NONUNIQUENESS OF SOLUTIONS OF INITIAL-VALUE PROBLEMS FOR PARABOLIC p-LAPLACIAN

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Abstract. We construct a positive solution to a quasilinear parabolic problem in a bounded spatial domain with the $p$-Laplacian and a nonsmooth reaction function. We obtain nonuniqueness for zero initial data. Our method is based on sub- and supersolutions and the weak comparison principle.

Using the method of sub- and supersolutions we construct a positive solution to a quasilinear parabolic problem with the $p$-Laplacian and a reaction function that is non-Lipschitz on a part of the spatial domain. Thereby we obtain nonuniqueness for zero initial data.

1. Introduction

The problem of uniqueness and nonuniqueness of solutions to various types of initial (and boundary) value problems for quasilinear parabolic equations has been an interesting research topic for several decades (see, e.g., Fujita and Watanabe [3] and the references therein, Guedda [4], Ladyzhenskaya and Ural’tseva [6], and Oleinik and Kruzhkov [10]).

In this work we focus on the following problem with the $p$-Laplacian and a (partly) nonsmooth reaction function:

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta_p u &= q(x)|u|^{\alpha-1}u \quad \text{for } (x,t) \in \Omega \times (0,T) ; \\
    u(x,t) &= 0 \quad \text{for } (x,t) \in \partial \Omega \times (0,T) , \\
    u(x,0) &= 0 \quad \text{for } x \in \Omega.
\end{aligned}
\]

(1.1)

Here, $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$ denotes the $p$-Laplacian for $1 < p < \infty$, $\alpha \in (0,1)$ is a given number, $0 < T < \infty$, and the potential $q$ satisfies

(Q) $q \in C(\Omega)$, $q \geq 0$, and $q(x_0) > 0$ for some $x_0 \in \Omega$.

We assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with a $C^{1+\mu}$-boundary $\partial \Omega$ where $\mu \in (0,1)$.

In particular, we deal with degenerate (singular) diffusion if $2 < p < \infty$ ($1 < p < 2$, respectively) and the reaction function $f(x,u) := q(x)|u|^{\alpha-1}u$. Notice that if $q(x_0) > 0$ then the function $u \mapsto f(x_0,u)$ satisfies neither a local Lipschitz nor
an Osgood (see [11]) condition near \( u = 0 \) provided \( \alpha \in (0, 1) \). The case \( p = 2 \)
(the Laplace operator) was treated in Fujita and Watanabe [3] by entirely different
methods based on the Green’s function for the heat equation. An important special
case, \( N = 1, 1 < p < \infty \), and \( q(x) \equiv \lambda > 0 \) (a constant), was treated in Guedda
[4] also by different methods. The main purpose of the present article is to fill
in the gap left open for \( 1 < p < \infty \) also by different methods. The main purpose of the present article is to fill
in the gap left open for \( 1 < p < \infty \), \( p \neq 2 \), and \( q \in C(\Omega) \), \( q \geq 0 \), where \( q \) is not
necessarily positive everywhere in \( \Omega \). Because of this possibly nonuniform positivity
of \( q \) over \( \Omega \), the method used in [4] cannot be applied here. We use a different
approach based on sub- and supersolutions and the weak comparison principle.

As a trivial consequence of the fact that problem (1.1) possesses a nontrivial non-
negative solution (see our main result, Theorem 1), we conclude that the weak
comparison principle does not hold for problem (1.1) considered with nontrivial
initial conditions, say, in \( W^{1,p}_0(\Omega) \).

Observe that our assumption (Q) implies that there exists \( R > 0 \) such that
\( q(x) \geq q_0 \equiv \text{const} > 0 \) for all \( x \in B_R(x_0) \) where
\[
B_R(x_0) := \{ x \in \mathbb{R}^N : |x - x_0| < R \} \subset \Omega.
\]

Let \( (\lambda_1, \varphi_{1,R}) \) denote the first eigenpair for the operator
\( -\Delta_p : W^{1,p}_0(B_R(x_0)) \rightarrow W^{-1,p'}(B_R(x_0)) \); that is,
\[
\begin{align*}
-\Delta_p \varphi_{1,R} &= \lambda_1 R \varphi_{1,R}^{-1} \quad \text{in } B_R(x_0); \\
\varphi_{1,R} &= 0 \quad \text{on } \partial B_R(x_0),
\end{align*}
\]
and \( \varphi_{1,R} \in W^{1,p}_0(B_R(x_0)) \) is normalized by \( \varphi_{1,R}(x_0) = 1 \). Note that this normalization yields
\( 0 < \varphi_{1,R}(x) \leq 1 \) for all \( x \in B_R(x_0) \). Moreover, we denote by
\[
\overline{\varphi}_{1,R}(x) := \begin{cases} 
\varphi_{1,R}(x) & \text{for } x \in B_R(x_0) \\
0 & \text{for } x \in \Omega \backslash B_R(x_0),
\end{cases}
\]
the natural zero extension of \( \varphi_{1,R} \) from \( B_R(x_0) \) to the whole of \( \Omega \). Our main
theorem is the following nonuniqueness result.

**Theorem 1.1.** Assume that \( 0 < \alpha < \min\{1, p - 1\} \) and (Q) are satisfied. Then
there exists \( T > 0 \) small enough, such that problem (1.1) possesses (besides the
trivial solution \( u \equiv 0 \)) a nontrivial, nonnegative weak solution
\[
u \in C([0, T] \rightarrow L^2(\Omega)) \cap L^p((0, T) \rightarrow W^{1,p}(\Omega))
\]
which is bounded below by a subsolution \( \underline{u} : \Omega \times (0, T) \rightarrow \mathbb{R}_+ \) of type
\[
\underline{u}(x, t) = \theta(t)\overline{\varphi}_{1,R}(x)^\beta \geq 0 \quad \text{in } \Omega \times (0, T),
\]
where \( \theta : [0, T] \rightarrow \mathbb{R}_+ \) is a strictly increasing, continuously differentiable function
with \( \theta(0) = 0 \), and \( \beta \in (1, \infty) \) is a suitable number.

In contrast with this nonuniqueness result, several uniqueness results have been
established in [2].

**Remark 1.2.** Assume that \( q \in L^\infty(\Omega) \) satisfies \( 0 \leq q(x) \leq \lambda_1 \) a.e. in \( \Omega \),
where \( \lambda_1 \) stands for the principal eigenvalue of \( -\Delta_p \) with zero Dirichlet boundary conditions
on \( \Omega \). Then the condition \( \alpha < p - 1 \) is essential for obtaining our nonuniqueness
result. Namely, if \( \alpha = p - 1 \) then \( u \equiv 0 \) is the unique weak solution of \( (1.1) \). The uniqueness follows directly from the following standard energy estimate:
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u(x,t)|^2 \, dx + \int_{\Omega} |\nabla u|^p \, dx = \int_{\Omega} q(x)|u|^p \, dx \leq \lambda_1 \int_{\Omega} |u|^p \, dx.
\]
By the variational characterization of \( \lambda_1 \) (Poincaré’s inequality in Lindqvist [8]), we get
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u(x,t)|^2 \, dx \leq -\int_{\Omega} |\nabla u|^p \, dx + \lambda_1 \int_{\Omega} |u|^p \, dx \leq 0,
\]
which implies \( u(x,t) \equiv 0 \) in \( \Omega \times (0,T) \), thanks to \( u(x,0) \equiv 0 \) in \( \Omega \).

A weaker result than our Theorem 1.1 [1] has recently been published in Merchán, Montoro, and Peral [9, Theorem 2.2, p. 248]. There, a very strong uniform positivity condition on the potential \( q \) is assumed, \( q_0 = \inf_{\Omega} q > 0 \). This means that it suffices to treat the constant case \( q(x) \equiv q_0 = \text{const} > 0 \) and then use the resulting solution as a subsolution for the general case \( q(x) \geq q_0 = \text{const} > 0 \). In contrast, our Theorem 1.1 above does not assume \( q_0 > 0 \); we assume only \( q \geq 0 \) and \( q \not\equiv 0 \) in \( \Omega \). Nevertheless, our proof of this result, especially our construction of a nonzero subsolution, is simpler than in [9].

2. Proof of Theorem 1.1

Note that \( \varphi_{1,R}^\beta \) defined in (1.3) is continuous on \( \overline{\Omega} \) and \( \varphi_{1,R}^\beta \) is continuously differentiable for any constant \( \beta > 1 \). We need to establish a few additional properties of \( \varphi_{1,R}(x) \equiv \varphi_{1,R}(|x-x_0|) = \varphi_{1,R}(r) \), with \( r = |x-x_0| \) and the usual harmless abuse of notation.

**Lemma 2.1.** If \( \beta \in (0, \infty) \) then
\[
-\Delta_p \left( \varphi_{1,R}^\beta \right) = \beta^{p-1} \varphi_{1,R}^{(p-1)(\beta-1)-1} \left[ \lambda_{1,R} \varphi_{1,R}^p - (p-1)(\beta-1)|\nabla \varphi_{1,R}|^p \right] \quad (2.1)
\]
holds pointwise a.e. in \( B_R(x_0) \). In particular, for \( \beta \geq 1 \) we have
\[
-\Delta_p \left( \varphi_{1,R}^\beta \right) \leq C \equiv \text{const} < \infty \quad \text{pointwise a.e. in } B_R(x_0). \quad (2.2)
\]

**Proof.** Any function \( u: B_R(x_0) \to \mathbb{R} \) that is radially symmetric around \( x_0 \) can be written as \( u(x) = u(r) \) where \( r = |x-x_0| \). Using this notation we obtain, by formal differentiation,
\[
\Delta_p u(|x-x_0|) = \text{div} \left( |u'(r)|^{p-2} u'(r) \frac{x-x_0}{r} \right)
= \left( |u'(r)|^{p-2} u'(r) \right)' + \frac{N-1}{r} |u'(r)|^{p-2} u'(r). \quad (2.3)
\]
It is well-known that the first eigenfunction \( \varphi_{1,R} \) is radially symmetric around \( x_0 \), positive, and \( C^2 \) in \( \overline{B_R(x_0)} \setminus \{x_0\} \), see e.g. [1]. Therefore, we get a.e. in \( B_R(x_0) \),
\[
\Delta_p \left( \varphi_{1,R}^\beta \right) = \beta^{p-1} \varphi_{1,R}^{(p-1)(\beta-1)} |\varphi_{1,R}'|^{p-2} \varphi_{1,R}'
+ \frac{N-1}{r} \beta^{p-1} \varphi_{1,R}^{(p-1)(\beta-1)} |\varphi_{1,R}'|^{p-2} \varphi_{1,R}'
\]
Proof.
We will show that the following inequality holds
\[
\frac{\partial u}{\partial t} - \Delta_p u \leq q(x)|u|^{p-1}u.
\]
Using \( 0 < \alpha < \min\{1, p - 1\} \), equation (2.5), and the continuity of \( \theta : [0, S] \to \mathbb{R}_+ \), we get
\[
\frac{d\theta}{dt} \leq -C\theta(t)^{p-1} + q_0\theta(t)^\alpha \quad \text{for all } t \in [0, \varrho],
\]
where $\sigma \in (0, \bar{\sigma})$ is small enough, such that $\theta(t)^{p-1-\alpha} \leq q_0/(2C)$ holds for all $t \in [0, \sigma]$.

Inserting the inequality
$$\varphi_{1,R}^\beta \Delta_p(\varphi_{1,R}^\beta) \geq -C \equiv \text{const}$$
in $\Omega$ from Lemma 2.1, inequality (2.2), into (2.6), we obtain
$$\frac{d\theta}{dt} \leq \varphi_{1,R}^{-\beta} \Delta_p(\varphi_{1,R}^\beta) \theta(t)^{p-1} + q_0 \theta(t)^\alpha$$
$$\leq \varphi_{1,R}^{-\beta} \Delta_p(\varphi_{1,R}^\beta) \theta(t)^{p-1} + q_0 \varphi_{1,R}^{(\alpha-1)\beta} \theta(t)^\alpha,$$
thanks{0} to the normalization $0 < \varphi_{1,R} \leq 1$ in $B_R(x_0)$ combined with $(\alpha - 1)\beta < 0$.

Finally, multiplying by $\varphi_{1,R}^\beta$, we arrive at
$$\frac{d\theta}{dt} \varphi_{1,R}^\beta \leq \Delta_p(\varphi_{1,R}^\beta) \theta(t)^{p-1} + q_0 \theta(t)^\alpha \varphi_{1,R}^\alpha$$
$$\leq \Delta_p(\varphi_{1,R}^\beta) \theta(t)^{p-1} + q(x) \theta(t)^\alpha \varphi_{1,R}^\alpha.$$ 

This inequality, combined with our definition of the function $\varphi_{1,R}$, guarantees that
$$\bar{u}(x, t) = \theta(t) \varphi_{1,R}(x)$$
is a subsolution to problem (1.1). □

**Proof of Theorem 1.1.** First, let us observe that $\bar{u}(x, t) = \|q\|_\infty^{1/\alpha} t$ is a supersolution of (1.1) for $0 < t \leq 1$. Indeed, a straightforward calculation shows that
$$\frac{\partial \bar{u}}{\partial t} - \Delta_p \bar{u} = \|q\|_\infty^{1/\alpha} \geq q(x) \left(\|q\|_\infty^{1/\alpha} t\right)^\alpha = q(x) \|\bar{u}\|^{\alpha-1} \bar{u}$$
holds for $0 < t \leq 1$, since $q \in C(\bar{\Omega})$, $q \geq 0$, and $\|q\|_\infty = \sup_{x \in \Omega} q(x)$.

Second, we show now that $\underline{u} \leq \bar{u}$ for all $x \in \Omega$ and all $t > 0$ sufficiently small, say, $0 < t \leq \bar{\sigma}$. Evidently,
$$\underline{u}(x, t) = \theta(t) \varphi_{1}(x) \beta = c_1 t^{1-\alpha} \varphi_{1}^\beta \leq c_1 t^{1-\alpha} \bar{u}(x, t) = \|q\|_\infty^{1/\alpha} t$$
for $0 < t \leq \bar{\sigma}$, where $\bar{\sigma}$ satisfies
$$\bar{\sigma}^\alpha \leq \|q\|_\infty/c_1^{1-\alpha}.$$

Now it remains to show the existence of weak solution $u$ for (1.1), such that
$$\underline{u} \leq u \leq \bar{u} \quad \text{in } \Omega \times (0, T), \quad \text{where } T := \min\{\bar{\sigma}, \bar{\sigma}\} > 0.$$

Let us define a sequence of functions $u_n: \Omega \times (0, T) \rightarrow \mathbb{R}$ recursively for $n = 1, 2, 3, \ldots$, such that $u_n$ is the unique weak solution of
$$\frac{\partial u_n}{\partial t} - \Delta_p u_n = q(x)|u_{n-1}|^{\alpha-1} u_{n-1}, \quad (x, t) \in \Omega \times (0, T),$$
$$u_n(x, 0) = 0, \quad x \in \Omega,$$
$$u_n(x, t) = 0, \quad (x, t) \in \partial \Omega \times (0, T),$$
with $u_0 = \underline{u}$. By a weak solution of (2.7), we mean a Lebesgue-measurable function $u_n: \Omega \times (0, T) \rightarrow \mathbb{R}$ that satisfies
$$u_n \in C([0, T] \rightarrow L^2(\Omega)) \cap L^p((0, T) \rightarrow W^{1,p}_0(\Omega))$$
and the equation
\[
\int_{\Omega} u_n(x,t)\phi(x,t) \, dx - \int_{0}^{t} \int_{\Omega} u_n(x,s) \frac{\partial \phi}{\partial t}(x,s) \, dx \, ds
+ \int_{0}^{t} \int_{\Omega} |\nabla u_n(x,s)|^{p-2} \nabla u_n(x,s) \cdot \nabla \phi(x,s) \, dx \, ds
= \int_{0}^{t} \int_{\Omega} q(x)|u_{n-1}(x,s)|^{\alpha-1}u_{n-1}(x,s)\phi(x,s) \, dx \, ds
\]
(2.8)
for every \( t \in (0,T) \) and every test function \( \phi \in C([0,T] \to L^2(\Omega)) \cap L^p((0,T) \to W_0^{1,p}(\Omega)) \cap W^{1,p'}((0,T) \to W^{-1,p'}(\Omega)) \).

The questions of existence and uniqueness of weak solutions of problems of type (2.7) obtained by monotone iterations have been discussed in [12, Appendix A, §A.1]. Let us deduce from the fact that \( u_0 = \underline{u} \) is a subsolution of (1.1) the inequalities \( u_{n-1} \leq u_n \) in \( \Omega \times (0,T) \) for every \( n = 1,2,3,\ldots \). The proof is by induction on \( n \). The first inequality, \( u_0 \leq u_1 \) in \( \Omega \times (0,T) \), holds by the Weak Comparison Principle (see [12, Lemma 4.9, p. 618]) and the fact that \( u_0 = \underline{u} \) is a subsolution of (1.1). Now assume that \( u_{n-1} \leq u_n \) in \( \Omega \times (0,T) \) for some \( n \in \mathbb{N} \). Then we have
\[
\frac{\partial u_n}{\partial t} - \Delta_p u_n = |u_{n-1}|^{\alpha-1}u_{n-1} \leq |u_n|^{\alpha-1}u_n = \frac{\partial u_{n+1}}{\partial t} - \Delta_p u_{n+1}
\]
in \( \Omega \times (0,T) \) and consequently \( u_n \leq u_{n+1} \) in \( \Omega \times (0,T) \) again, by [12, Lemma 4.9, p. 618]. Therefore, monotonicity holds: \( \underline{u} = u_0 \leq u_1 \leq u_2 \leq \cdots \leq \overline{u} \) in \( \Omega \times (0,T) \). The comparison with the supersolution \( \overline{u} \) is deduced again from the Weak Comparison Principle. Hence, \( u_n \) is uniformly bounded in \( \Omega \times (0,T) \) by \( \underline{u} \leq u \leq \overline{u} \). A global regularity result from [7, Theorem 0.1, p. 552] (cf. [12, Lemma 4.6, p. 617]) guarantees \( u_n \in C^{1+\gamma,\frac{\gamma}{2}}(\overline{\Omega} \times [0,T]) \) uniformly for \( n \in \mathbb{N} \), where \( \gamma \in (0,1) \) is independent of \( n \). We follow the notations and definitions of Hölder spaces of functions on \( \Omega \times [0,T] \) from [5] Chpt. 1, p. 7. Thus, by the Arzelà-Ascoli theorem, \( \{u_n\} \) is relatively compact in \( C^{1,0}(\overline{\Omega} \times [0,T]) \). Hence, the sequence \( \{u_n\} \) possesses a subsequence which converges to \( u \in C^{1,0}(\overline{\Omega} \times [0,T]) \). Therefore, in the weak formulation of (2.8) we may pass to the limit as \( n \to \infty \), thus verifying that the limit function \( u \) is a weak solution of (1.1) in \( \Omega \times (0,T) \), such that \( \underline{u} \leq u \leq \overline{u} \).

Acknowledgments. All authors were partially supported by a joint exchange program between the Czech Republic and Germany: By the Ministry of Education, Youth, and Sports of the Czech Republic under the grant No. 7AMB14DE005 (exchange program “MOBILITY”) and by the Federal Ministry of Education and Research of Germany under grant No. 57063847 (D.A.A.D. Program “PPP”).

The research of Peter Takáč was partially supported also by the German Research Society (D.F.G.), grant No. TA 213 / 15-1 and the research of Vladimir E. Bobkov was supported by the German Research Society (D.F.G.), grant No. TA 213 / 16-1 (doctoral fellow).

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