Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 34, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

ESTIMATES FOR SOLUTIONS TO A CLASS OF NONLINEAR TIME-DELAY SYSTEMS OF NEUTRAL TYPE

GENNADII V. DEMIDENKO, INESSA I. MATVEEVA

Abstract. We consider nonlinear time-delay systems of neutral type with constant coefficients in the linear terms,

$$\frac{d}{dt}\big(y(t)+Dy(t-\tau)\big)=Ay(t)+By(t-\tau)+F(t,y(t),y(t-\tau)).$$

We obtain estimates characterizing the exponential decay of solutions at infinity, and dependending on the norms of the powers of D.

1. Introduction

There is large number of works devoted to the study of delay differential equations, see for instance [1, 2, 12, 14, 15, 16, 17, 18, 19, 26]. The question of asymptotic stability is very important from the theoretical and practical viewpoints, because delay differential equations arise in many applied problems when describing the processes whose rates of change are defined by present and previous states; see [13, 20, 22] and the bibliography therein.

This article presents a continuation of our work on stability of solutions to delay differential equations [4, 5, 6, 7, 8, 9, 10, 11, 23, 24]. We consider the system of nonlinear delay differential equations

$$\frac{d}{dt}\big(y(t)+Dy(t-\tau)\big)=Ay(t)+By(t-\tau)+F(t,y(t),y(t-\tau)),\quad t>0,\tag{1.1}$$

where A, B, D are constant $(n\times n)$ matrices, $\tau>0$ is the time delay, and $F(t,u,v)$ is a real-valued vector function satisfying the Lipschitz condition with respect to u, and the inequality

$$\|F(t,u,v)\|\le q_1\|u\|+q_2\|v\|,\tag{1.2}$$

for some constants $q_1,q_2\ge 0$. When $D\neq 0$ this system is called one of neutral type [12].

Our aim is to obtain new estimates on the exponential decay of solutions to (1.1) without finding roots of characteristic quasipolynomials defined by the linear part of (1.1) (when $F(t,u,v)\equiv 0$). In recent years, the study in this direction has developed rapidly. For constant coefficients, there are a lot of works for linear delay differential equations including equations of neutral type. It should be noted

---

2000 *Mathematics Subject Classification.* 34K20.
Key words and phrases. Time-delay systems; neutral type; exponential stability; Lyapunov-Krasovskii functional.

Submitted November 3, 2014. Published February 6, 2015.

that various Lyapunov-Krasovskii functionals are used for obtaining exponential estimates (see the bibliography in [16]).

The case of nonlinear equations is of special interest and is more complicated in comparison with the case of linear equations. Along with estimates of exponential decay of solutions, a very important question is deriving estimates for attraction sets of nonlinear equations. The natural problem is to obtain such estimates by means of the Lyapunov-Krasovskii functionals used for exponential stability analysis of equations defined by the linear part. To the best of our knowledge, the first constructive estimates of attraction sets for the system

$$\frac{d}{dt}y(t) = Ay(t) + By(t-\tau) + F(t, y(t), y(t-\tau)), \tag{1.3}$$

using Lyapunov-Krasovskii functionals associated with the exponentially stable linear system

$$\frac{d}{dt}y(t) = Ay(t) + By(t-\tau), \tag{1.4}$$

were obtained in [4, 5, 6, 25].

To study the asymptotic stability of solutions to (1.4), the authors in [4] proposed to use the Lyapunov-Krasovskii functional

$$\langle Hy(t), y(t)\rangle + \int_{t-\tau}^{t} \langle K(t-s)y(s), y(s)\rangle\, ds, \tag{1.5}$$

where the real matrices H and $K(s)$ satisfy

$$\begin{gathered} H = H^* > 0, \quad K(s) = K^*(s) \in C^1[0,\tau], \\ K(s) > 0, \quad \frac{d}{ds}K(s) < 0, \quad s \in [0,\tau], \end{gathered} \tag{1.6}$$

where $H > 0$ means that H is postive definite.

The usage of (1.5) allowed us to obtain estimates for the exponential decay of solutions to the linear system (1.4). The authors in [4, 5] considered (1.3), with

$$\|F(t,u,v)\| \le q_1\|u\|^{1+\omega_1} + q_2\|v\|^{1+\omega_2}, \quad q_1 \ge 0,\ q_2 \ge 0,\ \omega_1 \ge 0,\ \omega_2 \ge 0.$$

Using the functional in (1.5), conditions of asymptotic stability of the zero solution were obtained, estimates characterizing the decay rate at infinity were established, and estimates of attraction sets of the zero solution were derived. Using a generalization of the functional in (1.5), analogous results were obtained for linear and nonlinear systems of delay differential equations with periodic coefficients in the linear terms, see [4, 5, 6, 23, 24].

To study the exponential stability of solutions to the systems of linear differential equations

$$\frac{d}{dt}(y(t) + Dy(t-\tau)) = Ay(t) + By(t-\tau), \tag{1.7}$$

the first author in [7] introduced the Lyapunov-Krasovskii functional

$$\begin{aligned} V(\varphi) &= \langle H\big(\varphi(0) + D\varphi(-\tau)\big), (\varphi(0) + D\varphi(-\tau))\rangle \\ &\quad + \int_{-\tau}^{0} \langle K(-s)\varphi(s), \varphi(s)\rangle\, ds, \quad \varphi(s) \in C[-\tau, 0]\,, \end{aligned} \tag{1.8}$$

where the matrices H and $K(s)$ satisfy (1.6). In particular, the following result was obtained.

Theorem 1.1. *Suppose that there exist matrices H and $K(s)$ satisfying* (1.6) *and that the matrix*

$$C = -\begin{pmatrix} HA + A^*H + K(0) & HB + A^*HD \\ B^*H + D^*HA & D^*HB + B^*HD - K(\tau) \end{pmatrix} \tag{1.9}$$

is positive definite. Then the zero solution to (1.7) *is exponentially stable.*

Using the functional in (1.8), the study of exponential stability of solutions to (1.1) was conducted in [7, 8, 9, 10, 11]. There, conditions for exponential stability of the zero solution, estimates for the exponential decay of solutions at infinity, and estimates of attraction sets of the zero solution were obtained.

Note that in [7, 8] the estimates of exponential decay of solutions to (1.1) were obtained when $\|D\| < 1$ (here and thereafter we use the spectral norm of matrices). In [9], for the linear case ($F(t,u,v) \equiv 0$) analogous estimates were established when the spectrum of the matrix D belongs to the unit disk $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$. However, in the case of $\|D\| < 1$, the estimates are weaker in comparison with the estimates obtained in [7]. More precise exponential estimates for the linear systems were obtained in [10, 11]. Moreover, in [11] the authors established estimates of exponential decay of solutions of the linear time-delay systems of neutral type with periodic coefficients.

In this article we consider the nonlinear time-delay system (1.1) when the spectrum of the matrix D belongs to the unit disk. Our aim is to obtain estimates characterizing exponential decay of solutions at infinity dependending on the norms $\|D^j\|$.

2. Estimates of solutions

Consider the initial value problem for (1.1),

$$\begin{gathered} \frac{d}{dt}(y(t) + Dy(t-\tau)) = Ay(t) + By(t-\tau) + F(t, y(t), y(t-\tau)), \quad t > 0, \\ y(t) = \varphi(t), \quad t \in [-\tau, 0], \\ y(0+) = \varphi(0), \end{gathered} \tag{2.1}$$

where $\varphi(t) \in C^1[-\tau, 0]$ is a given vector function.

Suppose that the conditions of Theorem 1.1 are satisfied. Using the matrices H and $K(s)$, we introduce

$$S = \begin{pmatrix} -HA - A^*H - K(0) & HAD + K(0)D - HB \\ D^*A^*H + D^*K(0) - B^*H & K(\tau) - D^*K(0)D \end{pmatrix}, \tag{2.2}$$

$$q = \Big(q_1 + \sqrt{q_1^2 + (q_1\|D\| + q_2)^2}\Big)\|H\|, \tag{2.3}$$

$$\begin{aligned} R = -HA - A^*H - K(0) - qI - (HAD + K(0)D - HB)\Big[K(\tau) \\ - D^*K(0)D - qI\Big]^{-1}(HAD + K(0)D - HB)^*, \end{aligned} \tag{2.4}$$

where I is the unit matrix. It is not hard to verify that the matrix C in (1.9) is positive definite if and only if the matrix S is positive definite. Note that R is positive definite if the matrix $S - qI$ is positive definite.

Theorem 2.1. *Let the conditions of Theorem 1.1 be satisfied. Suppose that the parameters* q_1, q_2 *are such that the matrix* $S - qI$ *is positive definite. Let* $k > 0$ *be the maximal number such that*

$$\frac{d}{ds}K(s) + kK(s) \le 0, \quad s \in [0, \tau]\,. \tag{2.5}$$

Let $r_{\min} > 0$ *be the minimal eigenvalue of the matrix* R. *Then, each solution to* (2.1) *satisfies*

$$\|y(t) + Dy(t-\tau)\| \le \sqrt{\frac{V(\varphi)}{h_{\min}}}\exp\Big(-\frac{\gamma t}{2\|H\|}\Big), \quad t > 0, \tag{2.6}$$

where $V(\varphi)$ *is defined by* (1.8), $h_{\min} > 0$ *is the minimal eigenvalue of the matrix* H, *and*

$$\gamma = \min\{r_{\min},\ k\|H\|\} > 0. \tag{2.7}$$

Proof. We follow the strategy in [4]. Let $y(t)$ be a solution to (2.1). Using the matrices H and $K(s)$ indicated in Theorem 1.1, we consider the Lyapunov-Krasovskii functional defined in (1.8). Introducing the conventional notation

$$y_t : \theta \to y(t+\theta), \quad \theta \in [-\tau, 0],$$

we have

$$\begin{aligned} V(y_t) &= \langle H(y_t(0) + Dy_t(-\tau)), (y_t(0) + Dy_t(-\tau))\rangle + \int_{-\tau}^{0} \langle K(-\theta)y_t(\theta), y_t(\theta)\rangle\, d\theta \\ &= \langle H(y(t) + Dy(t-\tau)), (y(t) + Dy(t-\tau))\rangle + \int_{t-\tau}^{t} \langle K(t-s)y(s), y(s)\rangle\, ds. \end{aligned}$$

The time derivative of this functional is

$$\begin{aligned} \frac{d}{dt}V(y_t) \equiv\ & \langle H(Ay(t) + By(t-\tau)), (y(t) + Dy(t-\tau))\rangle \\ &+ \langle H(y(t) + Dy(t-\tau)), (Ay(t) + By(t-\tau))\rangle \\ &+ \langle HF(t, y(t), y(t-\tau)), (y(t) + Dy(t-\tau))\rangle \\ &+ \langle H(y(t) + Dy(t-\tau)), F(t, y(t), y(t-\tau))\rangle \\ &+ \langle K(0)y(t), y(t)\rangle - \langle K(\tau)y(t-\tau), y(t-\tau)\rangle \\ &+ \int_{t-\tau}^{t} \big\langle \frac{d}{dt}K(t-s)y(s), y(s)\big\rangle ds. \end{aligned}$$

Using the matrix C defined in (1.9), we obtain

$$\begin{aligned} \frac{d}{dt}V(y_t) \equiv\ & -\Big\langle C\begin{pmatrix} y(t) \\ y(t-\tau)\end{pmatrix}, \begin{pmatrix} y(t) \\ y(t-\tau)\end{pmatrix}\Big\rangle \\ &+ \langle HF(t, y(t), y(t-\tau)), (y(t) + Dy(t-\tau))\rangle \\ &+ \langle H(y(t) + Dy(t-\tau)), F(t, y(t), y(t-\tau))\rangle \\ &+ \int_{t-\tau}^{t} \big\langle \frac{d}{dt}K(t-s)y(s), y(s)\big\rangle ds. \end{aligned} \tag{2.8}$$

Consider the first summand in the right-hand side of (2.8). Since

$$\begin{pmatrix} y(t) \\ y(t-\tau)\end{pmatrix} = \begin{pmatrix} I & -D \\ 0 & I\end{pmatrix}\begin{pmatrix} y(t) + Dy(t-\tau) \\ y(t-\tau)\end{pmatrix},$$

it follows that
$$\Big\langle C\begin{pmatrix} y(t)\\ y(t-\tau)\end{pmatrix},\begin{pmatrix} y(t)\\ y(t-\tau)\end{pmatrix}\Big\rangle \equiv \Big\langle S\begin{pmatrix} y(t)+Dy(t-\tau)\\ y(t-\tau)\end{pmatrix},\begin{pmatrix} y(t)+Dy(t-\tau)\\ y(t-\tau)\end{pmatrix}\Big\rangle,$$
where
$$S=\begin{pmatrix} I & 0\\ -D^* & I\end{pmatrix} C\begin{pmatrix} I & -D\\ 0 & I\end{pmatrix}=\begin{pmatrix} S_{11} & S_{12}\\ S_{12}^* & S_{22}\end{pmatrix}$$
which is defined in (2.2).

Now we consider the second and the third summands in the right-hand side of (2.8). In view of (1.2) we have
$$\begin{aligned}
&\langle HF(t,y(t),y(t-\tau)),(y(t)+Dy(t-\tau))\rangle\\
&+\langle H(y(t)+Dy(t-\tau)),F(t,y(t),y(t-\tau))\rangle\\
&\le 2\|H\|\big(q_1\|y(t)\|+q_2\|y(t-\tau)\|\big)\|y(t)+Dy(t-\tau)\|\\
&\le 2q_1\|H\|\|y(t)+Dy(t-\tau)\|^2+2(q_1\|D\|+q_2)\|H\|\|y(t-\tau)\|\|y(t)+Dy(t-\tau)\|\\
&\le q\big(\|y(t)+Dy(t-\tau)\|^2+\|y(t-\tau)\|^2\big),
\end{aligned}$$
where q is given in (2.3). Hence,
$$\begin{aligned}
&-\Big\langle C\begin{pmatrix} y(t)\\ y(t-\tau)\end{pmatrix},\begin{pmatrix} y(t)\\ y(t-\tau)\end{pmatrix}\Big\rangle+\langle HF(t,y(t),y(t-\tau)),(y(t)+Dy(t-\tau))\rangle\\
&+\langle H(y(t)+Dy(t-\tau)),F(t,y(t),y(t-\tau))\rangle\\
&\le -\Big\langle (S-qI)\begin{pmatrix} y(t)+Dy(t-\tau)\\ y(t-\tau)\end{pmatrix},\begin{pmatrix} y(t)+Dy(t-\tau)\\ y(t-\tau)\end{pmatrix}\Big\rangle.
\end{aligned}\tag{2.9}$$
By the conditions of Theorem 2.1, the matrix $S-qI$ is positive definite. Using the representation
$$\begin{aligned}
S-qI&=\begin{pmatrix} I & S_{12}(S_{22}-qI)^{-1}\\ 0 & I\end{pmatrix}\begin{pmatrix} S_{11}-qI-S_{12}(S_{22}-qI)^{-1}S_{12}^* & 0\\ 0 & S_{22}-qI\end{pmatrix}\\
&\quad\times\begin{pmatrix} I & 0\\ (S_{22}-qI)^{-1}S_{12}^* & I\end{pmatrix},
\end{aligned}$$
we have
$$\begin{aligned}
&\Big\langle (S-qI)\begin{pmatrix} y(t)+Dy(t-\tau)\\ y(t-\tau)\end{pmatrix},\begin{pmatrix} y(t)+Dy(t-\tau)\\ y(t-\tau)\end{pmatrix}\Big\rangle\\
&\ge\langle [S_{11}-qI-S_{12}(S_{22}-qI)^{-1}S_{12}^*](y(t)+Dy(t-\tau)),(y(t)+Dy(t-\tau))\rangle.
\end{aligned}$$
Since the matrix $S-qI$ is positive definite, the matrix
$$R=S_{11}-qI-S_{12}(S_{22}-qI)^{-1}S_{12}^*$$
is positive definite. Taking into account (2.2), the matrix R has the form (2.4). Consequently, from (2.9) we obtain
$$\begin{aligned}
&-\Big\langle C\begin{pmatrix} y(t)\\ y(t-\tau)\end{pmatrix},\begin{pmatrix} y(t)\\ y(t-\tau)\end{pmatrix}\Big\rangle\\
&+\langle HF(t,y(t),y(t-\tau)),(y(t)+Dy(t-\tau))\rangle\\
&+\langle H(y(t)+Dy(t-\tau)),F(t,y(t),y(t-\tau))\rangle\\
&\le -\langle R(y(t)+Dy(t-\tau)),(y(t)+Dy(t-\tau))\rangle\\
&\le -r_{\min}\|y(t)+Dy(t-\tau)\|^2,
\end{aligned}\tag{2.10}$$

where $r_{\min} > 0$ is the minimal eigenvalue of R. Using the matrix H, we have

$$\|y(t) + Dy(t-\tau)\|^2 \geq \frac{1}{\|H\|}\langle H(y(t) + Dy(t-\tau)), (y(t) + Dy(t-\tau))\rangle.$$

By (2.10), from (2.8) we derive

$$\begin{aligned}\frac{d}{dt}V(y_t) \leq &-\frac{r_{\min}}{\|H\|}\langle H(y(t) + Dy(t-\tau)), (y(t) + Dy(t-\tau))\rangle \\ &+ \int_{t-\tau}^{t} \Big\langle \frac{d}{dt}K(t-s)y(s), y(s)\Big\rangle ds.\end{aligned}$$

Using (2.5), we have

$$\begin{aligned}\frac{d}{dt}V(y_t) \leq &-\frac{r_{\min}}{\|H\|}\langle H(y(t) + Dy(t-\tau)), (y(t) + Dy(t-\tau))\rangle \\ &- k\int_{t-\tau}^{t} \langle K(t-s)y(s), y(s)\rangle ds.\end{aligned}$$

Taking into account the definition of the functional (1.8), we obtain

$$\frac{d}{dt}V(y_t) \leq -\frac{\gamma}{\|H\|}V(y_t),$$

where $\gamma = \min\{r_{\min}, k\|H\|\} > 0$. From this differential inequality we obtain the estimate

$$V(y_t) \leq V(\varphi)\exp\Big(-\frac{\gamma t}{\|H\|}\Big).$$

Clearly,

$$\|y(t) + Dy(t-\tau)\|^2 \leq \frac{1}{h_{\min}}\langle H(y(t) + Dy(t-\tau)), (y(t) + Dy(t-\tau))\rangle,$$

where $h_{\min}$ is the minimal eigenvalue of H. Then, using the definition of the functional in (1.8), we have

$$\|y(t) + Dy(t-\tau)\| \leq \sqrt{\frac{V(y_t)}{h_{\min}}} \leq \sqrt{\frac{V(\varphi)}{h_{\min}}}\exp\Big(-\frac{\gamma t}{2\|H\|}\Big).$$

The proof is complete. $\square$

In the next theorem, based on (2.6), we prove estimates for the solution to (2.1). These estismates will be used for proving our main results. We introduce the following values:

$$\alpha = \sqrt{\frac{V(\varphi)}{h_{\min}}}, \quad \beta = \frac{\gamma}{2\|H\|}, \quad \Phi = \max_{s\in[-\tau,0]}\|\varphi(s)\|. \tag{2.11}$$

Theorem 2.2. *Let the conditions of Theorem 2.1 be satisfied. Then, on each segment $t \in [k\tau, (k+1)\tau)$, $k = 0, 1, \dots$, the solution y to* (2.1) *satisfies*

$$\|y(t)\| \leq \alpha\sum_{j=0}^{k}\|D^j\|e^{-\beta(t-j\tau)} + \|D^{k+1}\|\Phi, \tag{2.12}$$

where α, β, and Φ are defined in (2.11).

Proof. Obviously, taking into account (2.11), by (2.6) for $t \in [0, \tau)$ we have the inequality

$$\|y(t)\| \le \alpha e^{-\beta t} + \|Dy(t-\tau)\| \le \alpha e^{-\beta t} + \|D\|\Phi$$

which gives us (2.12) for $k = 0$. Let $t \in [k\tau, (k+1)\tau)$, $k = 1, 2, \dots$. It is not hard to show the sequence of the inequalities

$$\begin{aligned}
\|y(t)\| &\le \alpha e^{-\beta t} + \|Dy(t-\tau)\| \\
&\le \alpha e^{-\beta t} + \|Dy(t-\tau) + D^2 y(t-2\tau)\| + \|D^2 y(t-2\tau) + D^3 y(t-3\tau)\| + \dots \\
&\quad + \|D^k y(t-k\tau) + D^{k+1} y(t-(k+1)\tau)\| + \|D^{k+1} y(t-(k+1)\tau)\| \\
&\le \alpha e^{-\beta t} + \|D\|\,\|y(t-\tau) + Dy(t-2\tau)\| + \|D^2\|\,\|y(t-2\tau) + Dy(t-3\tau)\| \\
&\quad + \dots + \|D^k\|\,\|y(t-k\tau) + Dy(t-(k+1)\tau)\| + \|D^{k+1}\|\,\|y(t-(k+1)\tau)\|.
\end{aligned}$$

By (2.6) we derive the estimate

$$\begin{aligned}
\|y(t)\| \le \alpha e^{-\beta t} &+ \alpha\|D\| e^{-\beta(t-\tau)} + \alpha\|D^2\| e^{-\beta(t-2\tau)} + \dots \\
&+ \alpha\|D^k\| e^{-\beta(t-k\tau)} + \|D^{k+1}\|\Phi
\end{aligned}$$

which implies (2.12). The proof is complete. □

Next we obtain estimates for solutions to (2.1) on the whole half-line $\{t > 0\}$. Analogy as in [7], we distinguish three cases allowing us to obtain more precise estimates. Since the spectrum of the matrix D belongs to the unit disk $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$, it follows that $\|D^j\| \to 0$ as $j \to \infty$. Let $l > 0$ be the minimal integer such that $\|D^l\| < 1$. In Theorems 2.3–2.5 below we establish estimates if

$$\|D^l\| < e^{-l\beta\tau}, \quad \|D^l\| = e^{-l\beta\tau}, \quad e^{-l\beta\tau} < \|D^l\| < 1,$$

respectively, where $\beta = \frac{\gamma}{2\|H\|}$, with γ defined in (2.7).

Theorem 2.3. *Assume that*

$$\|D^l\| < e^{-l\beta\tau}. \tag{2.13}$$

Then the solution to the initial value problem (2.1) *satisfies*

$$\|y(t)\| \le \Big[\alpha\big(1 - \|D^l\| e^{l\beta\tau}\big)^{-1} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \max\{\|D\| e^{\beta\tau}, \dots, \|D^l\| e^{l\beta\tau}\}\Phi\Big] e^{-\beta t}, \tag{2.14}$$

for $t > 0$, where α, β, and Φ are defined in (2.11).

Proof. Using (2.12), on each segment $t \in [k\tau, (k+1)\tau)$, $k = 0, 1, \dots$, one can write the inequality

$$\|y(t)\| \le \Big[\alpha \sum_{j=0}^{k} \|D^j\| e^{j\beta\tau} + \|D^{k+1}\| e^{(k+1)\beta\tau}\Phi\Big] e^{-\beta t}.$$

In view of the condition on $\|D^l\|$, we obtain the estimate on the whole half-line $\{t > 0\}$,

$$\|y(t)\| \le \Big[\alpha \sum_{j=0}^{\infty} \|D^j\| e^{j\beta\tau} + \max\big\{\|D\| e^{\beta\tau}, \dots, \|D^l\| e^{l\beta\tau}\big\}\,\Phi\Big] e^{-\beta t}. \tag{2.15}$$

Consider the series $\sum_{j=0}^{\infty}\|D^j\|e^{j\beta\tau}$. Obviously,

$$\begin{aligned}
&\sum_{j=0}^{\infty}\|D^j\|e^{j\beta\tau}\\
&=\sum_{j=0}^{l-1}\|D^j\|e^{j\beta\tau}+\sum_{j=l}^{2l-1}\|D^j\|e^{j\beta\tau}+\sum_{j=2l}^{3l-1}\|D^j\|e^{j\beta\tau}+\dots\\
&\le\sum_{j=0}^{l-1}\|D^j\|e^{j\beta\tau}+\|D^l\|e^{l\beta\tau}\sum_{j=0}^{l-1}\|D^j\|e^{j\beta\tau}+(\|D^l\|e^{l\beta\tau})^2\sum_{j=0}^{l-1}\|D^j\|e^{j\beta\tau}+\dots\\
&=\Big(1+\|D^l\|e^{l\beta\tau}+(\|D^l\|e^{l\beta\tau})^2+\dots\Big)\sum_{j=0}^{l-1}\|D^j\|e^{j\beta\tau}.
\end{aligned}$$

Since $\|D^l\|e^{l\beta\tau}<1$, by (2.13), we have

$$\sum_{j=0}^{\infty}\|D^j\|e^{j\beta\tau}\le\big(1-\|D^l\|e^{l\beta\tau}\big)^{-1}\sum_{j=0}^{l-1}\|D^j\|e^{j\beta\tau}.$$

Using this inequality, from (2.15) we derive the required estimate (2.14). □

Theorem 2.4. *Assume that*

$$\|D^l\|=e^{-l\beta\tau}. \tag{2.16}$$

Then the solution to the initial value problem (2.1) *satisfies*

$$\begin{aligned}
\|y(t)\|\le\Big[\alpha\big(1+\frac{t}{l\tau}\big)\sum_{j=0}^{l-1}\|D^j\|e^{j\beta\tau}+\max\{1,\|D\|e^{\beta\tau},\dots,\\
\|D^{l-1}\|e^{(l-1)\beta\tau}\}\Phi\Big]e^{-\beta t},\quad t>0,
\end{aligned}\tag{2.17}$$

where α, β, and Φ are defined in (2.11).

Proof. By Theorem 2.2, the solution to (2.1) satisfies (2.12) on each segment $t\in[k\tau,(k+1)\tau)$, $k=0,1,\dots$. Consequently,

$$\|y(t)\|\le\Big[\alpha\sum_{j=0}^{k}\|D^j\|e^{j\beta\tau}+\|D^{k+1}\|e^{(k+1)\beta\tau}\Phi\Big]e^{-\beta t}.$$

Taking into account condition (2.16) on $\|D^l\|$, we obtain

$$\|y(t)\|\le\Big[\alpha\sum_{j=0}^{k}\|D^j\|e^{j\beta\tau}+\max\{1,\|D\|e^{\beta\tau},\dots,\|D^{l-1}\|e^{(l-1)\beta\tau}\}\Phi\Big]e^{-\beta t}. \tag{2.18}$$

If $k\le l-1$, then (2.17) follows from (2.18) for $t\in[0,l\tau)$.

Let $l\le k\le 2l-1$; i.e., $1\le\frac{t}{l\tau}<2$. Consider the sum $\sum_{j=0}^{k}\|D^j\|e^{j\beta\tau}$. Clearly,

$$\begin{aligned}
\sum_{j=0}^{k}\|D^j\|e^{j\beta\tau}&=\sum_{j=0}^{l-1}\|D^j\|e^{j\beta\tau}+\sum_{j=l}^{k}\|D^j\|e^{j\beta\tau}\\
&\le\sum_{j=0}^{l-1}\|D^j\|e^{j\beta\tau}+\|D^l\|e^{l\beta\tau}\sum_{j=0}^{k-l}\|D^j\|e^{j\beta\tau}
\end{aligned}$$

$$= \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \sum_{j=0}^{k-l} \|D^j\| e^{j\beta\tau}.$$

Then we have

$$\sum_{j=0}^{k} \|D^j\| e^{j\beta\tau} \le \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \frac{t}{l\tau} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}.$$

By this inequality, (2.17) follows from (2.18) for $t \in [l\tau, 2l\tau)$.

Let $ml \le k \le (m+1)l - 1$, $m = 2, 3, \dots$; i.e., $m \le \frac{t}{l\tau} < m+1$. Consider the sum $\sum_{j=0}^{k} \|D^j\| e^{j\beta\tau}$. It is not difficult to see that

$$\begin{aligned}
&\sum_{j=0}^{k} \|D^j\| e^{j\beta\tau} \\
&= \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \sum_{j=l}^{2l-1} \|D^j\| e^{j\beta\tau} + \dots + \sum_{j=ml}^{k} \|D^j\| e^{j\beta\tau} \\
&\le \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \|D^l\| e^{l\beta\tau} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \dots + \|D^{ml}\| e^{ml\beta\tau} \sum_{j=0}^{k-ml} \|D^j\| e^{j\beta\tau} \\
&\le \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \dots + \sum_{j=0}^{k-ml} \|D^j\| e^{j\beta\tau} \\
&\le (1+m) \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}.
\end{aligned}$$

Consequently,

$$\sum_{j=0}^{k} \|D^j\| e^{j\beta\tau} \le \big(1 + \frac{t}{l\tau}\big) \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}.$$

In view of this estimate, (2.17) follows from (2.18) for $t \in [ml\tau, (m+1)l\tau)$. Owing to arbitrariness of m, (2.17) is valid for all $t > 0$. $\square$

Theorem 2.5. *Assume that*

$$e^{-l\beta\tau} < \|D^l\| < 1. \tag{2.19}$$

Then the solution to the initial value problem (2.1) *satisfies*

$$\begin{aligned}
\|y(t)\| \le &\Big[\alpha \|D^l\| e^{l\beta\tau} (\|D^l\| e^{l\beta\tau} - 1)^{-1} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} \\
&+ \|D^l\|^{\frac{1}{l}-1} \max\{1, \|D\|, \dots, \|D^{l-1}\|\} \Phi\Big] \exp\big(\frac{t}{l\tau} \ln \|D^l\|\big),
\end{aligned} \tag{2.20}$$

for $t > 0$, where α, β, and Φ are defined in (2.11).

Proof. In view of Theorem 2.2, a solution to (2.1) satisfies (2.12) on each segment $t \in [k\tau, (k+1)\tau)$, $k = 0, 1, \dots$.

At first we consider the first summand in the right-hand side of (2.12). For $k \le l-1$ we obviously have

$$\sum_{j=0}^{k} \|D^j\| e^{j\beta\tau} \le \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}.$$

Let $ml \le k \le (m+1)l-1$, $m = 1, 2, 3, \dots$. Clearly,

$$\begin{aligned}
&\sum_{j=0}^{k} \|D^j\| e^{j\beta\tau} \\
&= \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \sum_{j=l}^{2l-1} \|D^j\| e^{j\beta\tau} + \dots + \sum_{j=ml}^{k} \|D^j\| e^{j\beta\tau} \\
&\le \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \|D^l\| e^{l\beta\tau} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \dots + \|D^{ml}\| e^{ml\beta\tau} \sum_{j=0}^{k-ml} \|D^j\| e^{j\beta\tau} \\
&\le \big[1 + \|D^l\| e^{l\beta\tau} + \dots + \|D^l\|^m e^{ml\beta\tau}\big] \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\sum_{j=0}^{k} \|D^j\| e^{j\beta\tau} \\
&\le \|D^l\|^m e^{ml\beta\tau} [1 + (\|D^l\| e^{l\beta\tau})^{-1} + \dots + (\|D^l\| e^{l\beta\tau})^{-m}] \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} \\
&\le \|D^l\|^m e^{ml\beta\tau} [1 + (\|D^l\| e^{l\beta\tau})^{-1} + \dots + (\|D^l\| e^{l\beta\tau})^{-m} + \dots] \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}.
\end{aligned}$$

Since $\|D^l\| e^{l\beta\tau} > 1$ owing to (2.19),

$$\sum_{j=0}^{k} \|D^j\| e^{j\beta\tau} \le \|D^l\|^m e^{ml\beta\tau} \big[1 - (\|D^l\| e^{l\beta\tau})^{-1}\big]^{-1} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}.$$

Taking into account that $ml\tau \le t < (m+1)l\tau$, we obtain

$$\begin{aligned}
\sum_{j=0}^{k} \|D^j\| e^{-\beta(t-j\tau)} &\le \|D^l\|^m e^{-\beta(t-ml\tau)} \big[1 - \big(\|D^l\| e^{l\beta\tau}\big)^{-1}\big]^{-1} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} \\
&\le \|D^l\|^{\frac{t}{l\tau}} \big[1 - \big(\|D^l\| e^{l\beta\tau}\big)^{-1}\big]^{-1} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}.
\end{aligned}$$

As result, we derive the estimate for the first summand in (2.12) for every k,

$$\begin{aligned}
&\alpha \sum_{j=0}^{k} \|D^j\| e^{-\beta(t-j\tau)} \\
&\le \alpha \big[1 - \big(\|D^l\| e^{l\beta\tau}\big)^{-1}\big]^{-1} \Big(\sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}\Big) \exp\big(\frac{t}{l\tau} \ln \|D^l\|\big).
\end{aligned} \tag{2.21}$$

We now consider the second summand in the right-hand side of (2.12). Obviously, for $0 \le k \le l-2$, we have
$$\|D^{k+1}\| \le \max\{\|D\|, \dots, \|D^{l-1}\|\}.$$
Let $ml-1 \le k \le (m+1)l-2$, $m = 1, 2, \dots$. Hence,
$$\|D^{k+1}\| \le \|D^l\|^m \|D^{k+1-ml}\| \le \|D^l\|^m \max\left\{1, \|D\|, \dots, \|D^{l-1}\|\right\}.$$
Since $\|D^l\| < 1$ and $t < ((m+1)l-1)\tau$,
$$\|D^l\|^m \le \|D^l\|^{\frac{t-(l-1)\tau}{l\tau}} = \|D^l\|^{\frac{1}{l}-1} \exp\Big(\frac{t}{l\tau} \ln \|D^l\|\Big).$$
Owing to arbitrariness of m, we infer that
$$\|D^{k+1}\| \le \|D^l\|^{\frac{1}{l}-1} \max\left\{1, \|D\|, \dots, \|D^{l-1}\|\right\} \exp\big(\frac{t}{l\tau} \ln \|D^l\|\big)$$
for every k. Taking into account the estimate (2.21) for the first summand in the right-hand side of (2.12), we derive (2.20). $\square$

We remark that the results obtained above give us the assertions on robust stability for (1.7). Indeed, consider uncertain systems of the form
$$\frac{d}{dt}\left(y(t) + Dy(t-\tau)\right) = Ay(t) + By(t-\tau) + \Delta A(t)y(t) + \Delta B(t)y(t-\tau), \quad (2.22)$$
where $\Delta A(t)$ and $\Delta B(t)$ are unknown $(n \times n)$ matrices such that
$$\|\Delta A(t)\| \le q_1, \quad \|\Delta B(t)\| \le q_2.$$
Obviously, in this case the vector function
$$F(t, u, v) = \Delta A(t)u + \Delta B(t)v$$
satisfies (1.2). Then Theorem 2.1 gives us the conditions of robust exponential stability for (1.7). From Theorems 2.3–2.5 we have the estimates of exponential decay of solutions to (2.22).

3. Illustrative examples

Consider the system (1.1), where
$$D = \begin{pmatrix} -0.1 & 0 \\ 0 & -0.1 \end{pmatrix}, \quad A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix},$$
a is a real parameter, $F(t, u, v)$ is a real-valued vector function satisfying the Lipschitz condition with respect to u and the inequality (1.2).

First we consider the linear case ($F(t, u, v) \equiv 0$); i.e. $q_1 = q_2 = 0$. In [27], in the case of arbitrary positive τ, stability was shown for $|a| < 0.4$. The same system was studied in [21], where stability was established for $|a| < 0.533$. In [3] exponential stability was shown for $|a| \le 0.6213$. Moreover, in the case of $a = 0.6213$ and $\tau = 1$, the following estimate for solutions was obtained
$$\|y(t)\| \le \Big(c_1\|y(0)\| + c_2 \sup_{-1\le s\le 0} \|y(s)\| + c_3 \sup_{-1\le s\le 0} \|\frac{d}{ds} y(s)\|\Big) e^{-0.00001559\, t/2},$$
with $c_j > 0$. In the same case, using our results, we establish the following inequality
$$\|y(t)\| \le d \max_{-1\le s\le 0} \|y(s)\| e^{-0.147\, t/2}, \quad d > 0. \quad (3.1)$$

Indeed, we choose the matrices H and $K(s)$ as follows

$$H=\begin{pmatrix}0.3 & 0.2\\ 0.2 & 0.8\end{pmatrix},\quad K(s)=e^{-ks}K_0,\quad k=0.147,\quad K_0=\begin{pmatrix}0.8 & 0.2\\ 0.2 & 0.2\end{pmatrix}.$$

Obviously, these matrices satisfy (1.6) and (2.5). Since the matrix

$$C=\begin{pmatrix}0.6 & 0.2 & -0.19426 & -0.16639\\ 0.2 & 0.6 & -0.55704 & -0.16426\\ -0.19426 & -0.55704 & 0.7154872 & 0.2410018\\ -0.16639 & -0.16426 & 0.2410018 & 0.1975108\end{pmatrix}$$

is positive definite, then by Theorem 1.1 the zero solution to the system is exponentially stable. To establish (3.1) we need to calculate, for $q=0$, the matrix R, its minimal eigenvalue $r_{\min}$, $\|H\|$, and $\beta=\frac{1}{2}\min\{\frac{r_{\min}}{\|H\|},k\}$. In our case

$$R=\begin{pmatrix}0.4741402 & 0.1208201\\ 0.1208201 & 0.1704705\end{pmatrix},\quad r_{\min}=0.1282659,$$
$$\|H\|=0.8701562,\quad \beta=\frac{1}{2}\min\{0.1474056,0.147\}=\frac{0.147}{2}.$$

Since $\|D\|<e^{-\beta\tau}$, by Theorem 2.3 we have (3.1).

It should be noted that, using the same matrices H and K_0, it is not hard to establish exponential stability in the case of arbitrary positive τ for $-0.9\le a\le 0.78$. It is enough to take, for example, $k=0.015/\tau$. Changing slightly H and K_0, the boundaries for a may be enlarged.

We now consider the case of $F(t,u,v)\not\equiv 0$. Let $a=0.6213$, $\tau=1$, $q_1=0.01$, $q_2=0.02$. As mentioned above, in [8] the authors established estimates of exponential decay for solutions of systems of the form (1.1) in the case of $\|D\|<1$. Using [8, Theorem 2], one can write down the inequality

$$\|y(t)\|\le d_1\max_{-1\le s\le 0}\|y(s)\|e^{-\beta_1 t},\quad d_1>0, \tag{3.2}$$

where

$$\beta_1=\frac{1}{2}\min\Big\{\frac{c_{\min}}{(1+\|D\|^2)\|H\|}-\frac{\big(q_1+\|D\|q_2+\sqrt{(1+\|D\|^2)(q_1^2+q_2^2)}\big)}{(1+\|D\|^2)},k\Big\},$$

$c_{\min}$ is the minimal eigenvalue of C defined by (1.9). Choosing the same matrices H, K_0, and $k=0.1$, we have

$$C=\begin{pmatrix}0.6 & 0.2 & -0.19426 & -0.16639\\ 0.2 & 0.6 & -0.55704 & -0.16426\\ -0.19426 & -0.55704 & 0.7487219 & 0.2493105\\ -0.16639 & -0.16426 & 0.2493105 & 0.2058195\end{pmatrix},$$
$$c_{\min}=0.0660526,\quad \beta_1=\frac{1}{2}\min\{0.0410265,0.1\}=\frac{0.0410265}{2}.$$

At the same time, by Theorem 2.3 we have the estimate

$$\|y(t)\|\le d_2\max_{-1\le s\le 0}\|y(s)\|e^{-\beta t},\quad d_2>0, \tag{3.3}$$

where $\beta=0.0991651/2$. Indeed, in our case

$$R=\begin{pmatrix}0.4247753 & 0.1334548\\ 0.1334548 & 0.1389063\end{pmatrix},\quad r_{\min}=0.0862891.$$

Consequently,

$$\beta = \min\{\frac{r_{\min}}{\|H\|}, k\} = \frac{1}{2}\min\{0.0991651, 0.1\} = \frac{0.0991651}{2}.$$

Obviously, (3.3) is more strong than (3.2) because β characterizing the exponential decay rate of the solutions to (1.1) at infinity is larger than β_1.

All the numerical computations were performed by using Scilab 5.5.1.

Acknowledgments. The authors are grateful to the anonymous referee for the helpful comments and suggestions.

The authors were supported by the Russian Foundation for Basic Research (project no. 13-01-00329) and the Siberian Branch of the Russian Academy of Sciences (interdisciplinary project no. 80).

References

[1] R. P. Agarwal, L. Berezansky, E. Braverman, A. Domoshnitsky; *Nonoscillation Theory of Functional Differential Equations with Applications*, Springer, New York, 2012.

[2] N. V. Azbelev, V. P. Maksimov, L. F. Rakhmatullina; *Introduction to the Theory of Functional Differential Equations: Methods and Applications*, Contemporary Mathematics and Its Applications, **3**, Hindawi Publishing Corporation, Cairo, 2007.

[3] J. Baštinec, J. Diblík, D. Ya. Khusainov, A. Ryvolová; *Exponential stability and estimation of solutions of linear differential systems of neutral type with constant coefficients*, Bound. Value Probl. 2010, Art. ID 956121, 20 pp.

[4] G. V. Demidenko, I. I. Matveeva; *Asymptotic properties of solutions to delay differential equations*, Vestnik Novosib. Gos. Univ. Ser. Mat. Mekh. Inform., **5** (2005), 20–28 (Russian).

[5] G. V. Demidenko, I. I. Matveeva; *Matrix process modelling: Asymptotic properties of solutions of differential-difference equations with periodic coefficients in linear terms*, Proceedings of the Fifth International Conference on Bioinformatics of Genome Regulation and Structure (Novosibirsk, Russia, July 16–22, 2006). Novosibirsk: Institute of Cytology and Genetics, **3** (2006), 43–46.

[6] G. V. Demidenko, I. I. Matveeva; *Stability of solutions to delay differential equations with periodic coefficients of linear terms*, Siberian Math. J., **48** (2007), 824–836.

[7] G. V. Demidenko; *Stability of solutions to linear differential equations of neutral type*, J. Anal. Appl., **7** (2009), 119–130.

[8] G. V. Demidenko, T. V. Kotova, M. A. Skvortsova; *Stability of solutions to differential equations of neutral type*, Vestnik Novosib. Gos. Univ. Ser. Mat. Mekh. Inform., **10** (2010), 17–29 (Russian); English transl. in: J. Math. Sci., **186** (2012), 394–406.

[9] G. V. Demidenko, E. S. Vodop'yanov, M. A. Skvortsova; *Estimates of solutions to the linear differential equations of neutral type with several delays of the argument*, J. Appl. Indust. Math., **7** (2013), 472–479.

[10] G. V. Demidenko, I. I. Matveeva; *On exponential stability of solutions to one class of systems of differential equations of neutral type*, J. Appl. Indust. Math., **8** (2014), 510–520.

[11] G. V. Demidenko, I. I. Matveeva; *On estimates of solutions to systems of differential equations of neutral type with periodic coefficients*, Siberian Math. J., **55** (2014), 866–881.

[12] L. E. El'sgol'ts, S. B. Norkin; *Introduction to the Theory and Application of Differential Equations with Deviating Arguments*, Academic Press, New York, London, 1973.

[13] T. Erneux; *Applied Delay Differential Equations*, Surveys and Tutorials in the Applied Mathematical Sciences, **3**, Springer, New York, 2009.

[14] K. Gu, V. L. Kharitonov, J. Chen; *Stability of Time-Delay Systems*, Control Engineering, Boston, Birkhäuser, 2003.

[15] J. K. Hale; *Theory of Functional Differential Equations*, Springer-Verlag, New York, Heidelberg, Berlin, 1977.

[16] V. L. Kharitonov; *Time-Delay Systems. Lyapunov Functionals and Matrices*, Control Engineering, Birkhauser, Springer, New York, 2013.

[17] D. Ya. Khusainov, A. V. Shatyrko; *The Method of Lyapunov Functions in the Investigation of the Stability of Functional Differential Systems*, Kiev University Press, Kiev, 1997 (Russian).

[18] V. B. Kolmanovskii, A. D. Myshkis; *Introduction to the Theory and Applications of Functional Differential Equations*, Mathematics and its Applications, **463**, Kluwer Academic Publishers, Dordrecht, 1999.
[19] D. G. Korenevskii; *Stability of Dynamical Systems under Random Perturbations of Parameters. Algebraic Criteria*, Naukova Dumka, Kiev, 1989.
[20] Y. Kuang; *Delay Differential Equations with Applications in Population Dynamics*, Mathematics in Science and Engineering, **191**, Academic Press, Boston, 1993.
[21] X.-X. Liu, B. Xu; *A further note on stability criterion of linear neutral delay-differential systems*, J. Franklin Inst., **343** (2006), 630–634.
[22] N. MacDonald; *Biological Delay Systems: Linear Stability Theory*, Cambridge Studies in Mathematical Biology, **8**, Cambridge University Press, Cambridge, 1989.
[23] I. I. Matveeva, A. A. Shcheglova; *Some estimates of the solutions to time-delay differential equations with parameters*, J. Appl. Indust. Math., **5** (2011), 391–399.
[24] I. I. Matveeva; *Estimates of solutions to a class of systems of nonlinear delay differential equations*, J. Appl. Indust. Math., **7** (2013), 557–566.
[25] D. Melchor-Aguilar, S.-I. Niculescu; *Estimates of the attraction region for a class of nonlinear time-delay systems*, IMA J. Math. Control Inform., **24** (2007), 523–550.
[26] W. Michiels, S.-I. Niculescu; *Stability and Stabilization of Time-Delay Systems. An Eigenvalue-Based Approach*, Advances in Design and Control, **12**, Philadelphia, Society for Industrial and Applied Mathematics, 2007.
[27] Ju.-H. Park, S. Won; *A note on stability of neutral delay-differential systems*, J. Franklin Inst., **336** (1999), 543–548.

Gennadii V. Demidenko
Laboratory of Differential and Difference Equations, Sobolev Institute of Mathematics, 4, Acad. Koptyug avenue, Novosibirsk 630090, Russia.
Department of Mechanics and Mathematics, Novosibirsk State University, 2, Pirogov street, Novosibirsk 630090, Russia
E-mail address: `demidenk@math.nsc.ru`

Inessa I. Matveeva
Laboratory of Differential and Difference Equations, Sobolev Institute of Mathematics, 4, Acad. Koptyug avenue, Novosibirsk 630090, Russia.
Department of Mechanics and Mathematics, Novosibirsk State University, 2, Pirogov street, Novosibirsk 630090, Russia
E-mail address: `matveeva@math.nsc.ru`