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EXISTENCE OF POSITIVE SOLUTIONS FOR SUPERLINEAR p-LAPLACIAN EQUATIONS

TING-MEI GAO, CHUN-LEI TANG

ABSTRACT. We obtain a positive solution for a superlinear p-Laplacian equations with the Dirichlet boundary-value conditions. Our main tool is a variation of the mountain pass theorem.

1. Introduction and statement of the main result

We consider the nonlinear elliptic equation of p-Laplacian type

$$\begin{gathered} -\Delta_p u = f(x,u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator with $p > 1$, Ω is a bounded domain in $R^N (N \geq 1)$ with smooth boundary $\partial\Omega$. The function $f \in C(\overline{\Omega} \times R, R)$ satisfies the following conditions:

(F1) f is subcritical in t, that is, there is a $q \in (p, Np/(N-p))$ when $N > p$; $q \in (p, +\infty)$ when $N \leq p$ such that
$$\lim_{t\to+\infty} \frac{f(x,t)}{t^{q-1}} = 0 \quad \text{uniformly in a.e. } x \in \Omega.$$

(F2)
$$b_0 \leq \liminf_{t\to 0^+} \frac{f(x,t)}{t^{p-1}} \leq \limsup_{t\to 0^+} \frac{f(x,t)}{t^{p-1}} \leq a(x)$$
uniformly in a.e. $x \in \Omega$, where b_0 is a constant, $a \in L^\infty(\Omega)$ satisfies $a(x) \leq \lambda_1$ for all $x \in \overline{\Omega}$ and $a(x) < \lambda_1$ on some $\Omega_1 \subset \Omega$ with $|\Omega_1| > 0$, λ_1 is the first eigenvalue of $-\triangle_p$ and $|\Omega_1|$ is the measure of Ω_1.

(F3) $\lim_{t\to+\infty} f(x,t)/t^{p-1} = +\infty$ uniformly in a.e. $x \in \Omega$.

In this article, we study the existence of a positive solution to (1.1) under the above assumptions. Since (F3) hods, problem (1.1) is called superlinear in t at $+\infty$. In many studies involving this superlinear problem, to get a nontrivial solution of (1.1), a very famous theorem - Mountain pass theorem is a common tool, but in

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applying this theorem, usually, we have to suppose another condition, that is, for some $\mu > p, M > 0$

$$0 < \mu F(x,t) \le f(x,t)t \quad \text{for a.e. } x \in \overline{\Omega} \text{ and for all } |t| \ge M. \tag{1.2}$$

The condition (1.2) is convenient, but it is very restrictive, in particular, it implies (F3). To overcome this difficulty, many efforts have been made. Wang and Tang [8] proved the following existence theorem without condition (1.2).

Theorem 1.1. *Suppose f is subcritical in t and assume that* (F3) *and the following conditions hold:*

- (F2') $\limsup_{t\to 0^+} \frac{f(x,t)}{t^{p-1}} = c(x)$ *uniformly in a.e.* $x \in \Omega$, *where* $c \in L^\infty(\Omega)$ *satisfies* $c(x) \le \lambda_1$ *for all* $x \in \overline{\Omega}$ *and* $c(x) < \lambda_1$ *on some* $\Omega' \subset \Omega$ *with* $|\Omega'| > 0$.
- (F4') *There exists* $\theta \ge 1$ *such that* $\theta H(x,t) \ge H(x,\xi t)$ *for all* $x \in \overline{\Omega}, t \in R$ *and* $\xi \in [0,1]$, *where* $H(x,t) = f(x,t)t - pF(x,t)$ *and* $F(x,t) = \int_0^t f(x,s)ds$.
- (H1) $f(x,t) \ge 0$ *for all* $t \ge 0, x \in \overline{\Omega}$.

Then (1.1) *has at least one positive solution.*

Assumptions (F4') was first introduced by Jeanjean in [3] for $p = 2$, Liu and Li in [5] extended it for general $p > 1$, it is helpful for proving that the functional corresponding to problem (1.1) satisfies Cerami condition (C).

Recently by a monotony condition instead of (1.2), Iturriage and Lorca [2] obtained the following result.

Theorem 1.2. *Suppose $f(x,t)$ is subcritical in t and satisfies the following:*

- (G1) $f : \overline{\Omega} \times [0,+\infty) \to [0,+\infty)$ *is a Carathéodory function.*
- (G2) $\limsup_{t\to 0^+} \frac{f(x,t)}{t^{p-1}} = 0$ *uniformly in a.e.* $x \in \Omega$.
- (G3) *for any* $M > 0$, *there exists* c_0 *such that*
$$f(x,t) \ge M|t|^{p-1} - c_0, \quad \text{for all } t \ge 0 \text{ and } x \in \overline{\Omega}.$$
- (G4) *there exists* $R > 0$ *such that the map* $t \mapsto f(x,t)t^{1-p}$ *is non-decreasing if* $t > R$ *for a.e.* $x \in \overline{\Omega}$.

Then (1.1) *has a positive solution.*

Assumptions (G1)–(G3) and the subcritical condition ensure that the energy functional associated with (1.1) has mountain pass geometry. Assumption (G4) allow us to show the boundedness of the (PS)-sequence at the mountain pass level. Motivated by Theorem 1.1 and 1.2, we assume a new condition and obtain the following result.

Theorem 1.3. *Suppose* (F1)–(F3) *hold and assume*

- (F4) *There exist two constants* $\theta \ge 1, \theta_0 > 0$ *such that* $\theta H(x,s) \ge H(x,t) - \theta_0$ *for all* $x \in \overline{\Omega}, 0 \le t \le s$, *where* $H(x,t) = f(x,t)t - pF(x,t)$ *and* $F(x,t) = \int_0^t f(x,s)ds$.

Then (1.1) *has at least one positive solution.*

Remark 1.4. Theorem 1.3 unifies and generalizes Theorems 1.1 and 1.2 mainly in four aspects:

Firstly, it is obvious that the conditions (F2') in Theorem 1.1, and (G2) in Theorem 1.2 are stronger than our condition (F2).

Secondly, the assumption (G3) can imply our condition (F3).

Thirdly, it is easily proved that (F4') implies (F4) and we also can show (G4) implies (F4) (see Remark 1.5), that is to say, (F4) is weaker than (F4') and (G4).

In addition, as many studies, to prove that a nontrivial solution of problem (1.1) is positive, Theorems 1.1 and 1.2 require $f(x,t) \geq 0$ for all $t \geq 0, x \in \overline{\Omega}$, in this article, we do not need it any more, in fact, the condition $b_0 \leq \liminf_{t\to 0^+} \frac{f(x,t)}{t^{p-1}}$ in (F2) together with (F3) are sufficient for showing that a nontrivial solution of (1.1) is positive.

Moreover, we can find some functions that satisfy the conditions of Theorem 1.3, but not the conditions of Theorems 1.1 and 1.2. For instance,

$$f(x,t) = \begin{cases} a(x)t^{q-1} - b(x)t^{p-1}, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

with $a, b \in C(\overline{\Omega})$, $a > 0$, $b > 0$ and $q \in (p, Np/(N-p))$. It is obvious that f satisfies (F1)–(F3). Let $\theta = 1, \theta_0 = 0$, then we also can check that f satisfies (F4), and so f satisfies all the conditions of Theorem 1.3, but it does not satisfy the assumptions of Theorems 1.1 and 1.2 (in fact, if $t > 0$ is small enough, $f(x,t) < 0$).

Remark 1.5. Condition (G4) implies condition (F4).

Proof. (1) Claim: $H(x,t)$ is nondecreasing on $(R, +\infty)$ for a.e. $x \in \overline{\Omega}$. In fact, if we assume $0 < R < t < s$, then

$$\begin{aligned}
&H(x,s) - H(x,t)\\
&= p\Big[\frac{1}{p}(f(x,s)s - f(x,t)t) - (F(x,s) - F(x,t))\Big]\\
&= p\Big[\int_R^s \frac{f(x,s)}{s^{p-1}}\tau^{p-1}d\tau - \int_R^t \frac{f(x,t)}{t^{p-1}}\tau^{p-1}d\tau - \int_t^s \frac{f(x,\tau)}{\tau^{p-1}}\tau^{p-1}d\tau\\
&\quad + \frac{f(x,s)}{ps^{p-1}}R^p - \frac{f(x,t)}{pt^{p-1}}R^p\Big]\\
&= p\Big[\int_t^s \Big(\frac{f(x,s)}{s^{p-1}} - \frac{f(x,\tau)}{\tau^{p-1}}\Big)\tau^{p-1}d\tau + \int_R^t \Big(\frac{f(x,s)}{s^{p-1}} - \frac{f(x,t)}{t^{p-1}}\Big)\tau^{p-1}d\tau\\
&\quad + \frac{R^p}{p}\Big(\frac{f(x,s)}{s^{p-1}} - \frac{f(x,t)}{t^{p-1}}\Big)\Big] \geq 0,
\end{aligned}$$

which indicates that

$$H(x,s) \geq H(x,t)$$

for $s > t > R$. So $H(x,t)$ is nondecreasing on $(R, +\infty)$ for a.e. $x \in \overline{\Omega}$.

(2) Now, we prove (F4) holds. Suppose $\theta = 1, \theta_0 = 2\max_{\overline{\Omega}\times[0,R]}|H(x,s)|$. Since $H(x,t)$ is nondecreasing on $(R, +\infty)$ for a.e. $x \in \overline{\Omega}$, then

(i) for all $R \leq t \leq s, H(x,t) - \theta H(x,s) = H(x,t) - H(x,s) \leq 0 < \theta_0$.

(ii) for all $0 \leq t \leq s \leq R$,
$$H(x,t) - \theta H(x,s) = H(x,t) - H(x,s) \leq 2\max_{\overline{\Omega}\times[0,R]}|H(x,s)| = \theta_0.$$

(iii) for all $0 \leq t \leq R \leq s$,
$$H(x,t) - \theta H(x,s) = (H(x,t) - H(x,R)) + (H(x,R) - H(x,s)) \leq \theta_0 + 0.$$

So for all $x \in \overline{\Omega}$ and $0 \leq t \leq s$, $\theta H(x,s) \geq H(x,t) - \theta_0$. $\square$

2. Preliminaries

In this section, we give some preliminary knowledge and some important lemma which will be used to prove our theorem.

Let ϕ be a C^1-functional defined on a Banach space X, we say that ϕ satisfies the Cerami condition (C), if a sequence $\{u_n\} \subset X$ is such that $\{\phi(u_n)\}$ is bounded and $(1+\|u_n\|)\|I'(u_n)\| \to 0$ has a convergent subsequence; such a sequence is then called a Cerami sequence.

To obtain a positive solution of problem (1.1), we introduced the C^1-functional I, defined by

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega F(x,u^+)dx, \quad \forall u \in W_0^{1,p}(\Omega),$$

where $F(x,t) = \int_0^t f(x,s)ds$, and t^+ denotes the positive part of t. In the proof of our theorem, we shall use the following lemma.

Lemma 2.1. *If hypothesis* (F1)–(F4) *hold, then the functional I satisfies the Cerami condition.*

Proof. Let $\{u_n\} \subset W_0^{1,p}(\Omega)$ be a sequence such that

$$\begin{gathered} I(u_n) = \frac{1}{p}\int_\Omega |\nabla u_n|^p dx - \int_\Omega F(x,u_n^+)dx \to c \quad \text{as } n \to \infty, \\ (1+\|u_n\|)\|I'(u_n)\| \to 0 \quad \text{as } n \to \infty, \end{gathered} \tag{2.1}$$

then

$$\int_\Omega \Big(\frac{1}{p} f(x,u_n^+)u_n^+ - F(x,u_n^+)\Big)dx = c + o(1). \tag{2.2}$$

Next, we show that the sequence $\{u_n\}$ is bounded. Otherwise, there is a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) satisfying $\|u_n\| \to \infty$ as $n \to \infty$. Set $w_n = u_n/\|u_n\|$, then $\|w_n\| = 1$. Up to a subsequence, we assume that

$$\begin{gathered} w_n \rightharpoonup w \quad \text{in } W_0^{1,p}(\Omega); \quad w_n \to w \quad \text{in } L^r(\Omega)\ (1 \le r < p^*); \\ w_n(x) \to w(x) \quad \text{a.e.} x \in \Omega \end{gathered} \tag{2.3}$$

for some $w \in W_0^{1,p}(\Omega)$ as $n \to \infty$. It is easily that w^+ and w^- have the same convergence which is similar to (2.3), where $u^\pm = \max\{\pm u, 0\}$ for $u \in W_0^{1,p}(\Omega)$. We claim that $w^+ \equiv 0$. Let $\Omega_0 = \{x \in \Omega : w^+(x) = 0\}$, $\Omega^+ = \{x \in \Omega : w^+(x) > 0\}$. Since $\|u_n\| \to +\infty$, then $u_n^+ \to +\infty$ as $n \to +\infty$ for a.e. $x \in \Omega^+$. Since $\lim_{t\to+\infty} \frac{f(x,t)}{t^{p-1}} = +\infty$ by (F3), one has

$$\lim_{n\to\infty} \frac{f(x,u_n^+)}{(u_n^+)^{p-1}} = +\infty \quad \text{a.e. } x \in \Omega^+.$$

From (2.1), we obtain

$$|\langle I'(u_n), u\rangle| \le \varepsilon_n, \tag{2.4}$$

where $\varepsilon_n = (1+\|u_n\|)\|I'(u_n)\| \to 0$ as $n \to \infty$. It follows from (2.4) that

$$|\|u_n^+\|^p - \int_\Omega f(x,u_n^+)u_n^+ dx| \le \varepsilon_n,$$

which implies

$$\int_{\Omega^+} \frac{f(x,u_n^+)}{(u_n^+)^{p-1}}(w_n^+)^p dx \le 1 + \frac{\varepsilon_n}{\|u_n^+\|^p}. \tag{2.5}$$

If $|\Omega^+| > 0$, since $\|w_n^+\| = 1$, from (2.5), one obtains

$$+\infty \leftarrow \int_{\Omega^+} \frac{f(x,u_n^+)}{(u_n^+)^{p-1}}(w_n^+)^p dx \le 1 + \frac{\varepsilon_n}{\|u_n^+\|^p} \to 1,$$

which is a contradiction, so $|\Omega^+| = 0$ and $w^+ \equiv 0$.

By (F1) and (F2), we have

$$f(x,t) \le (a(x)+\varepsilon)\,|t|^{p-1} + A|t|^{q-1}, \quad \forall (x,t) \in \overline{\Omega} \times R,$$

where $A > 0$ is a constant, thus

$$F(x,t^+) \le \frac{1}{p}\,(a(x)+\varepsilon)\,|t|^p + A|t|^q, \quad \forall (x,t) \in \overline{\Omega} \times R. \tag{2.6}$$

Now set a sequence $\{t_n\}$ of real numbers such that $I(t_n u_n^+) = \max_{t\in[0,1]} I(tu_n^+)$. For any integer $m > 0$, since $w^+ \equiv 0$, then by (F2), (2.6), and the convergence of w_n^+, one has

$$\begin{aligned}
&\limsup_{n\to\infty} \int_\Omega F\left(x, (2pm)^{\frac{1}{p}} w_n^+\right) dx \\
&\le \limsup_{n\to\infty} \Big(\int_\Omega 2m(\lambda_1+\varepsilon)(w_n^+)^p dx + \int_\Omega A(2pm)^{\frac{q}{p}}(w_n^+)^q dx \Big) \\
&= \lim_{n\to\infty} (C_1\|w_n^+\|_p^p + C_2\|w_n^+\|_q^q) \\
&= C_1\|w^+\|_p^p + C_2\|w^+\|_q^q = 0,
\end{aligned}$$

where $C_1, C_2 > 0$ are constants. Since $\|u_n\| \to +\infty$ as $n \to \infty$, one has $0 \le (2pm)^{1/p}/\|u_n\| \le 1$ when n is big enough. By the definition of t_n, we obtain

$$I(t_n u_n^+) \ge I\left((2pm)^{\frac{1}{p}} w_n^+\right) \ge 2m - \int_\Omega F\Big(x, (2pm)^{\frac{1}{p}} w_n^+\Big) \ge m,$$

which implies

$$I(t_n u_n^+) \to +\infty \quad \text{as } n \to \infty. \tag{2.7}$$

Noting that $I(0) = 0$ and $I(u_n) \to c$, so $0 < t_n < 1$ when n is big enough. It follows that

$$\begin{aligned}
&\int_\Omega |\nabla(t_n u_n^+)|^p dx - \int_\Omega f(x, t_n u_n^+) t_n u_n^+ dx \\
&= \langle I'(t_n u_n^+), t_n u_n^+ \rangle = t_n \frac{dI(tu_n^+)}{dt}\Big|_{t=t_n} = 0.
\end{aligned} \tag{2.8}$$

But for $0 \le t_n \le 1$, $|t_n u_n| \le |u_n|$, then (F4), (2.7) and (2.8) imply

$$\begin{aligned}
&\int_\Omega \Big(\frac{1}{p} f(x,u_n^+)u_n^+ - F(x,u_n^+)\Big) dx \\
&= \frac{1}{p}\int_\Omega H(x,u_n^+) dx \ge \frac{1}{p\theta}\int_\Omega \big(H(x,t_n u_n^+) - \theta_0\big) dx \\
&= \frac{1}{\theta}\int_\Omega \Big(\frac{1}{p} f(x,t_n u_n^+) t_n u_n - F(x,t_n u_n^+)\Big) dx - \frac{\theta_0}{p\theta}|\Omega| \\
&= \frac{1}{\theta}\int_\Omega \Big(\frac{1}{p}|\nabla t_n u_n^+|^p - F(x,t_n u_n^+)\Big) dx - \frac{\theta_0}{p\theta}|\Omega| \\
&= \frac{1}{\theta} I(t_n u_n^+) - \frac{\theta_0}{p\theta}|\Omega| \to +\infty \quad (n \to \infty),
\end{aligned}$$

which contradicts to (2.2), so $\{u_n\}$ is bounded.

By the compactness of Sobolev embedding and the standard procedures, we know that $\{u_n\}$ has a convergence subsequence. That is to say, the functional I satisfies the (C) condition. □

3. Proof of the main result

Proof of Theorem 1.3. By critical point theory, for finding a positive solution of problem (1.1), we only need to find a nonzero critical point of the following C^1-functional

$$I(u)=\frac{1}{p}\int_\Omega|\nabla u|^p dx-\int_\Omega F(x,u^+)dx,\quad \forall u\in W_0^{1,p}(\Omega).$$

Since (F2) holds, there exists a positive constant $\alpha<1$ such that $\int_\Omega a(x)|u|^p dx<\alpha\int_\Omega|\nabla u|^p dx$ for all $u\in W_0^{1,p}(\Omega)$ (see [8, Lemma 2.2]). Let $\varepsilon>0$ be small enough such that $\alpha+\varepsilon/\lambda_1<1$. By (2.6), together with the Poincare inequality and Sobolev inequality, one obtains

$$\begin{aligned}
I(u)&\geq\frac{1}{p}\|u\|^p-\frac{1}{p}\int_\Omega(a(x)+\varepsilon)\,|u|^p dx-A\int_\Omega|u|^q dx\\
&\geq\frac{1}{p}\|u\|^p-\frac{1}{p}\int_\Omega\big(\alpha+\frac{\varepsilon}{\lambda_1}\big)|\nabla u|^p dx-C\|u\|^q\\
&=\frac{1}{p}\big(1-\alpha-\frac{\varepsilon}{\lambda_1}\big)\|u\|^p-C\|u\|^q,
\end{aligned}$$

where $C>0$ is a constant. Since $1-\alpha-\frac{\varepsilon}{\lambda_1}>0$ and $p<q$, when $\rho>0$ be small enough such that

$$\beta=\frac{1}{p}\Big(1-\alpha-\frac{\varepsilon}{\lambda_1}\Big)\rho^p-C\rho^q>0,$$

we have

$$I|_{\partial B_\rho}\geq\beta>0. \tag{3.1}$$

Since (F3) holds, then given $\varepsilon>0$, we can find $c(\varepsilon)>0$ such that

$$f(x,t)\geq\frac{t^{p-1}}{\varepsilon}-c(\varepsilon),\quad \forall(x,t)\in\overline{\Omega}\times R^+,$$

which implies

$$F(x,t^+)\geq\frac{1}{p\varepsilon}(t^+)^p-c(\varepsilon)t^+,\quad \forall(x,t)\in\overline{\Omega}\times R.$$

So, if $v_1>0$ is the eigenfunction of $(-\triangle_p,W_0^{1,p}(\Omega))$ corresponding to the first eigenvalue λ_1 with $\|v_1\|=1$, then

$$\int_\Omega\frac{F\left(x,(tv_1)^+\right)}{t^p}dx\geq\int_\Omega\Big(\frac{1}{p\varepsilon}(v_1^+)^p-\frac{c(\varepsilon){v_1}^+}{t^{p-1}}\Big)dx. \tag{3.2}$$

Letting $t\to+\infty$ in (3.2), it follows that

$$\liminf_{t\to+\infty}\int_\Omega\frac{F\left(x,(tv_1)^+\right)}{t^p}dx\geq\int_\Omega\frac{1}{p\varepsilon}(v_1^+)^p dx,$$

for all $\varepsilon>0$. For $\varepsilon>0$ is arbitrary, letting $\varepsilon\to 0$, we infer that

$$\lim_{t\to+\infty}\int_\Omega\frac{F\left(x,(tv_1)^+\right)}{t^p}dx=+\infty.$$

Consequently, one obtains

$$\frac{I(tv_1)}{t^p}=\frac{1}{p}\|v_1\|^p-\int_\Omega\frac{F\left(x,(tv_1)^+\right)}{t^p}dx=\frac{1}{p}-\int_\Omega\frac{F\left(x,(tv_1)^+\right)}{t^p}dx\to-\infty$$

as $t\to+\infty$. Hence, when t_0 is big enough, there exists $e=t_0v_1\in W_0^{1,p}(\Omega)\setminus\overline{B_\rho(0)}$ such that

$$I(e)\le 0. \tag{3.3}$$

Thus, Lemma 2.1 and (3.1), (3.3) permit the application of a variant of mountain pass theorem (see [1, p. 648]), so we get a critical point u of the functional I with $I(u)\ge\beta$. But from (F2), $f(x,0)=0$, then $I(0)=0$, that is $u\neq 0$. Since

$$0=\langle I'(u),u^-\rangle=\|u^-\|^p-\int_\Omega f(x,u^+)u^-dx=\|u^-\|^p\ge 0,$$

which implies that $\|u^-\|=0$, so $u\ge 0$. By the regularity results (see [4]), $u\in L^\infty(\Omega)$, and hence $u\in C^1(\Omega)$(see [6]). Since $u\in L^\infty(\Omega)$, it is easy to see that $\Delta_p u=-f(x,u)\in L^2_{loc}(\Omega)$. From $b_0\le\liminf_{t\to 0^+}\frac{f(x,t)}{t^{p-1}}$ by (F2), there exist a constant $\delta>0$ such that

$$f(x,t)\ge(b_0-1)t^{p-1},\quad\forall 0\le t\le\delta.$$

By (F3), we can find a positive constant M such that

$$f(x,t)\ge 0,\quad\forall t\ge M.$$

Because $f\in C(\overline{\Omega}\times R,R)$, then

$$|f(x,t)|\le B=B\delta^{-(p-1)}\delta^{p-1}\le B\delta^{-(p-1)}t^{p-1},\quad\forall\delta\le t\le M,$$

where $B>0$ is a constant, hence

$$f(x,t)\ge\left(-|b_0-1|-B\delta^{-(p-1)}\right)t^{p-1},\quad\forall t\ge 0.$$

Since $u\ge 0$, it follows that $f(x,u)\ge\left(-|b_0-1|-B\delta^{-(p-1)}\right)u^{p-1}=-Du^{p-1}$, where $D=|b_0-1|+B\delta^{-(p-1)}>0$. Therefore, $\Delta_p u=-f(x,u)\le Du^{p-1}$. Hence by the Strong maximum principle for p-Laplacian in [7] with $\beta(u)=D|u|^{p-1}$, one has $u>0$ a.e. on Ω. That is, u is a positive solution of problem (1.1). The proof is complete. □

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Ting-Mei Gao

School of Mathematics and Statistics, Southwest University, Chongqing 400715, China.
School of Mathematics and Computer Science, Shaanxi University of Technology, Hanzhong 723000, Shaanxi, China

E-mail address: gtmgtmgtm@126.com

Chun-Lei Tang (corresponding author)

School of Mathematics and Statistics, Southwest University, Chongqing 400715, China

E-mail address: tangcl@swu.edu.cn, tangcl8888@sina.com