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FIELDS OF RATIONAL CONSTANTS OF CYCLIC FACTORIZABLE DERIVATIONS

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Abstract. We describe all rational constants of a large family of four-variable cyclic factorizable derivations. Thus, we determine all rational first integrals of their corresponding systems of differential equations. Moreover, we give a characteristic of all four-variable Lotka-Volterra derivations with a nontrivial rational constant. All considerations are over an arbitrary field of characteristic zero. Our main tool is the investigation of the cofactors of strict Darboux polynomials. Factorizable derivations are important in derivation theory. Namely, we may associate the factorizable derivation with any given derivation of a polynomial ring and that construction helps to determine rational constants of arbitrary derivations. Besides, Lotka-Volterra systems play a significant role in population biology, laser physics and plasma physics.

1. Introduction

One of the main results of the paper is Theorem 4.1, which gives the description of the fields of rational constants of a family of four-variable Lotka-Volterra derivations. This is a generalization of Theorem 2.1, which describes the rings of polynomial constants. As an important consequence we obtain Corollary 4.2, which characterizes all four-variable Lotka-Volterra derivations with a nontrivial rational constant. Such a problem for three variables was studied by Moulin Ollagnier in [7]. We extend the results of Theorem 4.1 and Corollary 4.2 to cyclic factorizable derivations via diagonal automorphisms. All our considerations are over an arbitrary field k of characteristic zero.

Recall that if R is a commutative k-algebra, then a k-linear mapping $d : R \to R$ is called a *derivation* of R if for all $a, b \in R$

$$d(ab) = ad(b) + d(a)b.$$

We call $R^d = \ker d$ the *ring of constants* of the derivation d. Then $k \subseteq R^d$ and a *nontrivial* constant of d is an element of the set $R^d \setminus k$.

Let us fix some notation: $\mathbb{Q}_+$ - the set of positive rationals, $\mathbb{N}$ - the set of nonnegative integers, $\mathbb{N}_+$ - the set of positive integers, n - an integer ≥ 3, $k[X] := k[x_1, \dots, x_n]$, the ring of polynomials in n variables, $k(X) := k(x_1, \dots, x_n)$, the field of rational functions in n variables.

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If $f_1, \dots, f_n \in k[X]$, then there exists exactly one derivation $d : k[X] \to k[X]$ such that $d(x_1) = f_1, \dots, d(x_n) = f_n$. A derivation $d : k[X] \to k[X]$ is called *factorizable* if $d(x_i) = x_i f_i$, where the polynomials f_i are of degree 1 for $i = 1, \dots, n$. We may associate the factorizable derivation with any given derivation of $k[X]$ and that construction helps to obtain new facts on constants, especially rational constants, of the initial derivation (see, for instance, [6], [9]). A derivation $d : k[X] \to k[X]$ is said to be *cyclic factorizable* if $d(x_i) = x_i(A_i x_{i-1} + B_i x_{i+1})$, where $A_i, B_i \in k$ for $i = 1, \dots, n$ (we adopt the convention that $x_{n+1} = x_1$ and $x_0 = x_n$). Special cases of cyclic factorizable derivations are Lotka-Volterra derivations (see Section 2).

There is no general procedure for determining all constants of a derivation. Even for a given derivation the problem may be difficult, see for instance counterexamples to Hilbert's fourteenth problem (all of them are of the form $k[X]^d$, however it took more than a half century to find at least one of them, for more details we refer the reader to [8, 5]) or Jouanolou derivations (where the rings and fields of constants are trivial, see [6, 8]).

The main motivations of our study are the following:

- Lagutinskii's procedure of association of the factorizable derivation with any given derivation (for instance, [6], [9]);
- applications of Lotka-Volterra systems in population biology, laser physics and plasma physics (see, among many others, [1], [2], [3]);
- links to invariant theory, mainly to connected algebraic groups (see [8])

If δ is a derivation of $k(X)$ such that $\delta(x_i) = f_i$ for $i = 1, \dots, n$, then the set $k(X)^\delta \setminus k$ coincides with the set of all rational first integrals of a system of ordinary differential equations

$$\frac{dx_i(t)}{dt} = f_i(x_1(t), \dots, x_n(t)),$$

where $i = 1, \dots, n$ (for more details we refer the reader to [8]). Therefore, we describe both: all rational constants of a derivation and all rational first integrals of its corresponding system of differential equations.

2. Lotka-Volterra derivations and polynomial constants

Let $d : k[X] \to k[X]$ be a cyclic factorizable derivation of the form $d(x_i) = x_i(A_i x_{i-1} + B_i x_{i+1})$ where $A_i, B_i \in k$ for $i = 1, \dots, n$. Suppose that $A_i \neq 0$ for all i. Consider an automorphism $\sigma : k[X] \to k[X]$ defined by $\sigma(x_i) = A_{i+1}^{-1} x_i$ for $i = 1, \dots, n$. Then $\Delta = \sigma d \sigma^{-1}$ is also a derivation of the ring $k[X]$. Moreover, f is a nontrivial polynomial (respectively: rational, see Section 3) constant of a derivation d if and only if $\sigma(f)$ is a nontrivial polynomial (respectively: rational) constant of a derivation Δ. Clearly $\sigma^{-1}(x_i) = A_{i+1} x_i$ and a short computation shows that $\Delta(x_i) = x_i(x_{i-1} - C_i x_{i+1})$ for $C_i = -B_i A_{i+2}^{-1}$ (we allow $C_i = 0$) and $i = 1, \dots, n$. We can proceed similarly if $A_i = 0$ for some i but $B_i \neq 0$ for all i.

Let $C_1, \dots, C_n \in k$. From now on, $d : k[X] \to k[X]$ is a derivation of the form

$$d(x_i) = x_i(x_{i-1} - C_i x_{i+1})$$

for $i = 1, \dots, n$ (we still adhere to the convention that $x_{n+1} = x_1$ and $x_0 = x_n$). We call d a *Lotka-Volterra derivation* with parameters $C_1, \dots, C_n$.

Let $n = 4$. For arbitrary $C_1, C_2, C_3, C_4 \in k$ we may consider the four sentences:

$s_1 :\quad C_1 C_2 C_3 C_4 = 1.$

s_2 : $C_1, C_3 \in \mathbb{Q}_+$ and $C_1C_3 = 1$.
s_3 : $C_2, C_4 \in \mathbb{Q}_+$ and $C_2C_4 = 1$.
s_4 : $C_1C_2C_3C_4 = -1$ and $C_i = 1$ for two consecutive indices i.

In case s_2 let $C_1 = \frac{p}{q}$, where $p, q \in \mathbb{N}_+$ and $\gcd(p, q) = 1$. In case s_3 let $C_2 = \frac{r}{t}$, where $r, t \in \mathbb{N}_+$ and $\gcd(r, t) = 1$. In case s_4 we define the polynomial f_4, namely for $C_1 = C_2 = 1$ let

$$f_4 = x_1^2 + x_2^2 + x_3^2 + C_3^2x_4^2 + 2x_1x_2 - 2x_1x_3 - 2C_3x_1x_4 + 2x_2x_3 - 2C_3x_2x_4 + 2C_3x_3x_4,$$

for the other possibilities one has to rotate the indices appropriately.

Obviously sentences s_1 and s_4 are mutually exclusive. Note also that if $s_2 \wedge s_3$, then s_1. This means that we have ten cases to consider, depending on the truth values of the sentences s_1, s_2, s_3, s_4. Denote by $\neg s$ the negation of the sentence s.

Theorem 2.1 ([4, Theorem 2]). *Let $d : k[X] \to k[X]$ be a derivation of the form*

$$d = \sum_{i=1}^{4} x_i(x_{i-1} - C_ix_{i+1})\frac{\partial}{\partial x_i},$$

where $C_1, C_2, C_3, C_4 \in k$. Then the ring of constants of d is always finitely generated over k with at most three generators. In each case it is a polynomial ring, more precisely:

(1) *if $\neg s_1 \wedge \neg s_2 \wedge \neg s_3 \wedge \neg s_4$, then $k[X]^d = k$,*
(2) *if $s_1 \wedge \neg s_2 \wedge \neg s_3$, then $k[X]^d = k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4]$,*
(3) *if $\neg s_1 \wedge \neg s_2 \wedge \neg s_3 \wedge s_4$, then $k[X]^d = k[f_4]$,*
(4) *if $\neg s_1 \wedge \neg s_2 \wedge s_3 \wedge \neg s_4$, then $k[X]^d = k[x_2^t x_4^r]$,*
(5) *if $\neg s_1 \wedge s_2 \wedge \neg s_3 \wedge \neg s_4$, then $k[X]^d = k[x_1^q x_3^p]$,*
(6) *if $\neg s_1 \wedge \neg s_2 \wedge s_3 \wedge s_4$, then $k[X]^d = k[f_4, x_2^t x_4^r]$,*
(7) *if $\neg s_1 \wedge s_2 \wedge \neg s_3 \wedge s_4$, then $k[X]^d = k[f_4, x_1^q x_3^p]$,*
(8) *if $s_1 \wedge \neg s_2 \wedge s_3$, then $k[X]^d = k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4,\ x_2^t x_4^r]$,*
(9) *if $s_1 \wedge s_2 \wedge \neg s_3$, then $k[X]^d = k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4,\ x_1^q x_3^p]$,*
(10) *if $s_2 \wedge s_3$, then $k[X]^d = k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4,\ x_1^q x_3^p,\ x_2^t x_4^r]$.*

3. Darboux polynomials and rational constants

A polynomial $g \in k[X]$ is said to be *strict* if it is homogeneous and not divisible by the variables $x_1, \dots, x_n$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we denote by X^α the monomial $x_1^{\alpha_1} \dots x_n^{\alpha_n} \in k[X]$. Every nonzero homogeneous polynomial $f \in k[X]$ has a unique representation $f = X^\alpha g$, where X^α is a monomial and g is strict.

We call a nonzero polynomial $f \in k[X]$ a *Darboux polynomial* (or an *integral element*) of a derivation $\delta : k[X] \to k[X]$ if $\delta(f) = \Lambda f$ for some $\Lambda \in k[X]$. We will call Λ a *cofactor* of f. Since d is a homogeneous derivation of degree 1, the cofactor of each homogeneous form is a linear form. Denote by $k[X]_{(m)}$ the homogeneous component of $k[X]$ of degree m.

Lemma 3.1 ([11, Lemma 3.2]). *Let $n = 4$. Let $g \in k[X]_{(m)}$ be a Darboux polynomial of d with the cofactor $\lambda_1x_1 + \ldots + \lambda_4x_4$. Let $i \in \{1, 2, 3, 4\}$. If g is not divisible by x_i, then $\lambda_{i+1} \in \mathbb{N}$. More precisely, if $g(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_4) = x_{i+2}^{\beta_{i+2}}\overline{G}$ and $x_{i+2} \nmid \overline{G}$, then $\lambda_{i+1} = \beta_{i+2}$ and $\lambda_{i+3} = -C_{i+2}\lambda_{i+1}$.*

Corollary 3.2 ([11, Corollary 3.3])**.** *Let $n = 4$. If $g \in k[X]$ is a strict Darboux polynomial, then its cofactor is a linear form with coefficients in $\mathbb{N}$.*

For any derivation $\delta : k[X] \to k[X]$ there exists exactly one derivation $\bar{\delta} : k(X) \to k(X)$ such that $\bar{\delta}_{|k[X]} = \delta$. By a *rational constant* of the derivation $\delta : k[X] \to k[X]$ we mean the constant of its corresponding derivation $\bar{\delta} : k(X) \to k(X)$. The rational constants of δ form a field. For simplicity, we write δ instead of $\bar{\delta}$.

Lemma 3.3 ([10, Lemma 2])**.** *Let $n = 4$. The field $k(X)^d$ contains a nontrivial rational monomial constant if and only if at least one of the following two conditions is fulfilled:*

1. *$C_1, C_3 \in \mathbb{Q}$ and $C_1C_3 = 1$,*
2. *$C_2, C_4 \in \mathbb{Q}$ and $C_2C_4 = 1$.*

Proposition 3.4 ([8, Prop. 2.2.2])**.** *Let $\delta : k[X] \to k[X]$ be a derivation and let f and g be nonzero relatively prime polynomials from $k[X]$. Then $\delta(\frac{f}{g}) = 0$ if and only if f and g are Darboux polynomials of δ with the same cofactor.*

Proposition 3.5 ([8, Prop. 2.2.3])**.** *Let δ be a homogeneous derivation of $k[X]$ and let $f \in k[X]$ be a Darboux polynomial of δ with the cofactor $\Lambda \in k[X]$. Then Λ is homogeneous and each homogeneous component of f is also a Darboux polynomial of δ with the same cofactor Λ.*

Proposition 3.6 ([8, Prop 2.2.1])**.** *Let δ be a derivation of $k[X]$. Then $f \in k[X]$ is a Darboux polynomial of δ if and only if all factors of f are Darboux polynomials of δ. Moreover, if $f = f_1f_2$ is a Darboux polynomial, then sum of the cofactors of f_1 and f_2 equals the cofactor of f.*

4. Fields of rational constants of LV derivations

From now on, $n = 4$. For $C_1, C_2, C_3, C_4 \in k$ consider the sentences:

$$\tilde{s}_2 : \quad C_1, C_3 \in \mathbb{Q} \text{ and } C_1C_3 = 1.$$
$$\tilde{s}_3 : \quad C_2, C_4 \in \mathbb{Q} \text{ and } C_2C_4 = 1.$$

In case $\tilde{s}_2$ let $C_1 = \frac{p}{q}$, where $p, q \in \mathbb{Z}$, $q \neq 0$ and $\gcd(p, q) = 1$. In case $\tilde{s}_3$ let $C_2 = \frac{r}{t}$, where $r, t \in \mathbb{Z}$, $t \neq 0$ and $\gcd(r, t) = 1$. Note that these presentations of C_i are unique up to sign. Sentences s_1, s_2, s_3, s_4 and polynomial f_4 are as in Section 2.

Theorem 4.1. *Let $d : k(X) \to k(X)$ be a four-variable Lotka-Volterra derivation with parameters $C_1, C_2, C_3, C_4 \in k$. Then:*

1. *if $\neg s_1 \wedge \neg\tilde{s}_2 \wedge \neg\tilde{s}_3 \wedge \neg s_4$, then $k(X)^d = k$,*
2. *if $s_1 \wedge \neg\tilde{s}_2 \wedge \neg\tilde{s}_3$, then $k(X)^d = k(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4)$,*
3. *if $\neg s_1 \wedge \neg\tilde{s}_2 \wedge \neg\tilde{s}_3 \wedge s_4$, then $k(X)^d = k(f_4)$,*
4. *if $\neg s_1 \wedge \neg\tilde{s}_2 \wedge s_3 \wedge \neg s_4$, then $k(X)^d = k(x_2^t x_4^r)$,*
5. *if $\neg s_1 \wedge s_2 \wedge \neg\tilde{s}_3 \wedge \neg s_4$, then $k(X)^d = k(x_1^q x_3^p)$,*
6. *if $\neg s_1 \wedge \neg\tilde{s}_2 \wedge s_3 \wedge s_4$, then $k(X)^d = k(f_4, x_2^t x_4^r)$,*
7. *if $\neg s_1 \wedge s_2 \wedge \neg\tilde{s}_3 \wedge s_4$, then $k(X)^d = k(f_4, x_1^q x_3^p)$,*
8. *if $s_1 \wedge \neg\tilde{s}_2 \wedge s_3$, then $k(X)^d = k(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4,\ x_2^t x_4^r)$,*
9. *if $s_1 \wedge s_2 \wedge \neg\tilde{s}_3$, then $k(X)^d = k(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4,\ x_1^q x_3^p)$,*
10. *if $s_2 \wedge s_3$, then $k(X)^d = k(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4, x_1^q x_3^p, x_2^t x_4^r)$.*

Proof. All inclusions of the form $\supseteq$ follow from Theorem 2.1. Next we show the inclusions of the form $\subseteq$.

Let $\psi = \frac{f}{g} \in k(X)^d$, where $f, g \in k[X]\backslash\{0\}$ and $\gcd(f, g) = 1$. By Proposition 3.4 we have $d(f) = \Lambda f$ and $d(g) = \Lambda g$ for some $\Lambda \in k[X]$. Let $f = \sum f_j$ and $g = \sum g_j$, where f_j and g_j are homogeneous polynomials of degree j. By Proposition 3.5, since d is homogeneous, we have $d(f_j) = \Lambda f_j$ and $d(g_j) = \Lambda g_j$ for all $j \in \mathbb{N}$. Then, by Proposition 3.4 again, we have $d(\frac{f_j}{g_i}) = 0$ and $d(\frac{g_j}{g_i}) = 0$ for all i and j. Moreover, obviously

$$\frac{f}{g} = \frac{\sum_j \frac{f_j}{g_i}}{\sum_j \frac{g_j}{g_i}}$$

for some fixed i. Therefore it suffices to prove the assertion of Theorem 4.1 for homogeneous f and g.

Let $f = X^\alpha h$, where X^α is a monomial and h is strict (analogously we proceed for g). By Proposition 3.6 both X^α and h are Darboux polynomials of d. Let $\lambda = \lambda_1 x_1 + \ldots + \lambda_4 x_4$ be the cofactor of h. By Lemma 3.1 we have

$$\lambda_{i+3} = -C_{i+2}\lambda_{i+1} \tag{4.1}$$

for all i in the cyclic sense. Moreover, Corollary 3.2 gives $\lambda_i \in \mathbb{N}$ for $i = 1, \ldots, 4$.

Cases (1)–(3). Suppose that $\lambda_1 \neq 0$. Then (4.1) for $i = 2$ implies that also $\lambda_3 \neq 0$ and $C_4 = -\frac{\lambda_1}{\lambda_3} \in \mathbb{Q}$. Likewise, (4.1) for $i = 4$ gives $C_2 = -\frac{\lambda_3}{\lambda_1} \in \mathbb{Q}$. Therefore $C_2C_4 = 1$, which is a contradiction to $\neg\tilde{s}_3$. This proves that $\lambda_1 = 0$. Analogously we proceed for λ_2, λ_3 and λ_4. Hence we have $\lambda_1 = \ldots = \lambda_4 = 0$ and the only strict Darboux polynomials of d are constants of d. Note that $s_2 \Rightarrow \tilde{s}_2$. Hence $\neg\tilde{s}_2 \Rightarrow \neg s_2$. The same for s_3 and $\tilde{s}_3$. Thus, in view of Theorem 2.1, we have $h \in k$ or $h \in k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4]$ or $h \in k[f_4]$, respectively. Furthermore, by Proposition 3.6, the cofactor of X^α is equal to Λ, since the cofactor of h equals 0.

Similarly, $g = X^\beta l$, where $l \in k[X]^d$ and X^β is a Darboux monomial with the cofactor Λ. Then $\frac{X^\alpha}{X^\beta} \in k(X)^d$, by Proposition 3.4. In view of Lemma 3.3, $\frac{X^\alpha}{X^\beta} \in k$. Hence $\psi = c\frac{h}{l}$, where $c \in k$ and $h, l \in k[X]^d$. Thus $\psi \in k$ or $\psi \in k(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4)$ or $\psi \in k(f_4)$, respectively.

Cases (4), (6). As above $\lambda_2 = \lambda_4 = 0$. If, contrary to our claim, $\lambda_2 \neq 0$, then by (4.1) we have $\lambda_4 \neq 0$, $C_1 = -\frac{\lambda_2}{\lambda_4} \in \mathbb{Q}$ and $C_3 = -\frac{\lambda_4}{\lambda_2} \in \mathbb{Q}$, in contradiction with $\neg\tilde{s}_2$. By (4.1) for $i = 4$:

$$\lambda_3 = -C_2\lambda_1. \tag{4.2}$$

However, by Corollary 3.2 and by s_3, the left-hand side of (4.2) is nonnegative, whereas the right-hand side of (4.2) is nonpositive. Therefore $\lambda_3 = 0$ and, since $C_2 > 0$, we have $\lambda_1 = 0$. Then $h \in k[X]^d$. Since $\neg s_1 \wedge \neg\tilde{s}_2 \wedge s_3 \wedge \neg s_4 \Rightarrow \neg s_1 \wedge \neg s_2 \wedge s_3 \wedge \neg s_4$ and $\neg s_1 \wedge \neg\tilde{s}_2 \wedge s_3 \wedge s_4 \Rightarrow \neg s_1 \wedge \neg s_2 \wedge s_3 \wedge s_4$, we have case (4) or (6) of Theorem 2.1, respectively. Therefore we have $h \in k[x_2^t x_4^r]$ or $h \in k[f_4, x_2^t x_4^r]$, respectively. Moreover, the cofactor of X^α is equal to Λ.

Analogously, $g = X^\beta l$, where $l \in k[x_2^t x_4^r]$ and X^β is a Darboux monomial with the cofactor Λ. Then $\frac{X^\alpha}{X^\beta} \in k(X)^d$, by Proposition 3.4 again. Let $\frac{X^\alpha}{X^\beta} = x_1^a x_2^b x_3^c x_4^e$, where $a, b, c, e \in \mathbb{Z}$. Then

$$d\left(\frac{X^\alpha}{X^\beta}\right) = x_1^a x_2^b x_3^c x_4^e((b - eC_4)x_1 + (c - aC_1)x_2 + (e - bC_2)x_3 + (a - cC_3)x_4).$$

Since $d(\frac{X^\alpha}{X^\beta})=0$, we have two systems of linear equations:

$$\begin{aligned} b-eC_4&=0\\ e-bC_2&=0 \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} c-aC_1&=0\\ a-cC_3&=0. \end{aligned} \tag{4.4}$$

Since $\neg\tilde{s}_2$, we obtain $a=c=0$. Moreover, $e-b\frac{r}{t}=0$ implies $et=br$. Since $\gcd(r,t)=1$, we have $r\mid e$, and thus $e=jr$ for some $j\in\mathbb{Z}$. Therefore $br=jrt$, and since $r\neq 0$, we have $b=jt$. Consequently,

$$\frac{X^\alpha}{X^\beta}=x_2^{jt}x_4^{jr}=(x_2^tx_4^r)^j\in k(x_2^tx_4^r).$$

Thus $\psi=\frac{X^\alpha}{X^\beta}\frac{h}{l}$ belongs to $k(x_2^tx_4^r)$ or $k(f_4,x_2^tx_4^r)$, respectively.

Cases (5), (7). These two cases are completely analogous to cases (4) and (6), respectively.

Case (8). Similarly to case (4) we show that $\lambda_1=\dots=\lambda_4=0$. Therefore $h\in k[X]^d$. Since $s_1\wedge\neg\tilde{s}_2\wedge s_3\Rightarrow s_1\wedge\neg s_2\wedge s_3$, we have case (8) of Theorem 2.1. Hence, $h\in k[x_1+C_1x_2+C_1C_2x_3+C_1C_2C_3x_4,\ x_2^tx_4^r]$ and the cofactor of X^α is equal to Λ. Likewise, $g=X^\beta l$, where $l\in k[x_1+C_1x_2+C_1C_2x_3+C_1C_2C_3x_4,\ x_2^tx_4^r]$ and the cofactor of X^β equals Λ. In the same way as in case (4) we show that $\frac{X^\alpha}{X^\beta}\in k(x_2^tx_4^r)$. Finally, $\psi\in k(x_1+C_1x_2+C_1C_2x_3+C_1C_2C_3x_4,\ x_2^tx_4^r)$.

Case (9). It is entirely analogous to case (8).

Case (10). Since all C_i are positive, the the left-hand side of (4.1) is nonnegative and the right-hand side of (4.1) is nonpositive for $i=1,\dots 4$. Thus $\lambda_1=\dots=\lambda_4=0$ and $h\in k[X]^d$. Hence, by Theorem 2.1, we have $h\in k[x_1+C_1x_2+C_1C_2x_3+C_1C_2C_3x_4,\ x_1^qx_3^p,\ x_2^tx_4^r]$. Moreover, the cofactor of X^α equals Λ. Analogously, $g=X^\beta l$, where $l\in k[X]^d$ and the cofactor of X^β equals Λ. Then $\frac{X^\alpha}{X^\beta}\in k(X)^d$. If $\frac{X^\alpha}{X^\beta}=x_1^ax_2^bx_3^cx_4^e$, where $a,b,c,e\in\mathbb{Z}$, then we again obtain the systems of linear equations of the form (4.3) and (4.4). Similarly to case (4) we obtain $e=jr$, $b=jt$ for some $j\in\mathbb{Z}$ and $a=sq$, $c=sp$ for some $s\in\mathbb{Z}$. Thus $\frac{X^\alpha}{X^\beta}\in k(x_1^qx_3^p,\ x_2^tx_4^r)$. Consequently, $\psi\in k(x_1+C_1x_2+C_1C_2x_3+C_1C_2C_3x_4,\ x_1^qx_3^p,\ x_2^tx_4^r)$. $\square$

Note that Theorem 2.1 covers all the cases. Theorem 4.1 does not cover all the cases, however huge majority of them. Nevertheless, the following corollary covers all the cases.

Corollary 4.2. *If d is a four-variable Lotka-Volterra derivation, then $k(X)^d$ contains a nontrivial rational constant if and only if at least one of the following four conditions is fulfilled:*

1. $C_1C_2C_3C_4=1$,
2. $C_1,C_3\in\mathbb{Q}$ *and* $C_1C_3=1$,
3. $C_2,C_4\in\mathbb{Q}$ *and* $C_2C_4=1$,
4. $C_1C_2C_3C_4=-1$ *and* $C_i=1$ *for two consecutive indices* i.

Proof. If $\neg s_1\wedge\neg\tilde{s}_2\wedge\neg\tilde{s}_3\wedge\neg s_4$, then $k(X)^d=k$, by Theorem 4.1. If $\tilde{s}_2$ or $\tilde{s}_3$, then $k(X)^d\neq k$, by Lemma 3.3. If s_1 or s_4, then $k(X)^d\neq k$, by Theorem 4.1. $\square$

Note that if d is as in Theorem 4.1, then the field of rational constants equals the field of fractions of the ring of polynomial constants. Which is not true in general, even $k[X]$ may be trivial, while $k(X)$ nontrivial.

Example 4.3. Let $k \in \{\mathbb{R}, \mathbb{C}\}$. Let $d : k[X] \to k[X]$ be a derivation defined by

$$d(x_i) = x_i(x_{i-1} + x_{i+1}), \quad \text{for } i = 1, 3,$$
$$d(x_i) = x_i(x_{i-1} - \Pi x_{i+1}), \quad \text{for } i = 2, 4.$$

By Theorem 2.1, $k[X]^d = k$. Nevertheless, $\frac{x_1}{x_3} \in k(X)^d$.

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