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RIESZ BASIS AND EXPONENTIAL STABILITY FOR EULER-BERNOULLI BEAMS WITH VARIABLE COEFFICIENTS AND INDEFINITE DAMPING UNDER A FORCE CONTROL IN POSITION AND VELOCITY

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Abstract. This article concerns the Riesz basis property and the stability of a damped Euler-Bernoulli beam with nonuniform thickness or density, that is clamped at one end and is free at the other. To stabilize the system, we apply a linear boundary control force in position and velocity at the free end of the beam. We first put some basic properties for the closed-loop system and then analyze the spectrum of the system. Using the modern spectral analysis approach for two-points parameterized ordinary differential operators, we obtain the Riesz basis property. The spectrum-determined growth condition and the exponential stability are also concluded.

1. Introduction

We study the Riesz basis property and the stability of a flexible beam with nonuniform thickness or density, that is clamped at one end and is submitted to a linear boundary control force in position and velocity at the free end. The equations of motion of the system are given by

$$m(x)u_{tt}(x,t) + (EI(x)u_{xx}(x,t))_{xx} + \gamma(x)u_t(x) = 0, \quad 0 < x < 1,\ t > 0, \tag{1.1}$$

$$u(0,t) = u_x(0,t) = u_{xx}(1,t) = 0, \quad t > 0, \tag{1.2}$$

$$(EI(\cdot)u_{xx}(.,t))_x(1) = \alpha u(1,t) + \beta u_t(1,t), \quad t > 0, \tag{1.3}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad 0 < x < 1, \tag{1.4}$$

where α, β are two given positive constants, $u(x,t)$ stands for a transversal deviation of the beam at position x and time t; a subscript letter denotes the partial derivation with respect that variable. The length of the beam is chosen to be unity, $EI(x)$ is the stiffness of the beam, $m(x)$ is the mass density and $\gamma(x)$ is a continuous coefficient function of feedback damping. If $\gamma \geq 0$, it can be proven that the energy of the system decays exponential (see theorem 4.1).

A question was raised in Wang and al. [19]: Due to the nonuniform physical thickness and / or density of the Euler-Bernoulli beam with the variable coefficient damping $\gamma(x)$ in equation (1.1) , what kind of conditions on γ, can ensure that

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the system remains exponentially stable? In [19], the question is treated without boundary conditions. Our work is a continuation study of [19] and follows the same arguments. In this paper, we shall always assume that:

$$m(x), EI(x) \in \mathcal{C}^4(0,1) \quad \text{and} \quad m(x), EI(x) > 0. \tag{1.5}$$

With the assumption (1.5), we shall prove that system (1.1)–(1.4) is a Riesz spectral system in the sense that the generalized eigenfunctions of the system form a Riesz basis on the suitable Hilbert space (see [4]). The Riesz basis property, meaning that the generalized eigenvectors of the system form an unconditional basis for the state Hilbert space, is one of the fundamental properties of a linear vibrating system. For this kind of system, the stability is usually determined by the spectrum of the associated operator and one can also use the theory of nonharmonic Fourier series to obtain important properties such as the optimal decay rate of the energy.

There are two steps usually found in the study of linear systems with variable coefficients. The first is to transform the "dominant term" of the system under study into a uniform "dominant equation" by space scaling and state transformation where no variable coefficient is involved any longer. The second step is to approximate the eigenfunctions of the system by those of uniform "dominant equation". This fundamental idea comes essentially from Birkhoff's works [1] and [2] and Naimark [11] to estimate the eigenvalues. This approach has been used in dealing with the beam equations of variable coefficients (see Guo [7, 8], Wang [18] or [19] and the references therein).

Moreover, one of the methods on the verification of Riesz basis property well developed recently and applied successfully, is the classical Bari's theorem [3]. When $\alpha = 0$, the undamped case ($\gamma = 0$) has been studied in [7], where the author used a corollary of Bari's theorem on the Riesz basis property in [3]. Our work shall make use a result due to Wang [19], which deals with the eigenvalue problem of beams in the form of an ordinary differential equation $L(f) = \lambda f$ with λ-polynomial boundary conditions (Shkalikov [13], Tretter [15], Wang [16], [8] and the references therein). We establish conditions on the both positive feedback parameters α and β in order to get the Riesz basis property and the exponential stability for system (1.1)–(1.4).

The content of this article is as follows: in the next section, we convert system (1.1)–(1.4) into an abstract Cauchy problem in Hilbert state space, and discuss some basic properties of system. We show that system (1.1)–(1.4) can be associated to a $\mathcal{C}_0$ -semigroup, and the generator A_γ of C_0-semigroup has compact resolvents. Furthermore, we obtain an asymptotic expression for eigenvalues. In section 3, we discuss the Riesz basis property of the eigenfunctions as well as the exponential stability of the system. Through a bounded invertible transform $\mathcal{L}$, we establish the relationship between A_γ and $\mathbb{A}$ defined in (3.5) and obtain the Riesz basis property from the strong regularity of boundary conditions that has been verified in section 2. Incidentally, we also obtain conditions for the exponential stability of the system for indefinite damping.

2. Basic properties of the problem (1.1)–(1.4)

Let us introduce the following spaces:

$$H_E^2(0,1) = \{u(x) \in H^2(0,1) | u(0) = u_x(0) = 0\}, \tag{2.1}$$

$$H = H_E^2(0,1) \times L^2(0,1), \tag{2.2}$$

The superscript T stands for the transpose and the spaces $L^2(0,1)$ and $H^k(0,1)$ are defined as

$$L^2(0,1) = \big\{u : [0,1] \to \mathbb{C} : \int_0^1 |u|^2 dx < \infty\big\}, \tag{2.3}$$

$$H^k(0,1) = \{u : [0,1] \to \mathbb{C} : u, u^{(1)}, \dots, u^{(k)} \in L^2(0,1)\}. \tag{2.4}$$

In the space H, we define the inner-product

$$\langle u, v\rangle_H = \int_0^1 (m(x) f_2(x)\overline{g_2(x)} + EI(x) f_1''(x)\overline{g_1''(x)})dx + \alpha f_1(1)\overline{g_1(1)}, \tag{2.5}$$

where $u = (f_1, f_2)^T \in H$ and $v = (g_1, g_2)^T \in H$ and we denote by $\|\cdot\|_H$ the associated norm. Next, we define an unbounded linear operator $A_\gamma : D(A_\gamma) \subset H \to H$ as follows:

$$A_\gamma(f,g) = \Big(g(x), -\frac{1}{m(x)}((EI(x)f''(x))'' + \gamma(x)g(x))\Big)^T, \tag{2.6}$$

where $D(A_\gamma)$, the domain of operator A_γ is

$$\begin{aligned} D(A_\gamma) = \Big\{&(f,g)^T \in (H^4(0,1) \cap H_E^2(0,1)) \times H_E^2(0,1) : \\ &f''(1) = 0,\ (EI(\cdot)f''(\cdot))''(1) = \alpha f(1) + \beta g(1)\Big\}. \end{aligned} \tag{2.7}$$

With this notation, the set of equations (1.1)–(1.4) can be formally written as

$$\begin{gathered} \frac{dY(t)}{dt} = A_\gamma Y(t) \\ Y(0) = Y_0 \in H, \end{gathered} \tag{2.8}$$

where $Y(t) = (u(.,t), u_t(.,t))^T$, $Y(0) = (u_0, u_1)^T$. Here, it is clear that A_0 denotes the undamped case $\gamma(x) = 0$ which was studied in [21] and that

$$\Gamma_\gamma(f,g) = A_\gamma - A_0 = \Big(0, -\frac{\gamma(x)g(x)}{m(x)}\Big)$$

is a boundary linear operator on H. Therefore the following result follows immediately from the theory of operator semigroups (see Pazy [12, theorem 1.1]).

Theorem 2.1. *Let operators A_γ and A_0 be defined as before.Then A_0 is a m-dissipative operator and generates a C_0-group on H, and hence A_γ generates a C_0-group $e^{A_\gamma t}$ on H.*

Proof. In [21] we applied the Lumer-Phillips theorem, (see, e.g., [12, p.14]) to prove that operator A_0 is m-dissipative. Then using Hille-Yosida-Phillips theorem, we also obtained that operator A_0 is infinitesimal generator of a C_0-semigroup $S(t) = e^{A_0 t}$on H, satisfying

$$\|S(t)\| \le Me^{\omega t}.$$

Moreover we obtain $A_\gamma = \Gamma_\gamma + A_0$ where Γ_γ is a boundary linear operator on H. Then using the perturbation by bounded linear operator, we deduce that $A_\gamma = \Gamma_\gamma + A_0$ is infinitesimal generator of a C_0-semigroup $T(t) = e^{A_\gamma t}$, satisfying

$$\|T(t)\| \le Me^{(\omega + M\|\Gamma_\gamma\|)t}$$

(see A. Pazy [12, Theorem 1.1]). □

Theorem 2.2. *A_γ has compact resolvents and $0 \in \rho(A_\gamma)$. Therefore, the spectrum $\sigma(A_\gamma)$ consists entirely of isolated eigenvalues.*

Proof. Clearly, we only need to prove that $0 \in \rho(A_\gamma)$ and A_γ^{-1} is compact on H. For any $G = (g_1,\ g_2) \in H$, we need to find a unique $F = (f_1, f_2) \in D(A_\gamma)$ such that

$$A_\gamma F = G.$$

In other words such that the following equations are satisfied:

$$f_2(x) = g_1(x), \quad g_1 \in H_E^2(0,1) \tag{2.9}$$

$$-\frac{1}{m(x)}((EI(x)f_1''(x))'' + \gamma(x)f_2(x)) = g_2(x), \quad g_2 \in L^2(0,1) \tag{2.10}$$

$$f_1(0) = f_1'(0) = f_1''(1) = 0 \tag{2.11}$$

$$(EI(\cdot)f_1''(\cdot))'(1) = \alpha f_1(1) + \beta f_2(1). \tag{2.12}$$

Using (2.10) we obtain

$$(EI(x)f_1''(x))'' = -m(x)g_2(x) - \gamma(x)g_1(x)$$

By integrating we obtain

$$\int_x^1 (EI(r)f_1''(r))''dr = -\int_x^1 m(r)g_2(r) + \gamma(r)g_1(r)\,dr\,,$$

$$(EI(\cdot)f_1''(\cdot))'(1) - (EI(x)f_1''(x))' = -\int_x^1 m(r)g_2(r) + \gamma(r)g_1(r)\,dr\,.$$

Using the boundary condition (2.12) we obtain

$$\alpha f_1(1) + \beta g_1(1) - (EI(x)f_1''(x))' = -\int_x^1 m(r)g_2(r) + \gamma(r)g_1(r)\,dr\,,$$

$$(EI(x)f_1''(x))' - \alpha f_1(1) = \int_x^1 m(r)g_2(r) + \gamma(r)g_1(r)\,dr + \beta g_1(1)\,.$$

By integrating again we obtain

$$\begin{aligned}
&\int_x^1 (EI(\eta)f_1''(\eta))'d\eta - \alpha f_1(1)\int_x^1 d\eta \\
&= \int_x^1\int_\eta^1 m(r)g_2(r) + \gamma(r)g_1(r)\,dr\,d\eta + \beta g_1(1)\int_x^1 d\eta\,,
\end{aligned}$$

$$\begin{aligned}
&EI(1)f_1''(1) - EI(x)f_1''(x) - \alpha(1-x)f_1(1) \\
&= \int_x^1\int_\eta^1 m(r)g_2(r) + \gamma(r)g_1(r)\,dr\,d\eta + \beta(1-x)g_1(1)\,.
\end{aligned}$$

Since $f_1''(1) = 0$, we obtain

$$\begin{aligned}
&f_1''(x) + \alpha\frac{(1-x)}{EI(x)}f_1(1) \\
&= -\frac{1}{EI(x)}\int_x^1\int_\eta^1 m(r)g_2(r) + \gamma(r)g_1(r)\,dr\,d\eta - \beta\frac{(1-x)}{EI(x)}g_1(1)\,,
\end{aligned}$$

$$\int_0^x f_1''(\xi)d\xi + \alpha f_1(1)\int_0^x \frac{(1-\xi)}{EI(\xi)}d\xi$$

$$
\begin{aligned}
&= -\beta g_1(1)\int_0^x \frac{(1-\xi)}{EI(\xi)}d\xi - \int_0^x \frac{1}{EI(\xi)}\int_\xi^1\int_\eta^1 m(r)g_2(r)+\gamma(r)g_1(r)\,dr\,d\eta d\xi\,,\\
&f_1'(x)+\alpha f_1(1)\int_0^x \frac{(1-\xi)}{EI(\xi)}d\xi\\
&= \int_0^x -\frac{1}{EI(\xi)}\int_\xi^1\int_\eta^1 m(r)g_2(r)+\gamma(r)g_1(r)\,dr\,d\eta d\xi - \beta g_1(1)\int_0^x \frac{(1-\xi)}{EI(\xi)}d\xi\,,
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^x f_1'(s)ds+\alpha f_1(1)\int_0^x\int_0^s \frac{(1-\xi)}{EI(\xi)}\,d\xi\,ds\\
&= -\int_0^x\int_0^s \frac{1}{EI(\xi)}\int_\xi^1\int_\eta^1 m(r)g_2(r)+\gamma(r)g_1(r)\,dr\,d\eta\,d\xi\,ds\\
&\quad -\beta g_1(1)\int_0^x\int_0^s \frac{(1-\xi)}{EI(\xi)}\,d\xi\,ds\,.
\end{aligned}
$$

Using the boundary condition (2.11) we have

$$
\begin{aligned}
&f_1(x)+\alpha f_1(1)\int_0^x\int_0^s \frac{(1-\xi)}{EI(\xi)}\,d\xi\,ds\\
&= -\int_0^x\int_0^s \frac{1}{EI(\xi)}\int_\xi^1\int_\eta^1 m(r)g_2(r)+\gamma(r)g_1(r)\,dr\,d\eta\,d\xi\,ds\\
&\quad -\beta g_1(1)\int_0^x\int_0^s \frac{(1-\xi)}{EI(\xi)}\,d\xi\,ds\,.
\end{aligned}
$$

Next we determine $f(1)$. We obtain

$$
\begin{aligned}
&f_1(1)+\alpha f_1(1)\int_0^1\int_0^s \frac{(1-\xi)}{EI(\xi)}\,d\xi\,ds\\
&= -\int_0^1\int_0^s \frac{1}{EI(\xi)}\int_\xi^1\int_\eta^1 m(r)g_2(r)+\gamma(r)g_1(r)\,dr\,d\eta\,d\xi\,ds\\
&\quad -\beta g_1(1)\int_0^1\int_0^s \frac{(1-\xi)}{EI(\xi)}\,d\xi\,ds\,,
\end{aligned}
$$

$$
\begin{aligned}
&f_1(1)(1+\alpha\int_0^1\int_0^s \frac{(1-\xi)}{EI(\xi)}\,d\xi\,ds)\\
&= -\int_0^1\int_0^s \frac{1}{EI(\xi)}\int_\xi^1\int_\eta^1 m(r)g_2(r)+\gamma(r)g_1(r)\,dr\,d\eta\,d\xi\,ds\\
&\quad -\beta g_1(1)\int_0^1\int_0^s \frac{(1-\xi)}{EI(\xi)}\,d\xi\,ds\,,
\end{aligned}
$$

$$
\begin{aligned}
f_1(1) = \Big(&-\int_0^1\int_0^s \frac{1}{EI(\xi)}\int_\xi^1\int_\eta^1 m(r)g_2(r)+\gamma(r)g_1(r)\,dr\,d\eta\,d\xi\,ds\\
&-\beta g_1(1)\int_0^1\int_0^s \frac{(1-\xi)}{EI(\xi)}\,d\xi\,ds\Big)\Big/\Big(1+\alpha\int_0^1\int_0^s \frac{(1-\xi)}{EI(\xi)}\,d\xi\,ds\Big)\,,
\end{aligned}
$$

then

$$
f_1(x) = -K\int_0^x\int_0^s \frac{(1-\xi)}{EI(\xi)}\,d\xi\,ds - \beta g_1(1)\int_0^1\int_0^s \frac{(1-\xi)}{EI(\xi)}\,d\xi\,ds
$$

$$-\int_0^1\int_0^s \frac{1}{EI(\xi)}\int_\xi^1\int_\eta^1 m(r)g_2(r)+\gamma(r)g_1(r)\,dr\,d\eta\,d\xi\,ds\,,$$

with

$$K=\alpha\Big(-\int_0^1\int_0^s \frac{1}{EI(\xi)}\int_\xi^1\int_\eta^1 m(r)g_2(r)+\gamma(r)g_1(r)\,dr\,d\eta\,d\xi\,ds \\ -\beta g_1(1)\int_0^1\int_0^s\frac{(1-\xi)}{EI(\xi)}\,d\xi\,ds\Big)\Big/\Big(1+\alpha\int_0^1\int_0^s\frac{(1-\xi)}{EI(\xi)}\,d\xi\,ds\Big).$$

Obviously, $(f_1,\ f_2)\in D(A_\gamma)$, therefore

$$F=(f_1,\ f_2)=A_\gamma^{-1}G=(B(x),\ g_1)$$

where

$$B(x)=-K\int_0^x\int_0^s\frac{(1-\xi)}{EI(\xi)}\,d\xi\,ds-\beta g_1(1)\int_0^1\int_0^s\frac{(1-\xi)}{EI(\xi)}\,d\xi\,ds \\ -\int_0^1\int_0^s \frac{1}{EI(\xi)}\int_\xi^1\int_\eta^1 m(r)g_2(r)+\gamma(r)g_1(r)\,dr\,d\eta\,d\xi\,ds.$$

Finally we obtain that $0\in\rho(A_\gamma)$ and Sobolev's embedding theorem implies that A_γ^{-1} is a compact operator on H. Therefore, the spectrum $\sigma(A_\gamma)$ consists entirely of isolated eigenvalues. $\square$

3. Spectral analysis and the Riesz basis property

3.1. **Spectral analysis of operator A_γ.** In this section, we study the basic properties of system (1.1)–(1.4). Our work shall make use of the following result from [19], which deals with the eigenvalue problem of beams in the form of an ordinary differential equation $L(f)=\lambda f$ with λ-polynomial boundary conditions (see Shkalikov [13]; Tretter [15]). To begin, we recall some notations and definitions. Let $L(f)$ be an ordinary differential operator of order $n=2m\in\mathbb{N}$,

$$L(f)=f^{(n)}(x)+\sum_{\nu=1}^n f_\nu(x)f^{(n-\nu)}(x),\quad 0<x<1, \tag{3.1}$$

and let the boundary conditions defined at the two points $x=0$, and $x=1$ be

$$B_j(f)=\sum_{\nu=0}^{k_j}(\alpha_{j_\nu}f^{(k_j-\nu)}(0)+\beta_{j_\nu}f^{(k_j-\nu)}(1)),\quad 1\le j\le n, \tag{3.2}$$

where $k_j\in\mathbb{N}$, $1\le k_j\le n-1$ and $\alpha_{j_\nu},\ \beta_{j_\nu}\in\mathbb{C}$, $|\alpha_{j_0}|+|\beta_{j_0}|>0$. Suppose that the coefficient functions $f_\nu(x)$ $(1\le\nu\le n)$ in (3.1) are sufficiently smooth in $(0,1)$, and that the boundary conditions are normalized in the sense that $\kappa=\sum_{j=1}^n k_j$ is minimal with respect to all equivalent boundary conditions (see Naimark [11]).

Let $f_k(x,\rho)$ $(k=1,2,\dots,n)$ be the fundamental solutions for the equation:

$$L(f)+\rho^n f+\rho^m\mu(x)f(x)=0,\quad \rho\in\mathbb{C} \tag{3.3}$$

where $\mu(x)$ being continuous in $[0,1]$, and let ω_k $(k=1,2,\dots,n)$ be the n-th roots of $\omega^n+1=0$. If we denote by $\Delta(\rho)$ the characteristic determinant of (3.3) with respect to (3.2)

$$\Delta(\rho)=\det\big[B_j(f_k(.,\rho))\big]_{j,k=1,2,\dots,n}\,,$$

then $\Delta(\rho)$ can be expressed asymptotically in the form, for $(r \geq 1)$,

$$\Delta(\rho) = \rho^k \sum_{\mathbb{K}_k} e^{\rho\mu\mathbb{K}_k}[F^{\mathbb{K}_k}]_r \,, \tag{3.4}$$

whenever ρ is large enough (see Shkalikov [13] and Naimark [11]). Here, $\mathbb{K}_k$ is a k-elements subset of $\{1, 2, \dots, n\}$, $\mu_{\mathbb{K}_k} = \sum_{j\in\mathbb{K}_k} \omega_j$,

$$[F^{\mathbb{K}_k}]_r = F_0^{\mathbb{K}_k} + \rho^{-1}F_1^{\mathbb{K}_k} + \dots + \rho^{-r+1}F_{r-1}^{\mathbb{K}_k} + \mathcal{O}(\rho^{-r}),$$

and the sum runs over all possible selections of $\mathbb{K}_k$. Here and henceforth, $\mathcal{O}(\rho^{-r})$ means that $|\rho^r \times \mathcal{O}(\rho^{-r})|$ is bounded as $|\rho| \to \infty$.

Definition 3.1 ([19, p. 461])**.** The boundary problem (3.3) with (3.2) is said to be regular if the coefficients $F_0^{\mathbb{K}_k}$ in (3.4) are nonzero. Furthermore, the regular boundary problem (3.3) with (3.2) is said to be strongly regular if the zeros of $\Delta(\rho)$ are asymptotically simple and isolated one from another.

Let $W_2^m(0,1)$ be the usual Sobolev space of order m and let

$$V_E^m(0,1) = \{f(x) \in W_2^m(0,1) | B_j(f) = 0, \quad k_j < m\}.$$

Define a Hilbert space

$$\mathbb{H} = V_E^m(0,1) \times L^2(0,1),$$

with the norm

$$\|(f,g)\|_{\mathbb{H}}^2 = \|f\|_{W_2^m}^2 + \|g\|_2^2$$

and define the operator $\mathbb{A}$ in $\mathbb{H}$ by

$$\begin{gathered} \mathbb{A}(f,g) = (g, -L(f) - \mu(x)g) \\ D(\mathbb{A}) = \{(f,g) \in \mathbb{H} | \mathbb{A}(f,g) \in \mathbb{H},\ B_j(f) = 0,\ k_j \geq m\}. \end{gathered} \tag{3.5}$$

The following result used in [19] was presented in [16]. The reader can also be referred to [18, chapter 3].

Theorem 3.2 ([19, p. 461])**.** *If the ordinary differential system with parameter* $\lambda = \rho^m$

$$\begin{gathered} L(f,\lambda) = L(f) + \lambda^2 f + \lambda\mu(x) f \\ B_j(f) = 0, \quad 1 \leq j \leq 2m \end{gathered} \tag{3.6}$$

has strongly regular boundary conditions, then the generalized eigenfunction system of $\mathbb{A}$ *form a Riesz basis in the Hilbert space* $\mathbb{H}$.

Now we are ready to study the eigenvalue problem of A_γ. Let $\lambda \in \sigma(A_\gamma)$ and $\Phi = (\phi, \Psi)$ be an eigenfunction of A_γ corresponding to λ. Then $\Psi = \lambda\phi$ and ϕ satisfy the following equations:

$$\begin{gathered} \lambda^2 m(x)\phi(x) + (EI(x)\phi''(x))'' + \lambda\gamma(x)\phi(x) = 0, \quad 0 < x < 1, \\ \phi(0) = \phi'(0) = \phi''(1) = 0 \\ \phi'''(1) = \frac{1}{EI(1)}(\alpha + \beta\lambda)\phi(1). \end{gathered} \tag{3.7}$$

Expanding (3.7) yields

$$\begin{gathered}
\phi^{(4)}(x)+\frac{2EI'(x)}{EI(x)}\phi'''(x)+\frac{EI''(x)}{EI(x)}\phi''(x)\\
+\frac{\lambda^2 m(x)}{EI(x)}\phi(x)+\frac{\lambda\gamma(x)}{EI(x)}\phi(x)=0,\quad 0<x<1,\\
\phi(0)=\phi'(0)=\phi''(1)=0\\
\phi'''(1)=\frac{1}{EI(1)}(\alpha+\beta\lambda)\phi(1).
\end{gathered} \tag{3.8}$$

Two basic transformations are essential.

First, the "dominant term", $\phi^{(4)}(x)+\frac{\lambda^2 m(x)}{EI(x)}\phi(x)$ of (3.8), is transformed to become a uniform form by space scaling. In fact, set:

$$f(z)=\phi(x),\quad z=z(x)=\frac{1}{h}\int_0^x(\frac{m(\zeta)}{EI(\zeta)})^{1/4}d\zeta \tag{3.9}$$

where

$$h=\int_0^1(\frac{m(\zeta)}{EI(\zeta)})^{1/4}d\zeta \tag{3.10}$$

Then, (3.8) together with its boundary conditions can be transformed into

$$\begin{gathered}
f^{(4)}(z)+a(z)f'''(z)+b(z)f''(z)+c(z)f'(z)\\
+\lambda^2h^4f(z)+\frac{\lambda h^4\gamma(x)}{m(x)}f(x)=0,\quad 0<z<1,\\
f(0)=f'(0)=0,\\
z_x^2(1)f''(1)+z_{xx}(1)f'(1)=0,\\
f'''(1)+\frac{3z_{xx}(1)}{z_x^2(1)}f''(1)+\frac{z_{xxx}(1)}{z_x^3(1)}f'(1)-\frac{(\alpha+\lambda\beta)}{z_x^3(1)EI(1)}f(1)=0,
\end{gathered} \tag{3.11}$$

with

$$a(z)=\frac{6z_{xx}}{z_x^2}+\frac{2EI'(x)}{z_xEI(x)}, \tag{3.12}$$

$$b(z)=\frac{3z_{xx}^2}{z_x^4}+\frac{6z_{xx}EI'(x)}{z_x^3EI(x)}+\frac{EI''(x)}{z_x^2EI(x)}+\frac{4z_{xxx}}{z_x^3}, \tag{3.13}$$

$$c(z)=\frac{z_{xxxx}}{z_x^4}+\frac{2z_{xxx}EI'(x)}{z_x^4EI(x)}+\frac{z_{xx}EI''(x)}{z_x^4EI(x)}, \tag{3.14}$$

$$z_x=\frac{1}{h}\Big(\frac{m(x)}{EI(x)}\Big)^{1/4},\quad z_x^4=\frac{1}{h^4}\frac{m(x)}{EI(x)}, \tag{3.15}$$

$$z_{xx}=\frac{1}{4h}\Big(\frac{m(x)}{EI(x)}\Big)^{-3/4}\frac{d}{dx}\Big(\frac{m(x)}{EI(x)}\Big)^{1/4}. \tag{3.16}$$

If we definite

$$d(x)=\frac{\gamma(x)}{m(x)}, \tag{3.17}$$

then equation in (3.11) is

$$\begin{gathered}
f^{(4)}(z)+a(z)f'''(z)+b(z)f''(z)+c(z)f'(z)\\
+\lambda^2h^4f(z)+\lambda h^4d(z)f(z)=0,\quad 0<z<1.
\end{gathered} \tag{3.18}$$

Second, to cancel the term $a(z)f'''(z)$ in (3.11) as was done in Naimark [11], we make the invertible state transformation

$$g(z) = \exp\Big(\frac{1}{4}\int_0^z a(\zeta)d\zeta\Big)f(z), \quad 0 < z < 1,$$

and we arrive at the following eigenvalue problem that is equivalent to the original one:

$$\begin{gathered} g^{(4)}(z) + b_1(z)g''(z) + c_1(z)g'(z) + d_1(z)g(z) \\ + \lambda h^4 d(z)g(z) + \lambda^2 h^4 g(z) = 0, \quad 0 < z < 1, \\ g(0) = g'(0) = 0 \\ g''(1) + b_{11}g'(1) + b_{12}g(1) = 0 \\ g'''(1) + b_{21}g''(1) + b_{22}g'(1) + b_{23}g(1) = 0, \end{gathered} \tag{3.19}$$

where

$$b_1(z) = -\frac{3}{2}a'(z) - \frac{3}{8}a^2(z) + b(z), \tag{3.20}$$

$$c_1(z) = \frac{1}{8}a^3(z) - \frac{1}{2}a(z)b(z) - a''(z) + c(z), \tag{3.21}$$

$$\begin{aligned} d_1(z) = \frac{3}{16}a'^2(z) - \frac{1}{4}a'''(z) + \frac{3}{32}a'(z)a^2(z) - \frac{3}{256}a^4(z) \\ + b(z)(\frac{1}{16}a^2(z) - \frac{1}{4}a'(z)) - \frac{a(z)c(z)}{4}, \end{aligned} \tag{3.22}$$

$$b_{11} = -\frac{1}{2}a(1) + \frac{z_{xx}(1)}{z_x^2(1)}, \tag{3.23}$$

$$b_{12} = \frac{\frac{1}{16}z_x^2(1)a^2(1) - \frac{1}{4}z_x^2(1)a'(1) - \frac{1}{4}z_{xx}(1)a(1)}{z_x^2(1)}, \tag{3.24}$$

$$b_{21} = -\frac{3}{4}a(1) + \frac{3z_{xx}(1)}{z_x^2(1)}, \tag{3.25}$$

$$b_{22} = -\frac{3}{4}a'(1) + \frac{3}{16}a^2(1) - \frac{3z_{xx}(1)a(1)}{2z_x^2(1)} + \frac{z_{xxx}(1)}{z_x^3(1)}, \tag{3.26}$$

$$\begin{aligned} b_{23} = -\frac{1}{4}a''(1) + \frac{3}{16}a'(1)a(1) - \frac{1}{64}a^3(1) - \frac{3z_{xx}(1)a'(1)}{4z_x^2(1)} \\ + \frac{3z_{xx}(1)a^2(1)}{16z_x^2(1)} - \frac{z_{xxx}(1)a(1)}{4z_x^3(1)} - \frac{(\alpha+\lambda\beta)}{z_x^3(1)EI(1)}. \end{aligned} \tag{3.27}$$

To solve the eigenvalue problem (3.19), we follow the procedure in Birkhoff [1, 2] and Naimark [11], and divide the complex plane into eight distinct sectors,

$$S_k = \big\{z \in \mathbb{C} : \frac{k\pi}{4} \le \arg z \le \frac{(k+1)\pi}{4}\big\}, \quad k = 0, 1, 2, \dots, 7 \tag{3.28}$$

and let ω_1, ω_2, ω_3, ω_4 be the roots of equation $\theta^4 + 1 = 0$ that are arranged so that

$$\mathrm{Re}(\rho\omega_1) \le \mathrm{Re}(\rho\omega_2) \le \mathrm{Re}(\rho\omega_3) \le \mathrm{Re}(\rho\omega_4), \quad \forall \rho \in S_k. \tag{3.29}$$

Obviously, in sector S_1, we can choose

$$\omega_1 = \exp\big(i\frac{3}{4}\pi\big) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \quad \omega_2 = \exp\big(i\frac{1}{4}\pi\big) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i,$$

$$\omega_3 = \exp\big(i\frac{5}{4}\pi\big) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \quad \omega_4 = \exp\big(i\frac{7}{4}\pi\big) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i,$$

which satisfy the inequalities in (3.29) and choices can also be made for other sectors. In the rest of this section, we shall derive the asymptotic behavior of the eigenvalue of the sectors S_1 and S_2 because the same will hold for the other sectors with similar proofs.

Setting $\lambda = \rho^2/h^2$, in each sector S_k, we have the following result about the asymptotic fundamental solutions of system (3.19).

Lemma 3.3. *For $\rho \in S_k$ with ρ large enough, the equation*

$$\begin{aligned} &g^{(4)}(z) + b_1(z)g''(z) + c_1(z)g'(z) + d_1(z)g(z) \\ &+ \rho^2 h^2 d(z)g(z) + \rho^4 g(z) \quad =0, \quad 0 < z < 1, \end{aligned}$$

has four linearly independent asymptotic fundamental solutions,

$$\Phi_s(z,\rho) = e^{\rho\omega_s z}\big(1 + \frac{\Phi_{s,1}(z)}{\rho} + \mathcal{O}(\rho^{-2})\big), \quad s = 1,2,3,4 \tag{3.30}$$

and hence their derivatives for $s = 1,2,3,4$ and $j = 1,2,3$ are given by

$$\frac{d^j}{dz^j}\Phi_s(z,\rho) = (\rho\omega_s)^j e^{\rho\omega_s z}\Big(1 + \frac{\Phi_{s,1}(z)}{\rho} + \mathcal{O}(\rho^{-2})\Big) \tag{3.31}$$

where

$$\Phi_{s,1}(z) = -\frac{1}{4\omega_s}\int_0^z b_1(\zeta)d\zeta - \frac{h^2}{4\omega_s^3}\int_0^z d(\zeta)d\zeta, \quad \Phi_{s,1}(0) = 0, \tag{3.32}$$

for $s = 1,2,3,4$, and

$$\Phi_{s,1}(z) = \frac{\omega_s^2\mu_1 + \mu_2}{\omega_s^3}, \quad \text{with } \mu_1 = -\frac{1}{4}\int_0^1 b_1(\zeta)d\zeta, \quad \mu_2 = -\frac{h^2}{4}\int_0^1 d(\zeta)d\zeta. \tag{3.33}$$

Proof. The proof is a direct result in Birkhoff [1], [2] and Naimark [11]. Here we briefly present a simple calculation to find the asymptotic expansions of fundamental solutions in sector S_k. Let

$$\widetilde{\Phi}_s(z,\rho) = e^{\rho\omega_s z}\Big(\Phi_{s,0}(z) + \frac{\Phi_{s,1}(z)}{\rho} + \mathcal{O}(\rho^{-2})\Big), \quad s = 1,2,3,4$$

and

$$\begin{aligned} D(g) = g^{(4)}(z) &+ b_1(z)g''(z) + c_1(z)g'(z) + d_1(z)g(z) \\ &+ \rho^2 h^2 d(z)g(z) + \rho^4 g(z), \quad 0 < z < 1. \end{aligned}$$

Then, substituting $\widetilde{\Phi}_s(z,\rho)$ in the expression of $e^{-\rho\omega_s z}D(g)$, for $s = 1,2,3,4$, it yields

$$\begin{aligned} &e^{-\rho\omega_s z}D(\widetilde{\Phi}_s(z,\rho)) \\ &= (\rho\omega_s)^4\Big(\Phi_{s,0}(z) + \frac{\Phi_{s,1}(z)}{\rho}\Big) + 4(\rho\omega_s)^3\Big(\Phi'_{s,0}(z) + \frac{\Phi'_{s,1}(z)}{\rho}\Big) \\ &\quad + 6(\rho\omega_s)^2\Big(\Phi''_{s,0}(z) + \frac{\Phi''_{s,1}(z)}{\rho}\Big) + 4\rho\omega_s\Big(\Phi'''_{s,0}(z) + \frac{\Phi'''_{s,1}(z)}{\rho}\Big) + \Phi^{(4)}_{s,0}(z) + \frac{\Phi^{(4)}_{s,1}(z)}{\rho} \\ &\quad + b_1(z)(\rho\omega_s)^2\Big(\Phi_{s,0}(z) + \frac{\Phi_{s,1}(z)}{\rho}\Big) + 2b_1(z)\rho\omega_s\Big(\Phi'_{s,0}(z) + \frac{\Phi'_{s,1}(z)}{\rho}\Big) \\ &\quad + b_1(z)\Big(\Phi''_{s,0}(z) + \frac{\Phi''_{s,1}(z)}{\rho}\Big) + c_1(z)\rho\omega_s\Big(\Phi_{s,0}(z) + \frac{\Phi_{s,1}(z)}{\rho}\Big) \end{aligned}$$

$$
\begin{aligned}
&+ c_1(z)\Big(\Phi'_{s,0}(z) + \frac{\Phi'_{s,1}(z)}{\rho}\Big) + (\rho^4 + d_1(z) + \rho^2 h^2 d(z))\Big(\Phi_{s,0}(z) + \frac{\Phi_{s,1}(z)}{\rho}\Big) \\
&= \rho^4[\Phi_{s,0}(z) - \Phi_{s,0}(z)] + \rho^3[-\Phi_{s,1}(z) + 4\omega_s^3\Phi'_{s,0}(z) + \Phi_{s,1}(z)] \\
&\quad + \rho^2[4\omega_s^3\Phi'_{s,1}(z) + 6\omega_s^2\Phi''_{s,0}(z) + b_1(z)\omega_s^2\Phi_{s,0}(z) \\
&\quad + h^2 d(z)\Phi_{s,0}(z)] + \rho R_s(z,\rho),
\end{aligned}
$$

where $R_s(z,\rho)$ denote the remaining terms in the above equation and satisfy the following estimates for some positive constant M:

$$
R_s(z,\rho) \le M, \quad 0 < z < 1. \tag{3.34}
$$

Thus, by setting the coefficients of ρ^3 and ρ^2 to zero respectively, we obtain

$$
\begin{gathered}
\Phi'_{s,0}(z) = 0, \\
4\omega_s^3\Phi'_{s,1}(z) + 6\omega_s^2\Phi''_{s,0}(z) + b_1(z)\omega_s^2\Phi_{s,0}(z) + h^2 d(z)\Phi_{s,0}(z) = 0,
\end{gathered}
$$

which yield that $\Phi_{s,0}(z) = 1$ and $\Phi_{s,1}(z)$ given in (3.33) are linear independent solutions. Thus, as in the theorem in (Birkhoff [1], pp.225-226), we obtain the linearly independent fundamental solutions of

$$
\begin{gathered}
g^{(4)}(z) + b_1(z)g''(z) + c_1(z)g'(z) + d_1(z)g(z) \\
+ \rho^4 g(z) + \rho^2 h^2 d(z)g(z) = 0, \quad 0 < z < 1,
\end{gathered}
$$

given by $(s = 1,2,3,4)$

$$
\Phi_s(z,\rho) = \widetilde{\Phi}_s(z,\rho) + e^{\rho\omega_s z}\mathcal{O}(\rho^{-2}),
$$

from which we deduce the required results (3.30) and (3.31) $\square$

For convenience, we introduce the notation $[a]_2 = a + \mathcal{O}(\rho^{-2})$.

Lemma 3.4. *For $\rho \in S_1$, if we set $\delta = \sin(\pi/4) = \sqrt{2}/2$, then we have inequalities*

$$
\operatorname{Re}(\rho\omega_1) \le -|\rho|\delta, \quad \operatorname{Re}(\rho\omega_4) \ge |\rho|\delta, \quad e^{\rho\omega_1} = \mathcal{O}(\rho^{-2})
$$

as $|\rho| \to \infty$.

Furthermore, substituting (3.30) and (3.31) into the boundary conditions (3.19), we obtain asymptotic expressions for the boundary conditions for large enough $|\rho|$:

$$
U_4(\Phi_s,\rho) = \Phi_s(0,\rho) = 1 + \mathcal{O}(\rho^{-2}) = [1]_2, \quad s = 1,2,3,4, \tag{3.35}
$$

$$
U_3(\Phi_s,\rho) = \Phi'_s(0,\rho) = \rho\omega_s(1 + \mathcal{O}(\rho^{-2})), \tag{3.36}
$$

$$
U_3(\Phi_s,\rho) = \rho\omega_s[1]_2, \quad s = 1,2,3,4, \tag{3.37}
$$

$$
\begin{aligned}
&U_2(\Phi_s,\rho) \\
&= \Phi''_s(1,\rho) + b_{11}\Phi'_s(1,\rho) + b_{12}\Phi_s(1,\rho), \quad \text{for } s = 1,2,3,4, \\
&= (\rho\omega_s)^2 e^{\rho\omega_s}(1 + (\omega_s^2\mu_1 + \mu_2)\rho^{-1}\omega_s^{-3} + b_{11}\rho^{-1}\omega_s^{-1} + \mathcal{O}(\rho^{-2})),
\end{aligned} \tag{3.38}
$$

$$
U_2(\Phi_s,\rho) = (\rho\omega_s)^2 e^{\rho\omega_s}[1 + (\omega_s^2(\mu_1 + b_{11}) + \mu_2)\omega_s^{-3}\rho^{-1}]_2, \tag{3.39}
$$

$$
\begin{aligned}
&U_1(\Phi_s,\rho)\\
&=\Phi_s'''(1,\rho)+b_{21}\Phi_s''(1,\rho)+b_{22}\Phi_s'(1,\rho)+b_{23}\Phi_s(1,\rho)\\
&=(\rho\omega_s)^3e^{\rho\omega_s}(1+(\omega_s^2\mu_1+\mu_2)\rho^{-1}\omega_s^{-3}+b_{21}\rho^{-1}\omega_s^{-1}\\
&\quad+b_{23}(\rho\omega_s)^{-3}+\mathcal{O}(\rho^{-2}))\\
&=(\rho\omega_s)^3e^{\rho\omega_s}(1+(\omega_s^2(\mu_1+b_{21})+\mu_2)\omega_s^{-3}\rho^{-1}\\
&\quad-\frac{\beta\omega_s^{-3}\rho^{-1}}{z_x^3(1)EI(1)h^2}+\mathcal{O}(\rho^{-2}))
\end{aligned}\tag{3.40}
$$

$$
U_1(\Phi_s,\rho)=(\rho\omega_s)^3e^{\rho\omega_s}[1+(\omega_s^2(\mu_1+b_{21})+\mu_2)\omega_s^{-3}\rho^{-1}-\chi]_2,\tag{3.41}
$$

where

$$
\chi=\frac{\beta\omega_s^{-3}\rho^{-1}}{z_x^3(1)EI(1)h^2},\quad s=1,2,3,4,.
$$

Note that $\lambda=\rho^2/h^2\neq 0$ is the eigenvalue in (3.19) if and only if ρ satisfies the characteristic equation

$$
\Delta(\rho)=\begin{vmatrix}
U_4(\Phi_1,\rho) & U_4(\Phi_2,\rho) & U_4(\Phi_3,\rho) & U_4(\Phi_4,\rho)\\
U_3(\Phi_1,\rho) & U_3(\Phi_2,\rho) & U_3(\Phi_3,\rho) & U_3(\Phi_4,\rho)\\
U_2(\Phi_1,\rho) & U_2(\Phi_2,\rho) & U_2(\Phi_3,\rho) & U_2(\Phi_4,\rho)\\
U_1(\Phi_1,\rho) & U_1(\Phi_2,\rho) & U_1(\Phi_3,\rho) & U_1(\Phi_4,\rho)
\end{vmatrix}=0,\tag{3.42}
$$

so substituting (3.35)–(3.41) in (3.42)and using Lemma 3.4, we obtain that $\Delta(\rho)$ is the determinant of the matrix whose four columns are:

$$
\begin{gathered}
\begin{pmatrix}[1]_2\\ \rho\omega_1[1]_2\\ 0\\ 0\end{pmatrix},
\begin{pmatrix}[1]_2\\ \rho\omega_2[1]_2\\ (\rho\omega_2)^2e^{\rho\omega_2}[1+(\omega_2^2(\mu_1+b_{11})+\mu_2)\omega_2^{-3}\rho^{-1}]_2\\ (\rho\omega_2)^3e^{\rho\omega_2}[1+(\omega_2^2(\mu_1+b_{21})+\mu_2)\omega_2^{-3}\rho^{-1}-\chi]_2\end{pmatrix},\\
\begin{pmatrix}[1]_2\\ \rho\omega_3[1]_2\\ (\rho\omega_3)^2e^{\rho\omega_3}[1+(\omega_3^2(\mu_1+b_{11})+\mu_2)\omega_3^{-3}\rho^{-1}]_2\\ (\rho\omega_3)^3e^{\rho\omega_3}[1+(\omega_3^2(\mu_1+b_{21})+\mu_2)\omega_3^{-3}\rho^{-1}-\chi]_2\end{pmatrix},\\
\begin{pmatrix}0\\ 0\\ (\rho\omega_4)^2e^{\rho\omega_4}[1+(\omega_4^2(\mu_1+b_{11})+\mu_2)\omega_4^{-3}\rho^{-1}]_2\\ (\rho\omega_4)^3e^{\rho\omega_4}[1+(\omega_4^2(\mu_1+b_{21})+\mu_2)\omega_4^{-3}\rho^{-1}-\chi]_2\end{pmatrix}.
\end{gathered}
$$

Expanding the above determinant, we obtain the following expression:

$$
\begin{aligned}
&\Delta(\rho)\\
&=\rho^6e^{\rho\omega_4}\Big\{(\omega_2-\omega_1)\omega_3^2\omega_4^2e^{\rho\omega_3}[\omega_4-\omega_3+(\omega_3^{-1}\omega_4-\omega_3\omega_4^{-1})(\mu_1+b_{11})\rho^{-1}\\
&\quad+(\omega_4^{-2}-\omega_3^{-2})\mu_2\rho^{-1}+(\omega_4\omega_3^{-3}-\omega_3\omega_4^{-3})\mu_2\rho^{-1}+\frac{\beta\rho^{-1}}{z_x^3(1)EI(1)h^2}(\omega_3^{-2}-\omega_4^{-2})]\\
&\quad+(\omega_1-\omega_3)\omega_2^2\omega_4^2e^{\rho\omega_2}[\omega_4-\omega_2+(\omega_2^{-1}\omega_4-\omega_2\omega_4^{-1})(\mu_1+b_{11})\rho^{-1}\\
&\quad+(\omega_4^{-2}-\omega_2^{-2})\mu_2\rho^{-1}+(\omega_4\omega_2^{-3}-\omega_2\omega_4^{-3})\mu_2\rho^{-1}+\frac{\beta\rho^{-1}}{z_x^3(1)EI(1)h^2}(\omega_2^{-2}-\omega_4^{-2})]\\
&\quad+\mathcal{O}(\rho^{-2})\}
\end{aligned}
$$

In sector S_1, the choices are such that: $\omega_1^2 = -i$, $\omega_2^2 = i$, $\omega_3^2 = i$, $\omega_4^2 = -i$, $\omega_3^{-1}\omega_4 = i$, $\omega_2^{-1}\omega_4 = -i$, $\omega_3 = -\omega_2$, $\omega_4 - \omega_3 = \sqrt{2}$, $\omega_1 - \omega_3 = \sqrt{2}i$, $\omega_2 - \omega_1 = \sqrt{2}$, $\omega_4 - \omega_2 = -i\sqrt{2}$, $\omega_2^{-2} - \omega_4^{-2} = -2i$, $\omega_3^{-2} - \omega_4^{-2} = -2i$, $\omega_3^2\omega_4^2 = 1$, $\omega_2^2\omega_4^2 = 1$. A straightforward simplification will arrive at the following result, which is also true on all other sectors S_k (see Naimark, [11]).

Theorem 3.5. *Let $\Delta(\rho)$ be the characteristic determinant of the eigenvalue problem* (3.19)*. In sector S_1, an asymptotic expression of $\Delta(\rho)$ is given by:*

$$\Delta(\rho) = \rho^6 e^{\rho\omega_4}\{2e^{\rho\omega_2} + 2e^{-\rho\omega_2} + 2\mu_3\rho^{-1}e^{\rho\omega_2} + 2i\mu_4\rho^{-1}e^{-\rho\omega_2} + \mathcal{O}(\rho^{-2})\}, \tag{3.43}$$

where

$$\begin{gathered} \mu_3 = \sqrt{2}(\mu_1 + b_{11}) - \sqrt{2}\mu_2 + \sqrt{2}\frac{\beta}{z_x^3(1)EI(1)h^2} \\ \mu_4 = \sqrt{2}(\mu_1 + b_{11}) + \sqrt{2}\mu_2 - \sqrt{2}\frac{\beta}{z_x^3(1)EI(1)h^2} \end{gathered} \tag{3.44}$$

Thus, the boundary eigenvalue problem (3.19) *is strongly regular.*

Using (3.43), we can deduce an asymptotic expression for the eigenvalues of problem (3.19). The equation $\Delta(\rho) = 0$ and (3.43) imply that

$$2e^{\rho\omega_2} + 2e^{-\rho\omega_2} + 2\mu_3\rho^{-1}e^{\rho\omega_2} + 2i\mu_4\rho^{-1}e^{-\rho\omega_2} + \mathcal{O}(\rho^{-2}) = 0$$

which is equivalent to

$$e^{\rho\omega_2} + e^{-\rho\omega_2} + \mu_3\rho^{-1}e^{\rho\omega_2} + i\mu_4\rho^{-1}e^{-\rho\omega_2} + \mathcal{O}(\rho^{-2}) = 0 \tag{3.45}$$

and can be rewritten as

$$e^{\rho\omega_2} + e^{-\rho\omega_2} + \mathcal{O}(\rho^{-1}) = 0. \tag{3.46}$$

Note that the equation $e^{\rho\omega_2} + e^{-\rho\omega_2} = 0$ has solutions

$$\rho_n = \big(n + \frac{1}{2}\big)\frac{\pi i}{\omega_2}, \quad n = 1, 2, \dots \tag{3.47}$$

Let $\widetilde{\rho_n}$ be the solutions of (3.46). Applying Rouche's theorem see (Naimark [11, p.70]) to (3.46), we obtain the expression

$$\widetilde{\rho_n} = \rho_n + \alpha_n = (n + \frac{1}{2})\frac{\pi i}{\omega_2} + \alpha_n, \quad \alpha_n = \mathcal{O}(n^{-1}),\ n = N, N+1, \dots, \tag{3.48}$$

where N is a large positive integer. Substituting $\widetilde{\rho_n}$ into (3.45), and using the fact that $e^{\rho\omega_2} = -e^{-\rho\omega_2}$, we obtain

$$e^{\alpha_n\omega_2} - e^{-\alpha_n\omega_2} + \mu_3\widetilde{\rho_n}^{-1}e^{\alpha_n\omega_2} - i\mu_4\widetilde{\rho_n}^{-1}e^{-\alpha_n\omega_2} + \mathcal{O}(\widetilde{\rho_n}^{-2}) = 0.$$

Expanding the exponential function according to its Taylor series, we obtain

$$\alpha_n = -\frac{\mu_3}{2\omega_2\rho_n} + \frac{\mu_4}{2\omega_2\rho_n}i + \mathcal{O}(n^{-2}),\ n = N, N+1, \dots$$

Therefore,

$$\widetilde{\rho_n} = (n + \frac{1}{2})\frac{\pi i}{\omega_2} + \frac{\mu_3}{2(n + \frac{1}{2})\pi}i + \frac{\mu_4}{2(n + \frac{1}{2})\pi} + \mathcal{O}(n^{-2}),$$

for $n = N, N+1, \dots$. Note that $\lambda_n = \frac{\rho_n^2}{h^2} \neq 0$, $\omega_2 = e^{i\frac{\pi}{4}}$ and $\omega_2^2 = i$. So we have

$$\lambda_n = \frac{\sqrt{2}}{2h^2}(\mu_4 - \mu_3) + \frac{1}{h^2}\big[\frac{\sqrt{2}}{2}(\mu_4 + \mu_3) + (n + \frac{1}{2})^2\pi^2\big]i + \mathcal{O}(n^{-1}), \tag{3.49}$$

where $n = N, N+1, \dots$ with N large enough.

The same proof can be applied to sector S_2 because the eigenvalues of the problem (3.19) can be obtained by a similar calculation with the choices

$$\omega_1 = \exp(i\frac{1}{4}\pi) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \quad \omega_2 = \exp(i\frac{3}{4}\pi) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i,$$
$$\omega_3 = \exp(i\frac{7}{4}\pi) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \quad \omega_4 = \exp(i\frac{5}{4}\pi) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i,$$

so that (3.29) holds:

$$\operatorname{Re}(\rho\omega_1) \le \operatorname{Re}(\rho\omega_2) \le \operatorname{Re}(\rho\omega_3) \le \operatorname{Re}(\rho\omega_4), \quad \forall \rho \in S_2.$$

Hence, in sector S_2, we have the following asymptotic expression of $\Delta(\rho)$:

$$\Delta(\rho) = \rho^6 e^{\rho\omega_4}\{2e^{\rho\omega_2} + 2e^{-\rho\omega_2} - 2\mu_3\rho^{-1}e^{\rho\omega_2} + 2i\mu_4\rho^{-1}e^{-\rho\omega_2} + \mathcal{O}(\rho^{-2})\}.$$

By a direct calculation, we have

$$\widetilde{\rho_n} = (n+\frac{1}{2})\frac{\pi i}{\omega_2} - \frac{\mu_3}{2(n+\frac{1}{2})\pi}i + \frac{\mu_4}{2(n+\frac{1}{2})\pi} + \mathcal{O}(n^{-2}), \tag{3.50}$$

for $n = N, N+1, \dots$, with N large enough. Again, using $\lambda_n = \frac{\rho_n^2}{h^2} \neq 0$, $\omega_2 = e^{i\frac{3\pi}{4}}$ and $\omega_2^2 = -i$.

$$\lambda_n = \frac{\sqrt{2}}{2h^2}(\mu_4 - \mu_3) - \frac{1}{h^2}\big[\frac{\sqrt{2}}{2}(\mu_4+\mu_3) + (n+\frac{1}{2})^2\pi^2\big]i + \mathcal{O}(n^{-1}), \tag{3.51}$$

where $n = N, N+1, \dots$ with N large enough.

Here we should point out that the eigenvalues generated from the other sectors S_k coincide with those from S_1 and S_2. The detailed argument can be found in Naimark [11]. Combining with (3.49) and (3.51), we obtain the following result on the eigenvalues.

Theorem 3.6. *Let A_γ be defined by* (2.7) *and* (2.8)*, then an asymptotic expression of the eigenvalues of problem* (3.19) *is given by*

$$\lambda_n = \frac{\sqrt{2}}{2h^2}(\mu_4 - \mu_3) \pm \frac{1}{h^2}\big[\frac{\sqrt{2}}{2}(\mu_4+\mu_3) + (n+\frac{1}{2})^2\pi^2\big]i + \mathcal{O}(n^{-1}), \tag{3.52}$$

where $n = N, N+1, \dots$ with N large enough, and

$$\mu_4 - \mu_3 = 2\sqrt{2}\mu_2 - 2\sqrt{2}\frac{\beta}{z_x^3(1)EI(1)h^2} = 2\sqrt{2}\mu_2 - \frac{2\sqrt{2}\beta h}{EI(1)}(\frac{m(1)}{EI(1)})^{-3/4}, \tag{3.53}$$
$$d(z) = \frac{\gamma(x)}{m(x)}, \quad z_x = \frac{dz}{dx} = \frac{1}{h}(\frac{m(x)}{EI(x)})^{1/4},$$

so,

$$\mu_2 = -\frac{h^2}{4}\int_0^1 \frac{\gamma(x)}{m(x)}\frac{1}{h}(\frac{m(x)}{EI(x)})^{1/4}dx = -\frac{h}{4}\int_0^1 \frac{\gamma(x)}{m(x)}(\frac{m(x)}{EI(x)})^{1/4}dx.$$

Moreover, λ_n ($n = N, N+1, \dots$) with sufficiently large modulus are simple and distinct except for finitely many of them, and satisfy

$$\lim_{n\to+\infty}\operatorname{Re}\lambda_n = -\frac{1}{2h}\int_0^1 \frac{\gamma(x)}{m(x)}(\frac{m(x)}{EI(x)})^{1/4}dx - \frac{2\beta}{hEI(1)}(\frac{m(1)}{EI(1)})^{-3/4}. \tag{3.54}$$

3.2. **Riesz basis property of eigenfunctions of A_γ.** In this subsection, we discuss the Riesz basis property of the eigenfunctions of operator A_γ of the system (2.8). We follow an idea due to Wang (see [19] pp. 473-475). We begin with showing that the generalized eigenfunctions of A_γ form an unconditional basis in Hilbert state space H. For this aim, we introduce a transformation $\mathcal{L}$ via

$$\mathcal{L}(f,g) = (\phi,\psi)$$

where

$$\phi(x) = f(z), \quad \psi(x) = g(z), \quad z = \frac{1}{h}\int_0^x \Big(\frac{m(\zeta)}{EI(\zeta)}\Big)^{1/4} d\zeta, \tag{3.55}$$

with

$$h = \int_0^1 \Big(\frac{m(\zeta)}{EI(\zeta)}\Big)^{1/4} d\zeta. \tag{3.56}$$

It is easily seen that $\mathcal{L}$ is a bounded invertible operator on $\mathbb{H}$.

Now we define the ordinary differential operator:

$$\begin{gathered} L(f) = f^{(4)}(z) + a(z)f'''(z) + b(z)f''(z) + c(z)f'(z), \\ \mu(z) = h^2 d(z) = h^2\frac{\gamma(x)}{m(x)}, \\ B_1(f) = f(0) = 0, \quad B_2(f) = f'(0) = 0, \\ B_3(f) = z_x^2(1)f''(1) + z_{xx}(1)f'(1) = 0 \\ B_4(f) = f'''(1) + \frac{3z_{xx}(1)}{z_x^2(1)}f''(1) + \frac{z_{xxx}(1)}{z_x^3(1)}f'(1) - \frac{(\alpha+\lambda\beta)}{z_x^3(1)EI(1)}f(1) = 0, \end{gathered} \tag{3.57}$$

where the coefficients are given by (3.12) –(3.16). Let $\mathbb{A}$ be defined as in (3.5), $\eta \in \sigma(\mathbb{A})$ be an eigenvalue of $\mathbb{A}$ and (f,g) be an eigenfunction corresponding to η, then we have $g = \eta f$ and f will satisfy the equation

$$f^{(4)}(z) + a(z)f'''(z) + b(z)f''(z) + c(z)f'(z) + \eta\mu(z)f(z) + \eta^2 f(z) = 0,$$

with boundary conditions $B_j(f) = 0$, $j = 1,2,3,4$. Now by taking $\lambda = \frac{\eta}{h^2}$ and

$$\mathcal{L}(f,g) = (\phi(x),\psi(x))$$

we see that $\psi = \lambda\phi$ and ϕ satisfies the equation

$$\begin{gathered} \phi^{(4)}(x) + \frac{2EI'(x)}{EI(x)}\phi'''(x) + \frac{EI''(x)}{EI(x)}\phi''(x) \\ + \lambda\frac{\gamma(x)}{EI(x)}\phi(x) + \frac{\lambda^2 m(x)}{EI(x)}\phi(x) = 0, \quad 0<x<1, \\ \phi(0) = \phi'(0) = \phi''(1) = 0 \\ \phi'''(1) = \frac{1}{EI(1)}(\alpha+\beta\lambda)\phi(1). \end{gathered} \tag{3.58}$$

Hence we have that $\eta \in \sigma(\mathbb{A}) \Leftrightarrow \lambda \in \sigma(A_\gamma)$.

Theorem 3.7. *Let operator A_γ of the system* (2.8). *Then the eigenvalues of operator A_γ are all simple except for finitely many of them, and the generalized eigenfunctions of operator A_γ form a Riesz basis for the Hilbert state space H.*

Proof. From the previous subsection, we have shown that the boundary problem (3.19) is strongly regular because the coefficients of $F_0^{\mathbb{K}_k}$ are nonzero in $\Delta(\rho)$ (see Definition 3.1). Therefore the eigenvalues are separated and simple except for finitely many of them. Thus the first assertion is true. Next, according to Theorem 3.2, the strongly regular boundary conditions ensure that the generalized eigenfunction sequence $F_n = (f_n, \eta_n f_n)$ of operator $\mathbb{A}$ forms a Riesz basis for $\mathbb{H}$. Since $\mathcal{L}$ is bounded and invertible on $\mathbb{H}$, it follows that $\Psi_n = (\phi_n, \lambda_n \phi_n) = \mathcal{L} F_n$ also forms a Riesz basis on H. □

We are now in a position to investigate the exponential stability of system (2.8). Since the Riesz basis property implies the spectrum-determined growth condition (see Curtain and Zwart [4]) and (3.54) describes the asymptote of $\sigma(A_\gamma)$, for any small $\varepsilon > 0$ there are only finitely many eigenvalues of A_γ in the following half-plane:

$$\Sigma : \operatorname{Re}\lambda > -\frac{1}{2h}\int_0^1 \frac{\gamma(x)}{m(x)}\Big(\frac{m(x)}{EI(x)}\Big)^{1/4} dx - \frac{2\beta}{hEI(1)}\Big(\frac{m(1)}{EI(1)}\Big)^{-3/4} + \varepsilon. \tag{3.59}$$

The following are two stability results that describe how stability depend on the damping function γ.

4. Exponential stability

Following the idea in [7, Theorem 2.4], all the properties of operator A_γ found above, allow us to claim that for the semigroup $e^{A_\gamma t}$ generated by A_γ the spectrum-determined growth condition holds:

$$\omega(A_\gamma) = s(A_\gamma),$$

where

$$\omega(A_\gamma) = \lim_{t\to+\infty} \frac{1}{t}\|e^{A_\gamma t}\|_H$$

is the growth order of $e^{A_\gamma t}$ and

$$s(A_\gamma) = \sup\big\{ \operatorname{Re}\lambda : \lambda \in \sigma(A_\gamma)\big\}$$

is the spectral bound of A_γ.

The Theorem 3.7 is one of the fundamental properties of the evolutive system (1.1)–(1.4). Many other important properties of this system can be concluded from Theorem 3.7. The exponential stability stated below is one of such important property.

Theorem 4.1. *If γ is continuous and nonnegative, the system* (1.1)–(1.4) *is exponential stable for any $\beta > 0$ and $\alpha \geq 0$. That is, there are nonnegative constants M, ω such that the energy $E(t) = \frac{1}{2}\|(u, u_t)^T\|_H^2$ of system* (1.1)–(1.4) *satisfies*

$$E(t) \leq ME(0)e^{-\omega t}, \quad \forall t \geq 0,$$

for any initial condition $(u(x,0), u_t(x,0)) \in H$.

Proof. We have $\gamma(x) \geq 0$, and for any $(f, g) \in D(A_\gamma)$,

$$\begin{aligned}
&\langle A_\gamma(f,g),(f,g)\rangle \\
&= \Big\langle (g(x), -\frac{1}{m(x)}((EI(x)f''(x))'' + \gamma(x)g(x))), (f,\ g)\Big\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 [EI(x)g''(x)\overline{f''(x)} - (EI(x)f''(x))''\overline{g(x)} - \gamma(x)|g(x)|^2]dx + \alpha g(1)\overline{f(1)} \\
&= \int_0^1 EI(x)[g''(x)\overline{f''(x)} - f''(x)\overline{g(x)}]dx + \alpha\big(g(1)\overline{f(1)} - f(1)\overline{g(1)}\big) \\
&\quad - \beta|g(1)|^2 - \int_0^1 \gamma(x)|g(x)|^2dx;
\end{aligned}
$$

further

$$
\operatorname{Re}\langle A_\gamma(f,g),(f,g)\rangle = -\beta|g(1)|^2 - \int_0^1 \gamma(x)|g(x)|^2dx \le 0.
$$

Thus A_γ is dissipative and $e^{A_\gamma t}$ is a contraction semigroup on H. Moreover, the spectrum of A_γ has an asymptote

$$
\operatorname{Re}\lambda \sim -\frac{1}{2h}\int_0^1 \frac{\gamma(x)}{m(x)}(\frac{m(x)}{EI(x)})^{1/4}dx - \frac{2\beta}{hEI(1)}(\frac{m(1)}{EI(1)})^{-3/4}.
$$

If we can show that there is no eigenvalue on the imaginary axis, then the exponential stability holds. Let $\lambda = ir$ with $r \in \mathbb{R}^*$ be an eigenvalue of operator A_γ on the imaginary axis and $\Psi = (\phi,\psi)^T$ be the corresponding eigenfunction, then $\psi = \lambda\phi$. We have

$$
\begin{gathered}
\operatorname{Re}(\langle A_\gamma\Psi,\Psi\rangle_H) = -\beta|\psi(1)|^2 - \int_0^1 \gamma(x)|\psi(x)|^2dx, \\
0 = \|\Psi\|_H^2 \operatorname{Re}(\lambda) = \operatorname{Re}(\langle A_\gamma\Psi,\Psi\rangle_H) = -\beta|\psi(1)|^2 - \int_0^1 \gamma(x)|\psi(x)|^2dx,
\end{gathered}
$$

since $\beta > 0$, $\gamma(x) \ge 0$ and $\psi(x)$ are continuous, we obtain

$$
\psi(1) = 0 \quad \text{and} \quad \gamma(x)|\psi(x)|^2 = 0, \quad \forall x \in [0,1]. \tag{4.1}
$$

Then $\phi(1) = 0$ because $\psi = \lambda\phi$. We have the following equation satisfied by ϕ,

$$
\begin{gathered}
\lambda^2 m(x)\phi(x) + (EI(x)\phi''(x))'' + \lambda\gamma(x)\phi(x) = 0, \quad 0 < x < 1, \\
\phi(0) = \phi'(0) = \phi''(1) = \phi'''(1) = \phi(1) = 0
\end{gathered} \tag{4.2}
$$

The proof will be complete if we can show that there is only zero solution to 4.2. For this aim we follow a method used in [7, p.1917]. First, we claim that there is at least one zero of ϕ in $(0,1)$. In fact, by $\phi(0) = \phi(1) = 0$, it follows from Rolle's theorem that there is a $\xi_1 \in (0,1)$ such that $\phi'(\xi_1) = 0$, which, together with $\phi'(0) = 0$, claims that $(EI\phi'')(\xi_2) = 0$ for some $\xi_2 \in (0,\xi_1)$, and so $(EI\phi'')'(\xi_3) = 0$ for some $\xi_3 \in (\xi_2, 1)$ by condition $(EI\phi'')(1) = 0$. Hence there is a $\xi_4 \in (\xi_3, 1)$ such that $(EI\phi'')''(\xi_4) = 0$ by the condition $(EI\phi'')'(1) = 0$. However, $\lambda^2 m(\xi_4)\phi(\xi_4) + (EI(\xi_4)\phi'')''(\xi_4) + \lambda\gamma(\xi_4)\phi(\xi_4) = 0$. We have

$$
\lambda^2 m(\xi_4)\phi(\xi_4) + \lambda\gamma(\xi_4)\phi(\xi_4) = 0
$$

because $(EI(\xi_4)\phi'')''(\xi_4) = 0$. Then we obtain $\lambda m(\xi_4)\lambda\phi(\xi_4) + \gamma(\xi_4)\lambda\phi(\xi_4) = 0$. Since $\psi = \lambda\varphi$ we have

$$
\lambda m(\xi_4)\psi(\xi_4) + \gamma(\xi_4)\psi(\xi_4) = 0.
$$

Multiplying the conjugate of $\psi(\xi_4)$ on both sides of the above equation we obtain

$$
\lambda m(\xi_4)\psi(\xi_4)\overline{\psi(\xi_4)} + \gamma(\xi_4)\psi(\xi_4)\overline{\psi(\xi_4)} = 0.
$$

Then we obtain

$$\lambda m(\xi_4)|\psi(\xi_4)|^2 + \gamma(\xi_4)|\psi(\xi_4)|^2 = 0.$$

Using (4.1) we have $\lambda m(\xi_4)|\psi(\xi_4)|^2 = 0$, so $\psi(\xi_4) = 0$. Finally we conclude that $\phi(\xi_4) = 0$ since $\psi = \lambda\phi$ and λ is nonzero. Next, we show that if there are n different zeros of ϕ in $(0,1)$, then there are at least $n+1$ number of different zeros of ϕ in $(0,1)$. Indeed, suppose that

$$0 < \xi_1 < \xi_2 < \dots < \xi_n < 1, \quad \phi(\xi_i) = 0, \quad i = 1,2,3,\dots,n.$$

Since $\phi(0) = \phi(1) = 0$, it follows from Rolle's theorem that there are η_i, $i = 1,2,3,\dots,n+1$,

$$0 < \eta_1 < \xi_1 < \eta_2 < \xi_2 < \eta_3 < \xi_3 < \dots < \xi_n < \eta_{n+1} < 1,$$

such that $\phi'(\eta_i) = 0$. Noting that $\phi'(0) = 0$, there are α_i, $i = 1,2,3,\dots,n+1$,

$$0 < \alpha_1 < \eta_1 < \alpha_2 < \eta_2 < \alpha_3 < \eta_3 < \dots < \alpha_{n+1} < \eta_{n+1} < 1,$$

such that $(EI\phi'')(\alpha_i) = 0$. Since $(EI\phi'')(1) = 0$, using Rolle's theorem, we have β_i, $i = 1,2,3,\dots,n+1$,

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3 < \beta_3 < \dots < \alpha_{n+1} < \beta_{n+1} < 1,$$

such that $(EI\phi'')'(\beta_i) = 0$. Finally, by the condition $(EI\phi'')'(1) = 0$, we have θ_i, $i = 1,2,3,\dots,n+1$,

$$0 < \beta_1 < \theta_1 < \beta_2 < \theta_2 < \beta_3 < \theta_3 < \dots < \beta_{n+1} < \theta_{n+1} < 1,$$

such that $(EI\phi'')''(\theta_i) = 0$. Therefore

$$\phi(\theta_i) = 0, \quad i = 1,2,3,\dots,n+1$$

Third, by mathematical induction, there is an infinite number of different zeros $\{x_i\}_1^\infty$ of ϕ in $(0,1)$. Let $x_0 \in [0,1]$ be an accumulation point of $\{x_i\}_1^\infty$. It is obvious that

$$\phi^{(i)}(x_0) = 0, \quad i = 0,1,2,3.$$

Note that ϕ satisfies the linear differential equation (4.2). Therefore, $\phi = 0$ by uniqueness of the solution of linear ordinary differential equations. However $\Psi = (\phi,\psi)^T = (\phi,\lambda\phi)^T = 0$ contradicts Ψ being an eigenfunction and so there is no eigenvalue on the imaginary axis and we obtain $\mathrm{Re}(\lambda) < 0$. From Theorem 3.7 and the spectrum-determined growth condition, the system is exponentially stable. $\square$

Now we are ready to consider the case that $\gamma(x)$ is continuous and indefinite in $[0,1]$. We follow an idea due to Wang [19]. Let $\gamma(x) = \gamma^+(x) - \gamma^-(x)$ for all $x \in [0,1]$ with

$$\gamma^+(x) = \max\{\gamma(x), 0\} = \begin{cases} \gamma(x) & \text{if } \gamma(x) > 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\gamma^-(x) = \max\{-\gamma(x), 0\} = \begin{cases} -\gamma(x), & \text{if } \gamma(x) < 0 \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$A_{\gamma^+}(f,g) = \Big(g(x), -\frac{1}{m(x)}((EI(x)f''(x))'' + \gamma^+(x)g(x))\Big)^T,$$

for all $(f, g) \in D(A_{\gamma^+}) = D(A_\gamma)$, and

$$\Gamma_-(f, g) = (0, \frac{\gamma^-(x)}{m(x)} g(x))^T, \quad \forall (f, g) \in H.$$

Then A_γ can be written as $A_\gamma = A_{\gamma^+} + \Gamma_-$.

Theorem 4.2. *Let $s(A_{\gamma^+}) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A_{\gamma^+})\}$. If*

$$\max_{x\in[0,1]} \{\frac{\gamma^-(x)}{m(x)}\} < |s(A_{\gamma^+})|,$$

then system (2.8) *is exponentially stable.*

Proof. It is easy to verify that Γ_- is self-adjoint operator and

$$\|\Gamma_-\| = \max_{x\in[0,1]} \{\frac{\gamma^-(x)}{m(x)}\}. \tag{4.3}$$

By Theorem 4.1 and definition of operator A_{γ^+}, $e^{A_{\gamma^+} t}$ is a contraction semigroup and $s(A_{\gamma^+}) < 0$. Applying the perturbation theory of linear operators semigroup (see Pazy [12, Theorem 1.1 page 76]), we have $\lambda \in \rho(A_\gamma)$ whenever $\operatorname{Re}\lambda > s(A_{\gamma^+}) + \|\Gamma_-\|$. Again, Theorem 3.7 gives

$$\omega(A_\gamma) = s(A_\gamma) \le s(A_{\gamma^+}) + \|\Gamma_-\| < 0,$$

where $\omega(A_\gamma)$ denotes the growth bound of semigroup $e^{A_\gamma t}$. Therefore, system (2.8) is exponentially stable. $\square$

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