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A VARIATIONAL PRINCIPLE FOR BOUNDARY-VALUE PROBLEMS WITH NON-LINEAR BOUNDARY CONDITIONS

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ABSTRACT. In this article, we establish a variational principle for a class of boundary-value problems with a suitable non-linear boundary conditions. As an application of the variational principle, we study the existence of classical solutions for boundary-value problems.

1. Introduction

By using the variational principle, boundary-value problems have been studied by numerous mathematicians (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and references therein). In [1, 2], the authors studied equations with the boundary condition $u(0) = u(1) = 0$. In [4, 5, 6], the authors studied Sturm-Liouville boundary-value problems. In [3, 7], the authors studied Neumann boundary-value problems. In [8], Han studied the periodic boundary-value problems. In [9, 10, 11, 12, 13], the authors applied variational methods to impulsive differential equations. In all the references above, the boundary conditions are linear. In this article, we consider a boundary-value problem with non-linear boundary conditions:

$$\begin{gathered} x'' = f(t,x), \quad t \in [0,1], \\ H(x(0), x(1)) = 0, \\ \nabla H(x(0), x(1)) J[(x'(0), -x'(1)) - \nabla I(x(0), x(1))]^T = 0. \end{gathered} \tag{1.1}$$

Here, H and $I : R^2 \to R$ are continuously differentiable, and

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is the standard symplectic matrix. Also, we assume that the set $\mathcal{A} = \{(x,y) : H(x,y) = 0\}$ is nonempty. If $H(x,y) = x^2 + y^2$ and $I(x,y) = 0$, problem (1.1) becomes a Dirichlet boundary value problem. If $H(x,y) = x - y$ and $I(x,y) = 0$, problem (1.1) becomes a periodic boundary value problem. If $H(x,y) = x + y$ and $I(x,y) = 0$, then problem (1.1) becomes a antiperiodic boundary value problem.

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This article is organized as follows: in section 2, we construct a variational functional for (1.1). In section 3, we obtain sufficient conditions for (1.1) to have a solution.

2. Variational structure

Let W be the Sobolev space of functions $x : [0,1] \to R$ with a weak derivative $x' \in L^2(0,1;R)$. The inner product on W is

$$(x,y) = \int_0^1 [x'(t)y'(t) + x(t)y(t)]dt \tag{2.1}$$

and the corresponding norm is $\|\cdot\|$. For each $x \in W$, there exists a real number $\xi \in (0,1)$ such that

$$x(\xi) = \int_0^1 x(t)dt.$$

Then

$$\begin{aligned} |x(t)| &= |x(\xi) + \int_\xi^t x'(s)ds| \\ &\leq \Big(\int_0^1 x^2(t)dt\Big)^{1/2} + \Big(\int_0^1 (x'(t))^2 dt\Big)^{1/2} \leq \sqrt{2}\|x\|. \end{aligned} \tag{2.2}$$

To establish a variational principle for (1.1), we assume that f satisfies the condition

(H1) $f(t,x)$ is measurable in t for each $x \in \mathbb{R}$, continuous in x for almost every $t \in [0,1]$, and there exists $h_k \in L^1(0,1)$ for any $k > 0$ such that
$$|f(t,x)| \leq h_k(t)$$
for almost every $t \in [0,1]$ and all $|x| \leq k$.

Under this condition, we define the functional ϕ on W by

$$\phi(x) = \int_0^1 [\frac{1}{2}(x'(t))^2 + F(t,x(t))]dt + I(x(0),x(1)) \tag{2.3}$$

where $F(t,x) = \int_0^x f(t,u)du$. Then ϕ is continuously differentiable, weakly lower semi-continuous and

$$(\phi'(x),y) = \int_0^1 [x'(t)y'(t) + f(t,x(t))y(t)]dt + \nabla I(x(0),x(1))(y(0),y(1)) \tag{2.4}$$

for all $y \in W$, see [14]. Let Y be a C^1-manifold defined by

$$Y = \{x \in W : H(x(0),x(1)) = 0\}.$$

Then, Y is weakly closed since W can be compactly imbedded in $C[0,1]$. The following theorem is our main result.

Theorem 2.1. *Assume that f satisfies* (H1) *and that the following condition is satisfied,*

(H2) $\nabla H(x,y) \neq 0$ *for each (x,y) satisfying $H(x,y) = 0$, or $\mathcal{A}$ is a discrete set.*

If x is a critical point of the functional ϕ defined by (2.3) *on Y, then $x(t)$ is a solution of* (1.1).

Proof. For a given u in Y, let $DY(u)$ denote the tangent space to Y at u. If x is a critical point of the functional ϕ on Y, then for any $y \in DY(x)$ we have $(\phi'(x), y) = 0$. It follows from (2.4) that

$$\int_0^1 [x'(t)y'(t) + f(t, x(t))y(t)]dt + \nabla I(x(0), x(1)) \cdot (y(0), y(1)) = 0. \tag{2.5}$$

We define $\omega \in C(0, 1; R)$ by

$$\omega(t) = \int_0^t f(s, x(s))ds. \tag{2.6}$$

By Fubini's theorem and (2.5), we obtain that for any $y \in DY(x)$,

$$\begin{aligned}
&\int_0^1 [x'(t) - \omega(t)]y'(t)dt \\
&= -\int_0^1 f(t, x(t))y(t)dt - \nabla I(x(0), x(1)) \cdot (y(0), y(1)) \\
&\quad - \int_0^1 y'(t) \int_0^t f(s, x(s))dsdt \\
&= -y(1) \int_0^1 f(t, x(t))dt - \nabla I(x(0), x(1)) \cdot (y(0), y(1)).
\end{aligned} \tag{2.7}$$

We complete this proof by considering two cases. When $\nabla H(x(0), x(1)) \neq 0$, we have

$$DY(x) = \{y \in W : \nabla H(x(0), x(1)) \cdot (y(0), y(1)) = 0\}. \tag{2.8}$$

In (2.7), we can choose

$$y(t) = \sin(2n\pi t), \quad n = 1, 2, \dots,$$

and

$$y(t) = 1 - \cos(2n\pi t), \quad n = 1, 2, \dots.$$

It follows from (2.7) that

$$\int_0^1 [x'(t) - \omega(t)] \sin(2n\pi t)dt = \int_0^1 [x'(t) - \omega(t)] \cos(2n\pi t)dt = 0, \quad n = 1, 2, \dots.$$

A theorem for Fourier series implies that

$$x'(t) - \omega(t) = x'(0) \tag{2.9}$$

on $[0, 1]$. Thus, we have $x''(t) = f(t, x(t))$ and

$$\int_0^1 f(t, x(t))dt = x'(1) - x'(0). \tag{2.10}$$

Integrating both sides of (2.9) over $[0, 1]$, we have

$$x(1) - x(0) - \int_0^1 (1 - t)f(t, x(t))dt = x'(0). \tag{2.11}$$

Set $y(t) = \nabla H(x(0), x(1)) \cdot (t, t - 1)$. It is easy to show that $y \in DY(x)$ as $(y(0), y(1)) = J\nabla H(x(0), x(1))$. Inserting $y(t)$ into (2.5) we obtain

$$\Big[x(1) - x(0) - \int_0^1 (1 - t)f(t, x(t))dt\Big] \nabla H(x(0), x(1)) \cdot (1, 1)$$

$$+\nabla I(x(0),x(1))J\nabla H(x(0),x(1))+\int_0^1 f(t,x(t))dt\nabla H(x(0),x(1))\cdot(1,0)=0.$$

From (2.10) and (2.11), the above equality implies

$$\nabla H(x(0),x(1))J[(x'(0),-x'(1))-\nabla I(x(0),x(1))]^T=0.$$

When the $\mathcal{A}$ is a discrete set, $(x(0),x(1))$ is a isolated point of $\mathcal{A}$. Applying the implicit function theorem we obtain $\nabla H(x(0),x(1))=0$, so that

$$DY(x)=\{y\in W: y(0)=y(1)=0\}.$$

It is easy to show that $x(t)$ is a solution of problem (1.1). This completes the proof. □

3. Solutions to boundary-value problems

As an application of Theorem 2.1, we consider the existence of solutions for problem (1.1).

Theorem 3.1. *Assume that* (H1), (H2) *hold, and that the following conditions are satisfied:*

(H3) *The set $\mathcal{A}$ is bounded.*

(H4) *There is a positive constant l with $l<2$, and a positive function $c\in L^1(0,1)$ such that*

$$F(t,x)\geq -c(t)(1+|x|^l)$$

for almost every $t\in[0,1]$ and all $x\in\mathbb{R}$.

Then (1.1) *has at least one solution.*

Proof. Let y be in Y. By (H3), there exists a positive number M such that

$$y^2(0)+y^2(1)\leq M^2.$$

This implies

$$|y(t)|=|y(0)+\int_0^t y'(t)dt|\leq M+\int_0^1|y'(t)|dt\leq M+(\int_0^1[y'(t)]^2dt)^{1/2}. \quad (3.1)$$

Set

$$M_1=\min_{x^2+y^2\leq M^2} I(x,y).$$

Then, from (H4), (2.3) and (3.1), we have

$$\begin{aligned}\phi(y)&\geq \frac{1}{2}\int_0^1[y'(t)]^2dt-\int_0^1 c(t)(1+|y(t)|^l)dt+M_1\\&\geq \frac{1}{2}\int_0^1[y'(t)]^2dt+M_2(\int_0^1[y'(t)]^2dt)^{\frac{l}{2}}+M_3\end{aligned}$$

for some M_2 and M_3. It follows that

$$\lim_{\|y\|\to\infty}\phi(y)=+\infty,$$

since $\|y\|\to\infty$ if and only if $\int_0^1[y'(t)]^2dt\to\infty$. Hence, $\phi|Y$ is bounded from blow. Therefore, there exists a critical point of ϕ on Y. By Theorem 2.1, problem (1.1) has at least one solution. □

Theorem 3.2. *Assume that* (H1)–(H3) *hold, and that the following conditions are satisfied:*

(H5) *There is a positive function* $c \in L^1(0,1)$ *such that*

$$F(t,x) \geq -c(t)(1+x^2)$$

for almost every $t \in [0,1]$ *and all* $x \in \mathbb{R}$.

(H6) $2\int_0^1 c(t)dt < 1$.

Then (1.1) *has at least one solution.*

Proof. For each $y \in Y$, from (H5), (2.3) and (3.1), we obtain

$$\begin{aligned}\phi(y) &\geq \frac{1}{2}\int_0^1 [y'(t)]^2 dt - \int_0^1 c(t)(1+y^2(t))dt + M_1 \\ &\geq (\frac{1}{2} - \int_0^1 c(t)dt)\int_0^1 [y'(t)]^2 dt + M_4(\int_0^1 [y'(t)]^2 dt)^{1/2} + M_5\end{aligned}$$

for some M_4 and M_5. Assumption (H6) implies

$$\lim_{\|y\|\to\infty} \phi(y) = +\infty.$$

Therefore, problem (1.1) has at least one solution. □

Theorem 3.3. *Assume that* (H1), (H2) *hold, and that the following conditions are satisfied:*

(H7) *There is a positive function* $c \in L^1(0,1)$ *and positive constants* k_1, l *with* $l < 2$ *such that*

$$F(t,x) \geq k_1 x^2 - c(t)(1+|x|^l)$$

for almost every $t \in [0,1]$ *and all* $x \in \mathbb{R}$.

(H8) *There are positive constants* k_2 *and* k_3 *such that* $I(x,y) \geq -k_2 x^2 - k_3 y^2$.

(H9) $4(k_2+k_3) < \min\{1, 2k_1\}$.

Then (1.1) *has at least one solution.*

Proof. Assumptions (H7) and (H8), and (2.3) imply

$$\begin{aligned}\phi(y) &\geq \frac{1}{2}\int_0^1 [y'(t)]^2 dt + k_1\int_0^1 y^2(t)dt - \int_0^1 c(t)(1+|y(t)|^l)dt - k_2 y^2(0) - k_3 y^2(1) \\ &\geq (\frac{1}{2}\min\{1,2k_1\} - 2k_2 - 2k_3)\|y\|^2 - \int_0^1 c(t)(1+|y(t)|^l)dt).\end{aligned}$$

for each $y \in Y$. From (H9) we obtain

$$\lim_{\|y\|\to\infty} \phi(y) = +\infty.$$

Therefore (1.1) has at least one solution. □

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