(\tau_-)\Delta x\big)\zeta_{s,x}(\tau_-)\Delta x\Big]\nabla M_\tau\Big\|^2\\
&\leq 4N\mathbb{E}\int_s^T \big\|\big(g'_x(\tau, X_{s,x+\Delta x}(\tau_-))\\
&\quad - g'_x(\tau, X_{s,x}(\tau_-)) - g''_{xx}(\tau, X_{s,x}(\tau_-))Y_{s,\Delta x}(\tau_-)\big)\zeta_{s,x}(\tau_-)\Delta x\big\|^2\nabla\tau\\
&\quad + 4N\mathbb{E}\int_s^T \|g''_{xx}(\tau, X_{s,x}(\tau_-))(Y_{s,\Delta x}(\tau_-) - \zeta_{s,x}(\tau_-)\Delta x)\zeta_{s,x}(\tau_-)\Delta x\|^2\nabla\tau\\
&= o(\|\Delta x\|^4);
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\sup_{s\leq t\leq T}\Big\|\int_s^t \Big[\big(g'_x(\tau, X_{s,x+\Delta x}(\tau_-)) - g'_x(\tau, X_{s,x}(\tau_-))\big)\eta_{s,x}(\tau_-)(\Delta x)^2\Big]\nabla M_\tau\Big\|^2\\
&\leq 4N\mathbb{E}\int_s^T \big\|(g'_x(\tau, X_{s,x+\Delta x}(\tau_-)) - g'_x(\tau, X_{s,x}(\tau_-)))\eta_{s,x}(\tau_-)(\Delta x)^2\big\|^2\nabla\tau\\
&= o(\|\Delta x\|^4).
\end{aligned}
$$

Combining these results we obtain $\mathbb{E}\sup_{s\leq t\leq T}\|\psi_\Delta(t)\|^2 = o(\|\Delta x\|^4)$, which implies that

$$
\mathbb{E}\|\eta_{\Delta x}(t)\|^2 = o(\|\Delta x\|^4).
$$

Thus, $\frac{\|\eta_{\Delta x}(t)\|}{\|\Delta x\|^2} = 0$, or

$$
\frac{\partial^2}{\partial x^2}X_{s,x}(t) = \eta_{s,x}(t).
$$

The proof is complete. ☐

Lemma 4.8. *Let $p \geq 4$ and $2 \leq \beta \leq p$. Then, the map $F(\phi) : \phi \to \mathbb{E}|\phi|^\beta$ from $L_p(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbb{R}$ is twice differentiable at every $\phi_0 \neq 0$ and*

$$
F'(\phi_0)(\phi) = \beta\mathbb{E}[|\phi_0|^{\beta-1}\phi]; \quad F''(\phi_0)(\phi, \psi) = \beta(\beta-1)\mathbb{E}[|\phi_0|^{\beta-2}\phi\psi].
$$

Proof. We have

$$
\big|F(\phi_0 + \Delta\phi) - F(\phi_0) - \beta\mathbb{E}|\phi_0|^{\beta-1}\Delta\phi\big|
$$

$$
\begin{aligned}
&= \big|\mathbb{E}|\phi_0+\Delta\phi|^\beta - \mathbb{E}|\phi_0|^\beta - \beta\mathbb{E}|\phi_0|^{\beta-1}\Delta\phi\big| \\
&= \beta(\beta-1)\mathbb{E}[|\eta|^{\beta-2}(\Delta\phi)^2] \\
&\le \beta(\beta-1)[\mathbb{E}|\eta|^{m(\beta-2)}]^{1/m}[\mathbb{E}|\Delta\phi|^p]^{2/p},
\end{aligned}
$$

where $\eta \in (\phi_0, \phi_0+\Delta\phi)$ if $\phi_0+\Delta\phi > \phi_0$ and $\eta \in (\phi_0+\Delta\phi, \phi_0)$ if $\phi_0+\Delta\phi < \phi_0$. Hence, with $\frac{1}{m}+\frac{2}{p}=1$ we have

$$
\begin{aligned}
&\big|F(\phi_0+\Delta\phi) - F(\phi_0) - \beta\mathbb{E}|\phi_0|^{\beta-1}\Delta\phi\big| \\
&\le \beta(\beta-1)[\mathbb{E}|\eta|^{m(\beta-2)}]^{1/m}[\mathbb{E}|\Delta\phi|^p]^{2/p} \\
&\le \beta(\beta-1)[\mathbb{E}\max\{|\phi_0|, |\phi_0+\Delta\phi|\}^{m(\beta-2)}]^{1/m}[\mathbb{E}|\Delta\phi|^p]^{2/p}.
\end{aligned}
$$

The relation $\frac{1}{m}+\frac{2}{p}=1$ implies $m(\beta-2)<p$. Thus, $\mathbb{E}\max\{|\phi_0|, |\phi_0+\Delta\phi|\}^{m(\beta-2)} < \infty$. Therefore,

$$
\begin{aligned}
&|F(\phi_0+\Delta\phi) - F(\phi_0) - \beta\mathbb{E}|\phi_0|^{\beta-1}\Delta\phi| \\
&\le \beta(\beta-1)[\mathbb{E}|\eta|^{m(\beta-2)}]^{1/m}[\mathbb{E}(\Delta\phi)^p]^{2/p} \\
&= O(1)|\Delta\phi|_p^2 \quad \text{as } |\Delta\phi|_p \to 0.
\end{aligned}
$$

This means $F'(\phi_0)(\phi) = \beta\mathbb{E}|\phi_0|^{\beta-1}\phi$. The existence and continuity of the second derivative F'' con be proved by a similar way. $\square$

Lemma 4.9. *Let the coefficients of* (3.1) *be continuous in t, x and satisfy the conditions* (4.1)*. Suppose also that conditions of Lemma 4.7 are satisfied and $2 \le \beta \le p$. Then, for fixed $t > a$, the function $u(s,x) = \mathbb{E}\|X_{s,x}(t)\|^\beta$; $a < s < t$ is twice continuously differentiable with respect to x except perhaps at $x = 0$.*

Proof. The map $x \to X_{s,x}(t)$ is twice differentiable in x by Lemma 4.7. The map $X \to \|X\|$ from $\mathbb{R}^d$ to $\mathbb{R}$ and the map $F(\phi) = \mathbb{E}|\phi|^\beta$ from $L_p(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbb{R}$ are also twice differentiable. Therefore by chain rule, the map $u(s,x) = \mathbb{E}\|X_{s,x}(t)\|^\beta$ is twice differentiable. Further,

$$
u'_x(s,x)h = \beta\mathbb{E}[\|X_{s,x}(t)\|^{\beta-2} < X_{s,x}(t), \zeta_{s,x}(t)h >] \tag{4.15}
$$

$$
\begin{aligned}
u''_{xx}(s,x)h^2 = \beta\mathbb{E}\Big[&(\beta-2)\|X_{s,x}(t)\|^{\beta-4}\langle X_{s,x}(t), \zeta_{s,x}(t)h\rangle^2 \\
&+ \|X_{s,x}(t)\|^{\beta-2}\|\zeta_{s,x}(t)h\|^2 + \|X_{s,x}(t)\|^{\beta-2}\langle X_{s,x}(t), \eta_{s,x}(t)h^2\rangle\Big].
\end{aligned}
$$

The proof is complete. $\square$

Theorem 4.10. *Let M have independent increments and the conditions of Lemma 4.7 hold and $2 \le \beta \le p$. Suppose further that $\mathcal{A}V(t,x)$ is ld-continuous in (t,x) for all $V \in C^{1,2}(\mathbb{T}_a \times \mathbb{R}^d; \mathbb{R})$. Then, the function $u(s,x) = \mathbb{E}\|X_{s,x}(t)\|^\beta$, $a < s < t$ is ∇-differentiable in s, twice continuously differentiable with respect to x and satisfies the equation*

$$
u^{\nabla_s}(s,x) + \mathcal{A}u(s,x) = 0. \tag{4.16}
$$

Proof. By Lemma 4.7, $u(s,x)$ is twice differentiable in x. From (4.2), (4.8), (4.9), (4.12) and (4.15), it follows that $\int_s^t \mathcal{A}u(h, X_{s,x}(\tau_-))\nabla\tau$ is integrable. Therefore,

$$
u(h, X_{\rho(s),x}(r)) - u(h,x) - \int_{\rho(s)}^r \mathcal{A}u(h, X_{\rho(s),x}(\tau_-))\nabla\tau, \quad s \le r \le h \le t
$$

is an $\mathcal{F}_r$-martingale. In particular,

$$\mathbb{E}u(h, X_{\rho(s),x}(h)) - u(h,x) = \int_{\rho(s)}^{h} \mathbb{E}\mathcal{A}u(h, X_{\rho(s),x}(\tau_-))\nabla\tau.$$

Since M_t has independent increments, $X_{h,y}(t)$ is independent of $X_{\rho(s),x}(h)$ when $s \le h \le t$ and $y \in \mathbb{R}^d$, which implies that $\mathbb{E}u(h, X_{\rho(s),x}(h)) = u(\rho(s), x)$. Thus,

$$\frac{u(\rho(s),x) - u(h,x)}{\rho(s) - h} = \frac{1}{\rho(s) - h}\int_{\rho(s)}^{h} \mathbb{E}\mathcal{A}u(h, X_{\rho(s),x}(\tau_-))\nabla\tau.$$

If s is left-scattered, then

$$u^{\nabla_s}(s,x) = -\frac{1}{\nu(s)}\int_{\rho(s)}^{s} \mathcal{A}u(s, X_{\rho(s),x}(\tau_-))\nabla\tau = -\mathcal{A}u(s,x).$$

In the s is left-dense we let $h \to s$ to obtain

$$u^{\nabla_s}(s,x) = -\mathcal{A}u(s,x).$$

The proof is complete. □

Theorem 4.11. *Let the conditions in Theorem 4.10 hold. Suppose that for any fixed $T > 0$, there exist a function $\gamma_T : \mathbb{T} \to \mathbb{T}$ with $\gamma(T,s) \ge s + T$ for all $s \in \mathbb{T}$ such that $\gamma(T,s)$ and ∇-derivatives $\gamma^{\nabla_s}(T,s)$ are bounded. If the trivial solution of* (3.1) *is uniformly exponentially β-stable, then there exists a function $V(s,x) \in C^{1,2}(\mathbb{T}_a \times \mathbb{R}^d; \mathbb{R}_+)$ satisfying inequalities* (4.3), (4.4) *(with the power β).*

Proof. By Lemma 4.9 and Theorem 4.10, the function

$$V(s,x) = \int_{s}^{\gamma(T,s)} \mathbb{E}\|X_{s,x}(\tau_-)\|^{\beta}\nabla\tau, \tag{4.17}$$

is in class $C^{1,2}(\mathbb{T}_a \times \mathbb{R}^d; \mathbb{R}_+)$. From (4.2),

$$V(s_-, x) \le \int_{s_-}^{\gamma(T,s_-)} \Gamma\|x\|^{\beta} e_{\ominus\alpha}(\tau_-, s_-)\nabla\tau \le \alpha_1\|x\|^{\beta},$$

where $\alpha_1 = \frac{\Gamma(1+\nu_*\alpha)}{\alpha}$. By assumptions, the trivial solution of (3.1) is uniformly exponentially β-stable and $\gamma^{\nabla_s}(T,s)$ is bounded, we can choose $T > 0$ such that

$$\mathbb{E}\|X_{s,x}(\gamma(T,s))\|^{\beta} < \frac{1}{2}\|x\|^{\beta}, \quad \mathbb{E}\|X_{s,x}(\gamma(T,s))\|^{\beta}\gamma^{\nabla_s}(T,s) < \frac{1}{2}\|x\|^{\beta}. \tag{4.18}$$

Since f and g have bounded partial derivatives and $f(t,0) = 0,\ g(t,0) = 0$,

$$\|f(t,x)\| \le G\|x\|, \quad \|g(t,x)\| \le G\|x\|, \quad t \ge a,\ x \in \mathbb{R}^d.$$

Therefore,

$$\|\mathcal{A}[\|x\|^{\beta}](s,x)\| < c_1\|x\|^{\beta}, \tag{4.19}$$

for a certain constant c_1. Applying Itô's formula to the function $\|x\|^{\beta}$ and using (4.19) yields

$$\begin{aligned}
\mathbb{E}\|X_{s,x}(\gamma(T,s))\|^{\beta} - \|x\|^{\beta} &= \int_{s}^{\gamma(T,s)} \mathbb{E}\mathcal{A}(\|X_{s,x}(\tau_-)\|^{\beta})\nabla\tau \\
&\ge -c_1\int_{s}^{\gamma(T,s)} \mathbb{E}\|X_{s,x}(\tau_-)\|^{\beta}\nabla\tau = -c_1V(s,x).
\end{aligned}$$

Combining with (4.18) we obtain the inequality $V(s,x) > \alpha_2\|x\|^\beta$ with $\alpha_2 = \frac{1}{2c_1}$. Thus, the function V satisfies condition (4.3). Using [2, Theorem 5.80] to calculate ∇-differential of V with respect s and applying Theorem 4.10 we obtain

$$V^{\nabla_s}(s,x) + \mathcal{A}V(s,x) = \mathbb{E}\|X_{s_-,x}(\gamma(T,s_-))\|^\beta \gamma^{\nabla_s}(T,s_-) - \|x\|^\beta.$$

Using (4.18) again we have

$$V^{\nabla_s}(s,x) + \mathcal{A}V(s,x) \le -\frac{1}{2}\|x\|^\beta \le -\frac{1}{2\alpha_1}V(s_-,x).$$

Thus, the function V satisfies all conditions (4.3), (4.4) with $\alpha_3 = \frac{\alpha}{2\Gamma(1+\nu_*\alpha)}$. The proof is complete. □

Example 4.12. Consider the linear stochastic dynamic equation

$$\begin{gathered} d^\nabla X(t) = aX(t_-)d^\nabla t + bX(t_-)d^\nabla M(t) \quad \forall t \in \mathbb{T}_s \\ X(s) = x, \end{gathered} \tag{4.20}$$

where a, b are two constants, a is regressive and M is a square integrable martingale having independent increment. By direct calculation we have

$$\mathbb{E}X_{s,x}^2(t) = x^2 + \int_s^t q(\tau)\mathbb{E}X_{s,x}^2(\tau_-)\nabla\tau, \tag{4.21}$$

where

$$\begin{aligned} q(t) &= 2a + b^2\widehat{K}_t^c + a^2\nu(t) + 2b(1+a\nu(t))\int_{\mathbb{R}} u\Upsilon(t,du) \\ &\quad + b^2\int_{\mathbb{R}} u^2\Upsilon(t,du) - 2b\int_{\mathbb{R}} u\widehat{\Upsilon}(t,du) \\ &= 2a + b^2\widehat{K}_t^c + a^2\nu(t) + 2b\int_{\mathbb{R}} u\widetilde{\Upsilon}(t,du) \\ &\quad + b^2\int_{\mathbb{R}} u^2\Upsilon(t,du) + a\nu(s)\int_{\mathbb{R}} u\Upsilon(t,du). \end{aligned}$$

Since $\int_{\mathbb{R}} u\widetilde{\Upsilon}(t,du) = \mathbb{E}[M_t - M_{\rho(t)}|\mathcal{F}_{\rho(t)}] = 0$ and $\nu(t)\int_{\mathbb{R}} u\Upsilon(t,du) = 0$,

$$q(t) = 2a + b^2\widehat{K}_t^c + a^2\nu(t) + b^2\int_{\mathbb{R}} u^2\Upsilon(t,du). \tag{4.22}$$

We define the function $\bar{q}(t) = \lim_{\rho(s)\downarrow t} q(s)$. It is seen that $\bar{q}$ is rd-continuous and $\bar{q}(t) = q(\sigma(t))$ if t is right scattered. Since $\{t : \mu(t) > 0\}$ is countable and $\operatorname{meas}\{t : \mathbb{E}X_{s,x}^2(t_-) \ne \mathbb{E}X_{s,x}^2(t)\} = 0$,

$$\begin{aligned} \int_s^t q(\tau)\mathbb{E}X_{s,x}^2(\tau_-)\nabla\tau &= \int_{(s,t]} q(\tau)\mathbb{E}X_{s,x}^2(\tau_-)\,d\tau + \sum_{s<\tau\le t} q(\tau)\mathbb{E}X_{s,x}^2(\tau_-)\nu(\tau) \\ &= \int_{[s,t)} \bar{q}(\tau)\mathbb{E}X_{s,x}^2(\tau)\,d\tau + \sum_{s\le\tau<t} q(\sigma(\tau))\mathbb{E}X_{s,x}^2(\tau)\mu(\tau) \\ &= \int_s^t \bar{q}(\tau)\mathbb{E}X_{s,x}^2(\tau)\Delta\tau, \end{aligned}$$

from which it follows that

$$\mathbb{E}X_{s,x}^2(t) = x^2 e_{\bar{q}}(t,s), \quad t \ge s. \tag{4.23}$$

Further, it is known that

$$0 < e_{\overline{q}}(t,s) = \exp\Big\{\int_s^t \lim_{h\searrow\mu(\tau)} \frac{\ln(1+\overline{q}(\tau)h)}{h}\Delta\tau\Big\}.$$

Then, condition (4.2) implies

$$\int_s^t \lim_{h\searrow\mu(\tau)} \frac{\ln(1+\overline{q}(\tau)h)}{h}\Delta\tau \le \ln\Gamma - \theta(t-s) \quad \forall t > s.$$

Choose $T > 0$ such that $\ln\Gamma - \frac{\theta T}{2} < 0$ we obtain

$$\int_s^t \lim_{h\searrow\mu(\tau)} \frac{\ln(1+\overline{q}(\tau)h)}{h}\Delta\tau \le -\frac{\theta(t-s)}{2} \quad \forall t > s+T.$$

Thus, the exponential square stability of (4.20) implies

$$\sup\Big\{\frac{1}{t-s}\int_s^t \lim_{h\searrow\mu(\tau)} \frac{\ln(1+\overline{q}(\tau)h)}{h}\Delta\tau : t > s+T\Big\} < 0. \tag{4.24}$$

Conversely, supposing that (4.24) holds, there are $\alpha > 0, K^* > 0$ such that $0 < e_{\bar{q}}(t,s) \le K^* e_{-\alpha}(t,s)$. By using (4.23) we see that the trivial solution of is exponentially square stable. To illustrate the argument in the proof of Theorem 4.11 to construct a Lyapunov function we put

$$V(s,x) = x^2\int_s^\infty e_{\bar{q}}(\tau_-,s)\nabla\tau = x^2 Q(s).$$

By direct calculation we have

$$Q^{\nabla_s}(s) = -1 - q(s)Q(s).$$

Hence,

$$\begin{aligned} &V^{\nabla_s}(s,x) + \mathcal{A}V(s,x) = Q^{\nabla_s}(s)x^2 + q(s)Q(s)x^2 \\ &= (-q(s)Q(s)-1)x^2 + q(s)Q(s)x^2 = -x^2. \end{aligned} \tag{4.25}$$

Using (4.21) and the fact $\lim_{t\to\infty}\mathbb{E}X^2_{s,x}(t) = 0$ we can show that $V(s,x) \ge \alpha_1 x^2$ with $\alpha_1 = (\sup_t |q(t)|)^{-1}$. Further, $e_{\bar{q}}(t,s) \le K^* e_{-\alpha}(t,s)$. Thus, $V(s,x) \le \frac{K^*}{\alpha}x^2$. Combining (4.25) and these inequalities obtains

$$V^{\nabla_s}(s,x) + \mathcal{A}V(s,x) \le -\frac{\alpha}{K^*}V(s_-,x).$$

Thus, the conditions of Theorem 4.3 are satisfied. The proof is complete.

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