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A CRITICAL POINT THEOREM AND EXISTENCE OF MULTIPLE SOLUTIONS FOR A NONLINEAR ELLIPTIC PROBLEM

ABDEL RACHID EL AMROUSS, FOUAD KISSI

Abstract. In this article, we show the existence of multiple nontrivial solutions to a Dirichlet problem for the p-Laplacian. Our approach is based on a abstract critical point theorem.

1. Introduction

Let us consider the nonlinear elliptic problem

$$\begin{aligned} -\Delta_p u &= f(x,u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, Δ_p is the p-Laplacian operator defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < \infty$.

The growing attention in the study of the p-Laplace operator is motivated by the fact that it arises in various applications, e.g. non-Newtonian fluids, reaction-diffusion problems, flow through porus media, nonlinear elasticity, theory of superconductors, petroleum extraction, glacial sliding, astronomy, biology etc.

We assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the subcritical growth condition:

$$|f(x,t)| \le c(1 + |t|^{q-1}), \quad \forall t \in \mathbb{R}, \text{ a.e. } x \in \Omega,$$

for some $c > 0$, and $1 \le q < p^*$ where $p^* = \frac{Np}{N-p}$ if $1 < p < N$ and $p^* = +\infty$ if $N \le p$. The above condition implies that the functional $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$,

$$\Phi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx - \int_\Omega F(x,u)\, dx,$$

is well defined and of class C^1, where $F(x,t) = \int_0^t f(x,s)\, ds$. It is well known that the critical points of Φ are weak solutions of (1.1). In the previous decades, many existence and multiplicity results were obtained by applying the critical point theory to Φ.

If $f(x,0) = 0$, then the zero function $u = 0$ is a trivial solution of the problem (1.1). In this article we investigate the existence of nontrivial solutions for (1.1).

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For this purpose, some conditions on the nonlinearity near zero and near infinity are in order.

Let λ_1 and λ_2 be the first and the second eigenvalues of $-\Delta_p$ on $W_0^{1,p}(\Omega)$. It is well known that $\lambda_1 > 0$ is a simple eigenvalue, and that $\sigma(-\Delta_p) \cap (\lambda_1, \lambda_2) = \emptyset$, where $\sigma(-\Delta_p)$ is the spectrum of $-\Delta_p$ (cf. [3]).

In the semilinear case when $p = 2$, the existence of multiple solutions of the above problem has been studied by many authors, see for example [1, 6, 10, 15, 17]. The nonlinear case ($p \neq 2$), has been established by many authors under various conditions imposed on $f(x,t)$ or $F(x,t)$. In the case the nonlinearity $\frac{pF(x,t)}{|t|^p}$ stays asymptotically between the two first eigenvalues of $-\Delta_p$ and via direct variational methods or the minimax methods, existence of one solution were proved (cf. [8, 11]).

The existence of multiple solutions depends mainly on the local behavior of $f(x,t)$ or $F(x,t)$ near 0 and near infinity. In [13], a contribution was made when $\lim_{|t|\to\infty} \frac{pF(x,t)}{|t|^p} < \lambda_1$. Another contribution was made in [16], where the authors treated the resonance near zero at the first eigenvalue from the right and the non-resonance condition at infinity below λ_1. In [4], the authors obtained the existence of multiple nontrivial solutions for the case

$$\limsup_{|t|\to 0} \frac{pF(x,t)}{|t|^p} \leq \alpha < \lambda_1 < \beta \leq \liminf_{|t|\to+\infty} \frac{f(x,t)}{|t|^{p-2}t}.$$

As is well known, the Morse theory developed by Chang [7] or Mawhin and Willem [17] is very useful in studying the existence of multiple solutions for differential equations having the variational structure. Thus computation of critical groups may yield the existence and multiplicity of nontrivial solutions to our problem.

Before stating our main result, we state the following assumptions:

- (F0) $\sup_{|t|\leq R} |f(x,t)| \in L^\infty(\Omega)$ for $R > 0$.
- (F1) $\lambda_1 \leq \liminf_{|t|\to+\infty} \frac{f(x,t)}{|t|^{p-2}t} \leq \limsup_{|t|\to+\infty} \frac{f(x,t)}{|t|^{p-2}t} \leq \beta < \lambda_2$, uniformly for a.e. $x \in \Omega$.
- (F2) $L(x) = \liminf_{|t|\to+\infty} [pF(x,t) - tf(x,t)] \in L^1(\Omega)$ and $\int_\Omega L(x)\,dx > 0$.
- (F3) There exists $\delta > 0$ such that $0 < pF(x,t) \leq tf(x,t)$, for almost every $x \in \Omega$, and for every $0 < |t| \leq \delta$.
- (F4) There exist $\mu \in (0,p)$ and γ a constant non positive, such that
$$\liminf_{|t|\to 0} \frac{\mu F(x,t) - tf(x,t)}{|t|^p} \geq \gamma > \lambda_1(\frac{\mu}{p} - 1) \quad \text{uniformly a.e. } x \in \Omega.$$

The main result reads as follows.

Theorem 1.1. *Assume* (F0)–(F4) *hold, and that there exists* $t_0 \in]0, \delta[$ *such that* $f(x,t_0) = 0$ *a.e.* $x \in \Omega$. *Then* (1.1) *has at least three solutions.*

Example 1.2. Let us define the continuous function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$f(x,t) = \begin{cases} \frac{\lambda_1}{2}|t|^{p-2}t & \text{if } |t| \leq \delta/2, \\ \lambda_1|t|^{p-2}t + \frac{\operatorname{sign}(t)}{1+t^2} & \text{if } |t| \geq 2\delta. \end{cases}$$

The primitive F is such that

$$F(x,t) = \begin{cases} \frac{\lambda_1}{2p}|t|^p & \text{if } |t| \leq \delta/2, \\ \frac{\lambda_1}{p}|t|^p + \arctan(|t|) & \text{if } |t| \geq 2\delta. \end{cases}$$

A simple computation shows that

$$\liminf_{|t|\to+\infty}\frac{f(x,t)}{|t|^{p-2}t}=\lambda_1,\quad \liminf_{|t|\to+\infty}[pF(x,t)-tf(x,t)]=\frac{p\pi}{2}.$$

Hence the hypotheses of Theorem 1.1 are satisfied.

Note that our multiplicity result is not covered by the results mentioned in [4, 11, 13, 16, 19]. For the proof of our main result, we need to prove an abstract theorem which extends [18, Theorem 3.4].

Our work is organized as follows: as preliminaries, in section 2 we give the proof of the abstract theorem; in section 3 we prove our theorem.

2. An abstract critical point theorem

2.1. **Preliminaries.** Let X be a real Banach space endowed with the norm $\|\cdot\|$. Given a functional Φ of class C^1 on X, β, $c\in\mathbb{R}$, $\delta>0$ and $u\in X$, we adopt the notation:

$$\begin{gathered}\Phi^\beta=\{x\in X:\Phi(x)\le\beta\},\quad K=\{x\in X:\Phi'(x)=0\},\\ K_c=\{x\in K:\Phi(x)=c\},\quad (K_c)_\delta=\{x\in X:\operatorname{dist}(x,K_c)\le\delta\},\\ \tilde X=\{x\in X:\Phi'(x)\ne0\},\quad B_\delta(u)=\{x\in X:\|x-u\|\le\delta\}.\end{gathered}$$

The duality between X and its dual X' will be denoted by $\langle\cdot,\cdot\rangle$. Now, recall a generalization of the classical Palais-smale condition which has been introduced by the first author (see [9]).

Definition 2.1. Given $c\in\mathbb{R}$, we say that $\Phi\in C^1(X,\mathbb{R})$ satisfies the condition $(C)_c^{\alpha(\cdot)}$ if

(i) every bounded sequence $(u_n)\subset X$ such that $\Phi(u_n)\to c$ and $\Phi'(u_n)\to0$ possesses a convergent subsequence;

(ii) there exists $R>0$, $\sigma>0$, $\forall x\in\Phi^{-1}([c-\sigma,c+\sigma])$, $\|x\|\ge R$:

$$\|\Phi'(x)\|\ge\alpha(\|x\|),$$

where $\alpha:]0,\infty[\to]0,\infty[$ is C^1 and satisfies

$$\int_1^\infty\alpha(1+s)\,ds=+\infty.$$

If Φ satisfies the condition $(C)_c^{\alpha(\cdot)}$, for every $c\in\mathbb{R}$, we simply say that Φ satisfies $(C)^{\alpha(\cdot)}$.

Remark 2.2. Note that when $\alpha(s)$ is constant, $(\int_1^\infty\alpha(1+s)\,ds=\infty)$, the condition $(C)^{\alpha(\cdot)}$ is the classical Palais-Smale condition denoted (PS). And when $\alpha(s)=\frac{a}{s}$ where $a>0$, $(\int_1^\infty\alpha(1+s)\,ds=\infty)$, we get the condition (C) introduced by Cerami in [6].

Definition 2.3. We say that Φ satisfies the deformation condition (D_c) at $c\in\mathbb{R}$, if for any $\bar\varepsilon>0$ and any neighborhood N of K_c there exists $\varepsilon>0$ and a continuous deformation $\eta:[0,1]\times X\to X$ such that

(1) $\eta(0,.)=Id_X$,
(2) $\eta(t,x)=x$ if $x\in(X\backslash\Phi^{-1}([c-\bar\varepsilon,c+\bar\varepsilon]))$, $t\in[0,1]$,
(3) $\Phi(\eta(s,x))\le\Phi(\eta(t,x))$ if $s\ge t$,
(4) $\eta(1,\Phi^{c+\varepsilon}\backslash N)\subset\Phi^{c-\varepsilon}$.

Remark 2.4. The deformation condition (D_c) is a consequence of the above weak version of the Palais-Smale condition, see [9].

Next, we recall the notion of critical groups at an isolated critical point. For more details see [7, 17].

Definition 2.5. Suppose $u \in K_c$ is an isolated critical point of a functional $\Phi \in C^1(X, \mathbb{R})$. We define the q^{th} critical group of Φ at u with real coefficients $\mathbb{R}$ by

$$C_q(\Phi, u) = H_q(\Phi^c \cap U, (\Phi^c \setminus \{u\}) \cap U),$$

where U is a neighborhood of u such that $U \cap K_c = \emptyset$, and H_q denote the singular homology groups with coefficients in $\mathbb{R}$.

Furthermore, we have the following Morse relation between the critical groups and homological characterization of sub level sets. For details of the proof, we refer readers to [5, 7] for example.

Lemma 2.6. *If Φ satisfies the deformation condition (D_c) at $c \in \mathbb{R}$ then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$ we have:*

$$H_*(\Phi^{c+\varepsilon}, \Phi^{c-\varepsilon}) \cong H_*(\Phi^c \cup K_c, \Phi^c);$$
$$H_*(\Phi^{c+\varepsilon}, \Phi^{c-\varepsilon}) \cong 0 \quad \textit{if } K_c = \emptyset;$$
$$H_*(\Phi^{c+\varepsilon}, \Phi^{c-\varepsilon}) \cong \oplus_{i=1}^k C_*(\Phi, x_i) \textit{if} K_c = \{x_1, \dots, x_k\}.$$

Notice that this result implies that if $H_q(\Phi^{c+\varepsilon}, \Phi^c)$ is nontrivial for some q, then there exists a critical point $u \in K_c$ with $C_q(\Phi, u) \neq 0$. However, when $C_q(\Phi, 0) \cong 0$ for all q, we get that $u \neq 0$. We shall use the following lemma, which is proved in [9].

Lemma 2.7. *If $\Phi \in C^1(X, \mathbb{R})$, there exists a locally Lipschitz continuous function $V : \widetilde{X} \to X$ satisfying the conditions: $\|V(x)\| \leq 2$ and $\langle V(x), \Phi'(x) \rangle \geq \|\Phi'(x)\|$, $\forall x \in \widetilde{X}$.*

2.2. **A critical point result.** Our abstract critical point theorem can be stated as follows

Theorem 2.8. *Let X be a real Banach space and let $\Phi \in C^1(X, \mathbb{R})$. Assume Φ is not bounded below and the origin is an isolated critical point of Φ in X satisfying $C_1(\Phi, 0) = 0$. If Φ possesses a local minimum $u_0 \neq 0$ and Φ satisfies $(C)_c^{\alpha(\cdot)}$ for every $c \geq \Phi(u_0)$. Then, Φ possesses at least three critical points in X.*

Note that, in [18, Theorem 3.4] the authors establish the same result on real Hilbert spaces with the compactness Cerami condition $(C)_c$, satisfied for every $c \in \mathbb{R}$. The next lemma is essential in the proof of Theorem 2.8.

Lemma 2.9 (Deformation lemma)**.** *If $\Phi \in C^1(X, \mathbb{R})$ and satisfies $(C)_c^{\alpha(\cdot)}$ condition at $c \in \mathbb{R}$. Assume that K_c has isolated points. Then, given $\delta > 0$ and $\bar{\varepsilon} > 0$, there exist $\varepsilon \in (0, \bar{\varepsilon})$ and a continuous map $\eta : [0, 1] \times X \to X$ such that*

1. *$\eta(0, x) = x$, for every $x \in X$,*
2. *$\eta(1, x) = x$, for every $x \in \overline{X \backslash B_\delta}(u)$, where $u \in K_c$,*
3. *$\eta(t, x) = x$, for every $x \in (X \backslash \Phi^{-1}([c - \bar{\varepsilon}, c + \bar{\varepsilon}]))$, $t \in [0, 1]$,*
4. *$\eta(1, \Phi^{c+\varepsilon} \cap B_\delta(u)) \subset \Phi^{c-\varepsilon}$.*

Proof. It is easy to see that by the condition $(C)_c^{\alpha(\cdot)}$, that (K_c) is compact and hence let $R' > \max(R, \delta)$ such that $\bigcup_{v\in K_c} B_\delta(v) \subset B_{R'}(0)$. By the condition $(C)_c^{\alpha(\cdot)}$, we verify easily that there exist $\hat{\varepsilon} > 0$, with $\hat{\varepsilon} < \bar{\varepsilon}$ and $\beta > 0$, such that

$$\|\Phi'(x)\| \geq \beta \quad \text{for every} x \in [\Phi^{c+\hat{\varepsilon}} \backslash (\Phi^{c-\hat{\varepsilon}} \cup (K_c)_{\delta/2})] \cap B_{R'}(0). \tag{2.1}$$

Taking $0 < \varepsilon_1 < \hat{\varepsilon} < \bar{\varepsilon}$ and $\delta/2 < \mu < \delta$, we consider

$$A = X \backslash \Phi^{-1}([c - \hat{\varepsilon}, c + \hat{\varepsilon}], \quad B = \Phi^{-1}([c - \varepsilon_1, c + \varepsilon_1].$$

Define

$$f(x) = \frac{\operatorname{dist}(x, A)}{\operatorname{dist}(x, B) + \operatorname{dist}(x, A)},$$

$$g(x) = \frac{\operatorname{dist}(x, X\backslash B_\delta(u))}{\operatorname{dist}(x, B_\mu(u)) + \operatorname{dist}(x, X\backslash B_\delta(u))},$$

$$h(s) = \begin{cases} 1/\alpha(s) & \text{if } s > R' \\ 1/\alpha(R') & \text{if } s \leq R'. \end{cases}$$

From lemma 2.7, there exists a pseudo-gradient vector field V on $\tilde{X}$ associated with Φ. Put

$$W(x) = \begin{cases} -\alpha(R') f(x) g(x) h(\|x\|) V(x), & \text{if } x \in \tilde{X}, \\ 0, & \text{otherwise.} \end{cases}$$

By construction, W is locally Lipshitz continuous on X. Since $g = 0$ on $X\backslash B_\delta(u)$, one deduces that

$$0 \leq \|W(x)\| \leq 1, \quad \text{for every } x \in X.$$

Now, we consider the Cauchy problem

$$\begin{gathered} \frac{d\eta}{dt}(t, x) = W(\eta(t, x)), \\ \eta(0, x) = x. \end{gathered} \tag{2.2}$$

Clearly, (2.2) has a unique solution $\hat{\eta}(t, x)$ for all $t \geq 0$. Furthermore, $\hat{\eta} \in C([0, \infty) \times X, X)$.

Since $f = 0$ on A, $g = 0$ on $X\backslash B_\delta(u)$ and $\hat{\varepsilon} < \bar{\varepsilon}$, then η satisfies (1), (2) and (3).

Now, we verify (4). First, observe that the map $t \to \Phi(\eta(t, x))$ is decreasing. Indeed,

$$\begin{aligned} &\frac{d\Phi}{dt}(\eta(t, x)) = \langle \Phi'(\eta(t, x)), \frac{d\eta}{dt}(t, x) \rangle \\ &= -\alpha(R') f(\eta(t, x)) g(\eta(t, x)) h(\|\eta(t, x)\|) \langle \Phi'(\eta(t, x)), V(\eta(t, x)) \rangle \leq 0. \end{aligned}$$

Take $0 < \varepsilon < \min(\hat{\varepsilon}, \frac{\beta}{2})$ and let $x \in \Phi^{c+\varepsilon} \cap B_\delta(u)$, we will prove that

$$\Phi(\eta(1, x)) \leq c - \varepsilon. \tag{2.3}$$

By contradiction, we suppose that (2.3) does not holds. Then

$$c - \varepsilon < \Phi(\eta(1, x)) \leq \Phi(\eta(t, x)) \leq \Phi(x) \leq c + \varepsilon, \quad \forall t \in [0, 1].$$

So $f(\eta(t, x)) = 1$ for all $t \in [0, 1]$.

On the other hand, since $g = 0$ on $X\backslash B_\delta(u)$, $g = 1$ on $B_\mu(u)$, $R' > \delta$ and by (2.1), we have

$$\Phi(\eta(1, x)) - \Phi(x) = -\alpha(R') \int_0^1 g(\eta(t, x)) h(\|\eta(t, x)\|) \langle \Phi'(\eta(t, x)), V(\eta(t, x)) \rangle \, dt,$$

$$
\begin{aligned}
&= -\alpha(R')\int_0^1 g(\eta(t,x))h(\|\eta(t,x)\|)\|\Phi'(\eta(t,x))\|\chi_{\{t,\|\eta(t,x)\|\le R'\}}\,dt\\
&= -\int_0^1 g(\eta(t,x))\|\Phi'(\eta(t,x))\|\chi_{\{t,\eta(t,x)\in B_\delta(u)\}}\,dt\\
&\le -\int_0^1 \|\Phi'(\eta(t,x))\|\chi_{\{t,\eta(t,x)\in B_\mu(u)\setminus(K_c)_{\delta/2}\}}\,dt \le -\beta.
\end{aligned}
$$

Finally, we conclude that

$$\Phi(\eta(1,x)) \le c+\varepsilon-\beta < c-\varepsilon.$$

This is a contradiction. The proof is complete. □

Proof of Theorem 2.8. By contradiction, assume that the origin and u_0 are the only critical points of Φ. Let $c_0 = \Phi(u_0)$, and since u_0 is a local minimum of Φ, thus there exists $\rho_1 > 0$ such that

$$\Phi(u) \ge \Phi(u_0), \quad \forall u \in B_{\rho_1}(u_0). \tag{2.4}$$

Claim: There exist ρ, $\gamma > 0$ such that

$$\Phi(u) \ge \Phi(u_0)+\gamma, \quad \text{for all } u \in \partial B_\rho(u_0). \tag{2.5}$$

Indeed taking $\rho \in (0,\rho_1)$, we find $\gamma > 0$ satisfying (2.5). Otherwise, by Lemma 2.9, we obtain $\varepsilon > 0$ and a homeomorphism $\eta : X \to X$ such that

(1) $\eta(u) = u$, $\forall u \in \overline{X \setminus B_{\rho_1}(u_0)}$,
(2) $\eta(\Phi^{c_0+\varepsilon} \cap \partial B_\rho(u_0)) \subset \Phi^{c_0-\varepsilon}$.

Using these two conditions, we obtain $u \in B_{\rho_1}(u_0)$ so that $\Phi(u) < c_0$. But, that contradicts (2.4). The claim is proved.

Since Φ is not bounded below, there exists $e \in X$ such that

$$\|e\| \ge \rho \quad \text{and} \quad \Phi(e) < \Phi(u_0)+\gamma. \tag{2.6}$$

It is easy to see that (2.4) and (2.6) imply

$$\max(\Phi(u_0),\Phi(e)) < \inf_{\partial B_\rho} \Phi = b. \tag{2.7}$$

We define

$$c = \inf_{h\in\Gamma}\max_{t\in[0,1]} \Phi(h(t)),$$

where

$$\Gamma = \{h \in C([0,1],X) : h(0) = u_0, h(1) = e\}.$$

Thus, from (2.7), $c \ge b$ is a critical value of Φ. Let $\varepsilon > 0$ be such that $c-\varepsilon > \max(\Phi(u_0),\Phi(e))$ and suppose, without loss of generality, that c is the only critical value of Φ in $[c-\varepsilon, c+\varepsilon]$. Consider the exact sequence

$$\dots \to H_1(\Phi^{c+\varepsilon},\Phi^{c-\varepsilon}) \xrightarrow{\partial} H_0(\Phi^{c-\varepsilon},\emptyset) \xrightarrow{i_*} H_0(\Phi^{c+\varepsilon},\emptyset) \to \dots$$

where ∂ is the boundary homomorphism and i_* is induced by the inclusion mapping $i : (\Phi^{c-\varepsilon},\emptyset) \to (\Phi^{c+\varepsilon},\emptyset)$. The definition of c implies that u_0 and e are path connected in $\Phi^{c+\varepsilon}$ but not in $\Phi^{c-\varepsilon}$. Thus, $\ker i_* \ne \{0\}$ (cf. [17]) and, by exactness, $H_1(\Phi^{c+\varepsilon},\Phi^{c-\varepsilon}) \ne \{0\}$. Using lemmas 2.6, 2.9, we deduce that there exists u such that $\dim C_1(\Phi,u) \ge 1$. In view of $C_1(\Phi,0) = 0$ and $c \ge b$, we have $u \ne 0$ and $u \ne u_0$. The proof is complete. □

3. Proof of Theorem 1.1

In this section we shall use Theorem 2.8 for proving Theorem 1.1. The Sobolev space $W_0^{1,p}(\Omega)$ will be the Banach space X and the C^1 functional Φ will be

$$\Phi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega F(x,u)\,dx.$$

To apply Theorem 2.8, we need the following three lemmas.

Lemma 3.1. *Under assumptions* (F0)–(F2)*, the functional* Φ *satisfies the condition* $(C)_c^{\alpha(\cdot)}$ *for every* $c \geq 0$*, with* $\alpha(s) = \frac{1}{s}$*.*

Proof. We know that $-\Delta_p : X \to X'$ is bounded mapping of type (S^+) and $g' : X \to X'$, $g(u) = \int_\Omega F(x,u)\,dx$, is completely continuous, i.e. $u_n \rightharpoonup u$ implies $g'(u_n) \to g(u)$. From this, by a standard argument, the first assertion of definition 2.1 is verified.

Let us now prove that the second assertion of definition 2.1 is satisfied for every $c \geq 0$. By contradiction, assume that (ii) is false. Then, there exists $(u_n) \subset W_0^{1,p}(\Omega)$ such that

$$\Phi(u_n) \to c, \quad \Phi'(u_n)u_n \to 0, \quad \text{and} \quad \|u_n\| \to \infty. \tag{3.1}$$

From (F0) and (F1) it follows that there exists constants a and b such that

$$|f(x,t)| \leq a|t|^{p-1} + b, \quad \forall t \in \mathbb{R}, \text{a.e.} x \in \Omega.$$

Let us define $v_n = \frac{u_n}{\|u_n\|}$, $f_n = \frac{f(x,u_n)}{\|u_n\|^{p-1}}$, passing to subsequence of v_n (respectively f_n), still denoted by (v_n) (respectively f_n) we may assume that: $v_n \rightharpoonup v$ weakly in $W_0^{1,p}(\Omega)$, $v_n \to v$ strongly in $L^p(\Omega)$ and a.e. $x \in \Omega$, $f_n \rightharpoonup \tilde{f}$ in $L^{p'}(\Omega)$, where $p' = \frac{p}{p-1}$ is the conjugate exponent. We need to state the following claim.

Claim

1. $\tilde{f} = 0$ a.e. in $A = \{x \in \Omega | v(x) = 0 \text{a.e.}\}$;
2. $\lambda_1 \leq \frac{\tilde{f}}{|v|^{p-2}v} \leq \beta$ a.e. in $\Omega \backslash A$.

Indeed, define $\varphi(x) = \operatorname{sign}(\tilde{f}(x))\chi_A(x)$, where

$$\operatorname{sign}(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ -1, & \text{if } x < 0. \end{cases}$$

Thus, the inequality implies that

$$|f_n(x)\varphi(x)| \leq (a|v_n|^{p-1} + \frac{1}{\|u_n\|^{p-1}})\chi_A(x), \quad \text{a. e. } x \in \Omega.$$

Since $v_n \to v$ in $L^p(\Omega)$, it follows by passing to the limit that

$$f_n(x)\varphi(x) \to 0 \quad \text{in } L^{p'}(\Omega). \tag{3.2}$$

On the other hand, we have

$$\int_\Omega f_n\varphi\,dx \to \int_\Omega \tilde{f}\varphi\,dx = \int_\Omega |\tilde{f}|\chi_A\,dx = \int_A |\tilde{f}|\,dx.$$

It follows from (3.2) that $\int_A |\tilde{f}|\,dx = 0$. Thus the first assertion of claim is proved.

Now, we show the second assertion of claim. Put

$$\begin{aligned} B = &\big\{x \in \Omega\backslash A : \lambda_1|v(x)|^p > v(x)\tilde{f}(x) \text{ a.e.}\big\} \\ &\cup \big\{x \in \Omega\backslash A : \beta|v(x)|^p < v(x)\tilde{f}(x) \text{ a.e.}\big\}. \end{aligned}$$

It suffices to prove that meas$(B) = 0$. Indeed, by (F1), for all $\varepsilon > 0$, there exist a_ε, $b_\varepsilon \in L^{p'}(\Omega)$ such that

$$a_\varepsilon(x) + (\lambda_1 - \varepsilon)|t|^p \leq tf(x,t) \leq b_\varepsilon(x) + (\beta+\varepsilon)|t|^p, \quad \text{a.e. } x \in \Omega,\ \forall t \in \mathbb{R}.$$

This implies

$$\frac{a_\varepsilon(x)}{\|u_n\|^p} + (\lambda_1 - \varepsilon)|v_n|^p \leq v_n f_n(x) \leq \frac{b_\varepsilon(x)}{\|u_n\|^p} + (\beta+\varepsilon)|v_n|^p, \quad \text{a.e. } x \in \Omega. \tag{3.3}$$

Multiplying (3.3) by χ_B and integrating over Ω, we obtain

$$\begin{aligned}
&\int_\Omega \frac{a_\varepsilon(x)}{\|u_n\|^p}\chi_B(x)\,dx + (\lambda_1 - \varepsilon)\int_\Omega |v_n|^p \chi_B(x)\,dx \\
&\leq \int_\Omega v_n f_n(x)\chi_B(x)\,dx \\
&\leq \int_\Omega \frac{b_\varepsilon(x)}{\|u_n\|^p}\chi_B(x)\,dx + (\beta+\varepsilon)\int_\Omega |v_n|^p\chi_B(x)\,dx.
\end{aligned}$$

So letting $n \to \infty$ in this inequality, we obtain

$$(\lambda_1 - \varepsilon)\int_\Omega |v(x)|^p\chi_B(x)\,dx \leq \int_\Omega v(x)\tilde{f}(x)\chi_B(x)\,dx \leq (\beta+\varepsilon)\int_\Omega |v(x)|^p\chi_B(x)\,dx.$$

Since $\varepsilon > 0$ is arbitrary,

$$\lambda_1 \int_\Omega |v(x)|^p\chi_B(x)\,dx \leq \int_\Omega v(x)\tilde{f}(x)\chi_B(x)\,dx \leq \beta\int_\Omega |v(x)|^p\chi_B(x)\,dx. \tag{3.4}$$

It is clear that this inequality (3.4) is verified if and only if meas$(B) = 0$.

Letting, $m(x) = \frac{\tilde{f}(x)}{|v(x)|^{p-2}v(x)}$ if $v(x) \neq 0$ and $m(x) = \frac{1}{2}(\lambda_1 + \beta)$ if $v(x) = 0$. By (3.1) we have

$$\frac{|\langle \Phi'(u_n), u_n\rangle|}{\|u_n\|^p} = |1 - \int_\Omega \frac{f(x,u_n)}{\|u_n\|^{p-1}}v_n(x)\,dx| \leq \frac{\varepsilon_n}{\|u_n\|^{p-1}}.$$

Hence, we conclude that

$$\int_\Omega \frac{f(x,u_n)}{\|u_n\|^{p-1}}v_n(x)\,dx \to 1$$

and passing to the limit, we obtain $\int_\Omega \tilde{f}(x)v(x)\,dx = 1$, so that $v \neq 0$.

On the other hand, for any $w \in W_0^{1,p}(\Omega)$ we have

$$\frac{|\langle \Phi'(u_n), w\rangle|}{\|u_n\|^{p-1}} = |\int_\Omega |\nabla v|^{p-2}\nabla v\nabla w\,dx - \int_\Omega \frac{f(x,u_n)}{\|u_n\|^{p-1}}w\,dx| \leq \varepsilon_n\frac{\|w\|}{\|u_n\|^{p-1}}$$

So, passing to the limit, we conclude that

$$\int_\Omega |\nabla v|^{p-2}\nabla v\nabla w\,dx - \int_\Omega \tilde{f}(x)w(x)\,dx = 0;$$

that is,

$$\int_\Omega |\nabla v|^{p-2}\nabla v\nabla w\,dx - \int_\Omega m(x)|v|^{p-2}vw\,dx = 0, \forall w \in W_0^{1,p}(\Omega).$$

In other words, v is a weak solution of the problem

$$\begin{gathered}
-\Delta_p u = m(x)|u|^{p-2}u \quad \text{in } \Omega,\\
u = 0 \quad \text{on } \partial\Omega.
\end{gathered}$$

The result above and the claim imply

$$1 \in \sigma(-\Delta_p, m(\cdot)) \quad \text{and} \quad \lambda_1 \le m(\cdot) \le \beta < \lambda_2. \tag{3.5}$$

If we assume that $\lambda_1 < m(\cdot)$ on a subset of Ω of positive measure, then by the second part of (3.5), the strict monotonicity of λ_1 (cf. [12]) and the strict partial monotonicity of λ_2 (cf. [3]), we have

$$\lambda_1(m(\cdot)) < \lambda_1(\lambda_1(1)) = 1 \quad \text{and} \quad \lambda_2(m(\cdot)) > \lambda_2(\lambda_2(1)) = 1.$$

Therefore, it result that

$$\lambda_1(m(\cdot)) < 1 < \lambda_2(m(\cdot)). \tag{3.6}$$

Since $\sigma(-\Delta_p, m(\cdot)) \cap]\lambda_1(m(\cdot)), \lambda_2(m(\cdot))[= \emptyset$ (cf. [3]), the first part of (3.5) and (3.6) are in contradiction, hence $m(\cdot) = \lambda_1$ and v is a λ_1 eigenfunction. So, it follows that

$$|u_n(x)| \to +\infty \quad \text{a.e. } x \in \Omega. \tag{3.7}$$

On the other hand,

$$\lim_{n\to+\infty} \int_\Omega pF(x,u_n) - u_n f(x,u_n)\,dx = -pc. \tag{3.8}$$

Combining (3.7) and (F2), Fatou's lemma yields

$$\int_\Omega L(x)\,dx \le \liminf_{n\to+\infty} \int_\Omega pF(x,u_n) - u_n f(x,u_n)\,dx.$$

Via (3.8) we obtain

$$\int_\Omega L(x)\,dx \le -pc \le 0,$$

which contradicts (F2). Thus the lemma follows. □

Now, we show that the critical groups of Φ at zero are trivial.

Lemma 3.2. *Assume* (F0)–(F1), (F3), (F4). *Then* $C_q(\Phi,0) \cong 0$ *for all* $q \in \mathbb{Z}$.

Proof. Let $B_\rho = \{u \in W_0^{1,p}(\Omega), \|u\| \le \rho\}$, $\rho > 0$ which is to be chosen later. The idea of the proof is to construct a retraction of $B_\rho \setminus \{0\}$ to $B_\rho \cap \Phi^0 \setminus \{0\}$ and to prove that $B_\rho \cap \Phi^0$ is contractible in itself. For this purpose, we need to analyze the local properties of Φ near zero. Thus some technical affirmations must be proved.

Claim 1. Under (F0), (F1) and (F3), zero is local maximum for the functional $\Phi(su)$, $s \in \mathbb{R}$, for $u \ne 0$. In fact, it follows from the condition (F3), there exists a constant $c_0 > 0$ such that

$$F(x,t) \ge c_0|t|^p, \quad \text{for } x \in \Omega,\ |t| \le \delta. \tag{3.9}$$

Using (F0), (F1) and (3.9), we obtain

$$F(x,t) \ge c_0|t|^p - c_1|t|^q, \quad x \in \Omega,\ t \in \mathbb{R} \tag{3.10}$$

for some $q \in (p,p^*)$ and $c_1 > 0$. Then, for $u \in W_0^{1,p}(\Omega)$, $u \ne 0$ and $s > 0$, we have

$$\begin{aligned} \Phi(su) &= \frac{1}{p}s^p \int_\Omega |\nabla u|^p\,dx - \int_\Omega F(x,su)\,dx \\ &\le \frac{s^p}{p}\|u\|^p - \int_\Omega (c_0|su|^p - c_1|su|^q)\,dx \\ &\le \frac{s^p}{p}\|u\|^p - c_0 s^p \|u\|_p^p + c_1 s^q \|u\|_q^q. \end{aligned} \tag{3.11}$$

Since $p < q$ and by (3.11), there exists a $s_0 = s_0(u) > 0$ such that
$$\Phi(su) < 0, \quad \text{for all } 0 < s < s_0. \tag{3.12}$$

Claim 2. There exists $\rho > 0$ such that
$$\frac{d}{ds}\Phi(su)|_{s=1} > 0, \tag{3.13}$$
for every $u \in W_0^{1,p}(\Omega)$ with $\Phi(u) = 0$ and $0 < \|u\| \le \rho$.

Indeed, let $u \in W_0^{1,p}(\Omega)$ be such that $\Phi(u) = 0$. In turn, for (F4) and (F0)–(F1) respectively, we have for $\varepsilon > 0$ sufficiently small that there exists $r = r(\varepsilon) > 0$ such that
$$\mu F(x,u) - f(x,u)u \ge (\gamma - \varepsilon)|u|^p, \quad \text{a.e. } x \in \Omega \text{ and } |u| \le r,$$
and
$$\mu F(x,u) - f(x,u)u \ge -c_\varepsilon |u|^q, \quad \text{a.e. } x \in \Omega \text{ and } |u| > r,$$
for some $q \in (p, p^*)$ and $c_\varepsilon > 0$.

Define $\Omega_r(u) = \{x \in \Omega : |u| > r\}$ and $\Omega^r(u) = \{x \in \Omega : |u| \le r\}$. Then, since $\Phi(u) = 0$ and by the Poincaré inequality, we write
$$\begin{aligned}
\frac{d}{ds}\Phi(su)|_{s=1} &= \langle \Phi'(su), u\rangle|_{s=1} \\
&= \int_\Omega |\nabla u|^p\,dx - \int_\Omega f(x,u)u\,dx, \\
&= (1 - \frac{\mu}{p})\int_\Omega |\nabla u|^p\,dx + \int_{\Omega^r(u)} (\mu F(x,u) - f(x,u)u)\,dx \\
&\quad + \int_{\Omega_r(u)} (\mu F(x,u) - f(x,u)u)\,dx, \\
&\ge (1 - \frac{\mu}{p})\|u\|^p + (\gamma - \varepsilon)\int_{\Omega^r(u)} |u|^p\,dx - c_\varepsilon \int_{\Omega_r(u)} |u|^q\,dx, \\
&\ge \theta\|u\|^p - C_\varepsilon \|u\|^q,
\end{aligned}$$
for some $C_\varepsilon > 0$, where $\theta = (1 - \frac{\mu}{p} + \frac{\gamma}{\lambda_1} - \frac{\varepsilon}{\lambda_1})$. Since $p < q$, the inequality (3.13) follows for ε small enough such that $\theta > 0$.

Claim 3. For all $u \in W_0^{1,p}(\Omega)$ with $\Phi(u) \le 0$ and $\|u\| \le \rho$, we have
$$\Phi(su) \le 0, \quad \text{for all } s \in (0,1). \tag{3.14}$$
Indeed, given $\|u\| \le \rho$ with $\Phi(u) \le 0$, assume by contradiction that there exists some $s_0 \in (0,1]$ such that $\Phi(s_0 u) > 0$. Thus, by the continuity of Φ, there exists an $s_1 \in (s_0, 1]$ such that $\Phi(s_1 u) = 0$. Choose $s_2 \in (s_0, 1]$ such that $s_2 = \min\{s \in [s_0, 1]; \Phi(su) = 0\}$. It is easy to see that $\Phi(su) \ge 0$ for each $s \in [s_0, s_2]$. Taking $u_1 = s_2 u$, it is clear that
$$\Phi(su) - \Phi(s_2 u) \ge 0 \text{ implies } \frac{d}{ds}\Phi(su)|_{s=s_2} = \frac{d}{ds}\Phi(su_1)|_{s=1} \le 0.$$
This is a contradiction with (3.13). The proof of the claim is complete.

Let us fix $\rho > 0$ such that zero is the unique critical point of Φ in B_ρ. First, by taking the mapping $h : [0,1] \times (B_\rho \cap \Phi^0) \to B_\rho \cap \Phi^0$ as $h(s,u) = (1-s)u$, we have that $B_\rho \cap \Phi^0$ is contractible in itself.

Now, we prove that $(B_\rho \cap \Phi^0) \setminus \{0\}$ is contractible in itself too. For this purpose, define a mapping $T : B_\rho \setminus \{0\} \to (0,1]$ by

$$\begin{gathered} T(u) = 1, \quad \text{for } u \in (B_\rho \cap \Phi^0) \setminus \{0\}, \\ T(u) = s, \quad \text{for } u \in B_\rho \setminus \Phi^0 \text{ with } \Phi(su) = 0, s < 1. \end{gathered} \tag{3.15}$$

From the relations (3.12)-(3.14), the mapping T is well defined and if $\Phi(u) > 0$ then there exists an unique $T(u) \in (0,1)$ such that

$$\begin{gathered} \Phi(su) < 0, \forall s \in (0, T(u)), \\ \Phi(T(u)u) = 0, \\ \Phi(su) > 0, \forall s \in (T(u), 1)). \end{gathered} \tag{3.16}$$

Thus, using (3.13) and (3.15) and the Implicit Function Theorem, the mapping T is continuous.

Next, we define a mapping $\eta : B_\rho \setminus \{0\} \to (B_\rho \cap \Phi^0) \setminus \{0\}$ by

$$\begin{gathered} \eta(u) = T(u)u, u \in B_\rho \setminus \{0\} \text{ with } \Phi(u) \geq 0, \\ \eta(u) = u, u \in B_\rho \setminus \{0\} \text{ with } \Phi(u) < 0. \end{gathered}$$

Since $T(u) = 1$ as $\Phi(u) = 0$, the continuity of η follows from the continuity of T.

Obviously, $\eta(u) = u$ for $u \in (B_\rho \cap \Phi^0) \setminus \{0\}$. Thus, η is a retraction of $B_\rho \setminus \{0\}$ to $(B_\rho \cap \Phi^0) \setminus \{0\}$. Since $W_0^{1,p}(\Omega)$ is infinite dimensional, $B_\rho \setminus \{0\}$ is contractible in itself. By the fact that retracts of contractible space are also contractible, $(B_\rho \cap \Phi^0) \setminus \{0\}$ is contractible in itself. From the homology exact sequence, one has

$$H_q(B_\rho \cap \Phi^0, (B_\rho \cap \Phi^0) \setminus \{0\}) = 0, \quad \forall q \in \mathbb{Z}.$$

Hence

$$C_q(\Phi, 0) = H_q(B_\rho \cap \Phi^0, (B_\rho \cap \Phi^0) \setminus \{0\}) = 0, \forall q \in \mathbb{Z}.$$

□

Lemma 3.3. *Under the conditions of Theorem 1.1, Φ possesses a local minimum u_0 non trivial such that $\Phi(u_0) = 0$.*

Proof. Define the cut-off functional $\tilde{\Phi} : W_0^{1,p}(\Omega) \to \mathbb{R}$ as

$$\tilde{\Phi}(u) = \frac{1}{p}\|u\|^p - \int_\Omega \tilde{F}(x,u)\,dx,$$

where $\tilde{f}(x,t) = f(x,t)$ if $0 \leq t \leq t_0$, $f(x,t) = 0$ otherwise, and $\tilde{F}(x,t) = \int_0^t \tilde{f}(x,s)\,ds$.

Note that $\tilde{\Phi} \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ and From (F0) and (F1), there exists $M \in \mathbb{R}$ such that

$$\tilde{\Phi}(u) \geq \frac{1}{p}\|u\|^p - M, \quad \forall u \in W_0^{1,p}(\Omega).$$

This implies that $\tilde{\Phi}$ is coercive on $W_0^{1,p}(\Omega)$ and satisfies (PS). Hence, $\tilde{\Phi}$ is bounded below. Let $u_0 \in W_0^{1,p}(\Omega)$ a local minimum of $\tilde{\Phi}$. Thus, u_0 is a solution of the problem

$$\begin{gathered} -\Delta_p u_0 = \tilde{f}(x, u_0), \quad \text{in } \Omega, \\ u_0 = 0, \quad \text{on } \partial\Omega. \end{gathered}$$

By the theory regularity in [2], $u_0 \in C^1(\bar{\Omega})$. Considering the domain

$$\Omega_0 = \{x \in \Omega : u_0(x) < 0 \text{ or } u_0(x) > t_0\},$$

we have

$$\begin{gathered} -\Delta_p u_0 \leq 0, \quad \text{in } \Omega_0, \\ 0 \leq u_0 \leq t_0, \quad \text{on } \partial\Omega_0. \end{gathered}$$

From the maximum principle, we get $0 < u_0 < t_0$ in Ω_0, and hence $\Omega_0 = \emptyset$, i.e. $0 \leq u_0 \leq t_0$ in Ω_0.

Since u_0 is a local minimizer of $\tilde{\Phi}$ in $C_0^1(\Omega)$, it is also of Φ in $C_0^1(\Omega)$. Then, by [14, Theorem 2.1], u_0 is a local minimizer of Φ in $W_0^{1,p}(\Omega)$ and

$$C_q(\Phi, u_0) = \delta_{q,0}\mathbb{R}.$$

From Lemma 3.2, u_0 is nontrivial.

Now, we prove that $\Phi(u_0) = 0$. Indeed, since $0 < u_0 < t_0$, we obtain

$$\Phi(u_0) = \tilde{\Phi}(u_0) = \inf_{u \in W_0^{1,p}(\Omega)} \tilde{\Phi}(u) \leq \tilde{\Phi}(0) = 0.$$

Since $\tilde{\Phi}'(u_0).u_0 = 0$, we have

$$\tilde{\Phi}(u_0) = \frac{1}{p}\int_\Omega |\nabla u_0|^p - \int_\Omega \tilde{F}(x, u_0) = \frac{1}{p}\int_\Omega \tilde{f}(x, u_0)u_0 - \int_\Omega \tilde{F}(x, u_0).$$

However, from (F3), we obtain $\Phi(u_0) = \tilde{\Phi}(u_0) \geq 0$. □

Proof of Theorem 1.1. From (F0) and (F1), for some $\varepsilon > 0$ small, it follows that there is a constant $C > 0$ such that

$$F(x,t) \geq \frac{1}{p}(\lambda_1 + \varepsilon)|t|^p + C, \quad \forall t \in \mathbb{R}, \text{ a.e. } x \in \Omega.$$

Therefore, by the Poincaré inequality, for $u \in W_0^{1,p}(\Omega)$,

$$\Phi(u) \leq \frac{-\varepsilon}{p\lambda_1}\|u\|^p - C|\Omega|.$$

Hence Φ is not bounded below. By Lemmas 3.1, 3.2 and 3.3, we can apply Theorem 2.8 and we obtain that Φ possesses at least three critical points in $W_0^{1,p}(\Omega)$. This completes the proof. □

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Abdel Rachid El Amrouss
University Mohamed I, Faculty of sciences, Department of Mathematics, Oujda, Morocco
E-mail address: elamrouss@hotmail.com

Fouad Kissi
University Mohamed I, Faculty of sciences, Department of Mathematics, Oujda, Morocco
E-mail address: kissifouad@hotmail.com