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LARGE TIME BEHAVIOR FOR $p(x)$-LAPLACIAN EQUATIONS WITH IRREGULAR DATA

XIAOJUAN CHAI, HAISHENG LI, WEISHENG NIU

Abstract. We study the large time behavior of solutions to $p(x)$-Laplacian equations with irregular data. Under proper assumptions, we show that the entropy solution of parabolic $p(x)$-Laplacian equations converges in $L^q(\Omega)$ to the unique stationary entropy solution as t tends to infinity.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$. We consider the asymptotic behavior of the following nonlinear initial-boundary value problem with irregular data,

$$
\begin{gathered}
u_t - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{q-1}u = g \quad \text{in } \Omega \times \mathbb{R}^+, \\
u = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\
u(x,0) = u_0(x) \quad \text{in } \Omega,
\end{gathered} \tag{1.1}
$$

where $q \geq 1$, $p \in C(\overline{\Omega})$ with $1 < p^- = \min_{x\in\overline{\Omega}} p(x) \leq p^+ = \max_{x\in\overline{\Omega}} p(x) < \infty$. By irregular data, we mean that $u_0, g \in L^1(\Omega)$.

Equation in problem (1.1) could be viewed as a generalization of the usual p-Laplacian equations. It is a rather typical nonlinear problem with variable exponents. Problems of this kind are interesting from the purely mathematical point of view. Besides, they have potential applications in various fields such as electrorheological fluids (an essential class of non-Newtonian fluids) [30, 29], nonlinear elasticity [40] flow through porous media [1], image processing [14], etc. Perhaps for these reasons, such a field has attracted more and more attention and has undergone an explosive development in recent years, see the monograph [15] and the large amounts of references therein.

As an essential model involving variable exponents, problem (1.1) has been studied in various contexts by different authors. In [4], Antontsev and Shmarev investigated the existence and uniqueness results for some anisotropic parabolic equations involving variable exponents. Then with more general assumptions on the variable exponents, in [3, 17, 18], some existence results were obtained for the parabolic $p(x)$-Laplacian equations in different frameworks. In [8, 38], existence and uniqueness results were addressed for the parabolic $p(x)$-Laplacian equations with L^1-data.

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The asymptotic behavior for the parabolic $p(x)$-Laplacian equations has also been studied largely. In [5, 6, 7], the extinction, decay and blow up of solutions for some anisotropic parabolic equations with variable exponents were investigated. In [2], Akagi and Matsuura studied the convergence to stationary states for the solutions of the parabolic $p(x)$-Laplacian equations. In [19, 22] and [33]–[37], the large time behaviors for certain kinds of $p(x)$-Lalacian equations were investigated and described by means of global attractors.

The considerations in [19, 22], [33]–[37] were mainly focused on $p(x)$-Laplacian equations with regular data (the initial data and forcing terms were assumed to be L^2 integrable or even essentially bounded). In [12], we considered the existence of global attractors for some $p(x)$-Laplacian equations involving L^1 or even measure data. As we see, the less regularity of the data influences the regularity of the solutions greatly, and which in turn causes some crucial difficulties in investigating the asymptotic behaviors of the solutions, see also [21, 23, 25, 26, 27, 41].

In this article, we shall continue the study on the large time behavior of solutions to $p(x)$-Laplacian equations with irregular data as in [12]. But from a different point of view, here we investigate the convergence of the solutions to the stationary states as t tends to infinity. Under proper assumptions, we shall prove that the unique entropy solution $u(t)$ to the $p(x)$-Laplacian problem (1.1) converges in $L^1(\Omega)$ to the unique entropy solution v of the corresponding elliptic problem (2.2) as t tends to infinity.

Our work is largely motivated by the works of Petitta and his coauthors in [21, 25, 26, 27]. By using the comparison principle and some compactness results successfully, the authors have obtained the convergence of solutions to the stationary states for several type of parabolic equations (with constant exponent) involving irregular data. Yet, the variable exponent problem treated here exhibits some stronger nonlinearity and inhomogeneity, which require the analysis to be more delicate.

Next, we first provide some preliminaries in Section 2. Then in Section 3, the last section, we investigate the large time behavior of the entropy solution to problem (1.1). Throughout the paper, we denote $\Omega \times (0,T)$ by Q_T for any $T>0$, and we use C to denote some positive constant, which may distinguish with each other even in the same line and that only depends on.

2. Preliminaries

Let us begin with the definitions and some basic properties of the generalized Lebesgue and Sobolev spaces. Interested readers may refer to [15, 16, 20] for more details.

For a variable exponent $p \in C(\overline{\Omega})$ with $p^- > 1$, define the Lebesgue space $L^{p(\cdot)}(\Omega)$ as

$$L^{p(\cdot)}(\Omega) = \{u : \Omega \to \mathbb{R}; u \text{ is measurable and } \int_\Omega |u|^{p(x)} dx < \infty\}$$

with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \le 1\}.$$

We have

$$\min\{\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}\} \le \int_\Omega |u|^{p(x)}dx \le \max\{\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}\}.$$

As $p^- > 1$, the space is a reflexive Banach space with dual $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$. Let $r_i \in C(\overline{\Omega})$ with $r_i^- > 1$, $i = 1, 2$. Then if $r_1(x) \le r_2(x)$ for any $x \in \Omega$, the imbedding $L^{r_2(\cdot)}(\Omega) \hookrightarrow L^{r_1(\cdot)}(\Omega)$ is continuous, of which the norm does not exceed $|\Omega| + 1$. Besides, for any $u \in L^{p(\cdot)}(\Omega), v \in L^{p'(\cdot)}(\Omega)$, we have Hölder's inequality

$$\int_\Omega |uv|dx \le \Big(\frac{1}{p^-} + \frac{1}{(p^-)'}\Big)\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p(\cdot)}(\Omega)}.$$

For a positive integer k, the generalized Sobolev space $W^{k,p(\cdot)}(\Omega)$ is defined as

$$W^{k,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega), |\alpha| \le k\}$$

with norm

$$\|u\|_{W^{k,p(\cdot)}} = \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^{p(\cdot)}(\Omega)}.$$

Such a space is also a separable and reflexive Banach space.

For constant $1 \le m < \infty$, the time dependent spaces $L^m(0, T; W_0^{1,p(\cdot)}(\Omega))$ consists of all strongly measurable functions $u : [0, T] \to W_0^{1,p(\cdot)}(\Omega)$ with

$$\|u\|_{L^m(0,T;W_0^{1,p(\cdot)}(\Omega))} = (\int_0^T \|u\|^m_{W^{k,p(\cdot)}} dt)^{1/m} < \infty.$$

In this article, we assume that there exists a positive constant C such that

$$|p(x) - p(y)| \le -\frac{C}{\log|x - y|}, \text{ for every } x, y \in \Omega \text{ with } |x - y| < \frac{1}{2}. \tag{2.1}$$

This condition ensures that smooth functions are dense in the generalized Sobolev spaces. Then $W_0^{k,p(\cdot)}(\Omega)$ can naturally be defined as the completion of $C_c^\infty(\Omega)$ in $W^{k,p(\cdot)}(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,p(\cdot)}}$, and one has $W_0^{k,p(\cdot)}(\Omega) = W^{k,p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega)$. For $u \in W_0^{1,p(\cdot)}(\Omega)$, the Poincaré type inequality holds, i.e.,

$$\|u\|_{L^{p(\cdot)}(\Omega)} \le C\|\nabla u\|_{L^{p(\cdot)}(\Omega)},$$

where the positive constant C depends on p and Ω. So $\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$ is an equivalent norm in $W_0^{1,p(\cdot)}(\Omega)$.

Let $s(\cdot)$ be a measurable function on Ω such that $\operatorname{ess\,inf}_{x\in\Omega} s(x) > 0$. Define the Marcinkiewicz space $M^{s(\cdot)}(\Omega)$ as the set of measurable functions v such that

$$\int_{\Omega\cap\{|v|>k\}} k^{s(x)}dx < C,$$

for some positive constant C and all $k > 0$ [31]. It is obvious that if $s(x) \equiv s$ constant, the above definition coincides with the classical definition of Marcinkiewicz spaces. Thanks to Proposition 2.5 in [31], we have

Lemma 2.1. *Let $r(\cdot), s(\cdot) \in C(\overline{\Omega})$ such that $s^- > 0, (r - s)^- > 0$ and let $u(x, t)$ be a function defined on Q_T. If $u \in M^{r(\cdot)}(Q_T)$, then $|u|^{s(x)} \in L^1(Q_T)$. In particular, $M^{r(\cdot)}(Q_T) \subset L^{s(\cdot)}(Q_T)$ for all $s(\cdot), r(\cdot) \ge 1$ such that $(r - s)^- > 0$.*

Consider the following elliptic equation corresponding to (1.1)

$$\begin{gathered} -\operatorname{div}(|\nabla v|^{p(x)-2}\nabla v)+|v|^{q-1}v=g \quad \text{in } \Omega, \\ v=0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.2}$$

where $g \in L^1(\Omega)$. Let $T_k(s)$ be the usual truncating function defined as $T_k(\sigma)=\max\{-k,\min\{k,\sigma\}\}$. Denote $\Phi_k(\sigma)$ as its primitive function,

$$\Phi_k(\sigma)=\int_0^\sigma T_k(r)dr=\begin{cases}\sigma^2/2 & \text{if } |\sigma|<k,\\ k|\sigma|-k^2/2 & \text{if } |\sigma|\ge k.\end{cases}$$

Definition 2.2 ([13, 31, 39])**.** A measurable function v is called an entropy solution to problem (2.2), if $v \in L^q(\Omega)$ and for every $k>0, T_k(v)\in W_0^{1,p(\cdot)}(\Omega)$,

$$\int_\Omega |\nabla v|^{p(\cdot)-2}\nabla v\nabla T_k(v-\varphi)dx+\int_\Omega |v|^{q-1}vT_k(v-\varphi)dx \le \int_\Omega T_k(v-\varphi)gdx \tag{2.3}$$

holds for any $\varphi \in W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$.

A function v such that $T_k(v)\in W_0^{1,p(\cdot)}(\Omega)$, for all $k>0$, does not necessarily belong to $W_0^{1,1}(\Omega)$. Thus ∇v in the equation is defined in a very weak sense [9, 31]:

> For every measurable function $v:\Omega\to\mathbb{R}$ such that $T_k(v)\in W_0^{1,p(\cdot)}(\Omega)$ for all $k>0$, there exists a unique measurable function $w:\Omega\to\mathbb{R}^N$, which we call the very weak gradient of v and denote $w=\nabla v$, such that
>
> $\nabla T_k(v)=w\chi_{\{|v|<k\}}$, almost everywhere in Ω and for every $k>0$,
>
> where χ_E denotes the characteristic function of a measurable set E. Moreover, if v belongs to $W_0^{1,1}(\Omega)$, then w coincides with the weak gradient of v.

Theorem 2.3 ([13])**.** *Assume that $g\in L^1(\Omega)$, and (2.1) holds. Then problem (2.2) admits a unique entropy solution v.*

Definition 2.4 ([38])**.** A function u is called an entropy solution of (1.1), if for any $T>0$, $u\in C([0,T];L^1(\Omega))\cap L^q(Q_T)$ such that $T_k(u)\in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))$, $\nabla T_k(u)\in (L^{p(\cdot)}(Q_T))^N$, and

$$\begin{aligned} &\int_\Omega \Phi_k(u-\varphi)(T)dx-\int_\Omega \Phi_k(u_0-\varphi(0))dx+\int_0^T\langle\varphi_t,T_k(u-\varphi)\rangle dt\\ &+\int_{Q_T}|\nabla u|^{p(x)-2}\nabla u\cdot\nabla T_k(u-\varphi)\,dx\,dt+\int_{Q_T}|u|^{q-1}uT_k(u-\varphi)\,dx\,dt\\ &\le \int_{Q_T} gT_k(u-\varphi)dx, \end{aligned} \tag{2.4}$$

holds for any $k>0$ and any $\varphi\in C^1(\overline{Q}_T)$ with $\varphi=0$ on $\partial\Omega\times(0,T)$. Here $\langle\cdot,\cdot\rangle$ denotes the duality product between $W_0^{1,p(\cdot)}(\Omega)$ and its dual space $W^{-1,p'(\cdot)}(\Omega)$.

Remark 2.5. Similar to Definition 2.2, the gradient of u in Definition 2.2 is also defined in a very weak sense [38]. On the other hand, let

$$X=\{\phi|\ \phi\in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega)),\nabla\phi\in(L^{p(\cdot)}(Q_T))^N\}$$

with norm $\|\phi\|_X = \|\nabla\phi\|_{L^{p(\cdot)}(\Omega)} + \|\phi\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))}$. We can choose $\varphi \in X \cap L^\infty(Q_T)$ with $\varphi_t \in X^* + L^1(Q_T)$ as a test function in the definition above, see [8].

Remark 2.6. Let v be an entropy solution to problem (2.2). Since it is independent of time, we have, for any $\varphi \in C^1(\overline{Q}_T)$ with $\varphi = 0$ on $\partial\Omega \times (0,T)$,

$$\begin{aligned}
&\int_\Omega \Phi_k(v-\varphi)(T)dx - \int_\Omega \Phi_k(v-\varphi(0))dx \\
&= \int_0^T \langle (v-\varphi)_t, T_k(v-\varphi)\rangle dt \\
&= -\int_0^T \langle \varphi_t, T_k(v-\varphi)\rangle dt.
\end{aligned}$$

Thus we find that v is actually an entropy solution to (1.1) with initial data $u_0 = v$.

Theorem 2.7. *Assuming that $u_0, g \in L^1(\Omega)$ and (2.1) holds, problem (1.1) admits a unique entropy solution u.*

Proof. The proof is rather similar to [38] (see also [10, 28]), thus we just sketch it in a rather concise way. Consider the approximate problem

$$\begin{gathered}
u_t^n - \operatorname{div}(|\nabla u^n|^{p(x)-2}\nabla u^n) + |u^n|^{q-1}u^n = g^n \quad \text{in } \Omega\times\mathbb{R}^+, \\
u^n = 0 \quad \text{on } \partial\Omega\times\mathbb{R}^+, \\
u^n(x,0) = u_0^n \quad \text{in } \Omega,
\end{gathered} \tag{2.5}$$

where $\{g^n\}_{n\in\mathbb{N}}$, $\{u_0^n\}_{n\in\mathbb{N}}$ are smooth approximations of the data g and u_0 with

$$\|u_0^n\|_{L^1(\Omega)} \le \|u_0\|_{L^1(\Omega)}, \quad \|g^n\|_{L^1(\Omega)} \le \|g\|_{L^1(\Omega)}.$$

Similar to [38, Lemma 2.5], with rather minor modifications, we can prove that problem (2.5) admits a unique weak solution u^n for each n.

Performing the calculations as in [38, pp. 1384 Step 1] (see also [28, Claim 1]), we obtain that, up to a subsequence, $\{u^n\}$ converges to a function u in $C([0,T];L^1(\Omega))$, and hence almost everywhere in Q_T, for any given $T>0$. Using Vitali's convergence theorem, see for example [12], we can prove that $|u^n|^{q-1}u^n$ converges to $|u|^{q-1}u$ in $L^1(Q_T)$. Performing the calculations as Step 2 in [38], we can deduce that $\nabla T_k(u^n)$ converges to $\nabla T_k(u)$ strongly in $(L^{p(\cdot)}(Q_T))^N$. Taking $T_k(u^n-\varphi)$ as a test function in (2.5) and passing to the limit, it is easy to obtain that u is an entropy solution to problem (1.1). Thanks to the monotonicity of the term $|u|^{q-1}u$, the uniqueness can be proved in the same way as [38, pp1396-1398]. $\square$

Remark 2.8. Similar to [38], we can prove that (2.4) actually can hold as an equality. Yet, the inequality is enough to ensure the uniqueness, see [28] for the constant exponent case.

3. Asymptotic behavior

In this section, we consider the asymptotic behavior of the entropy solution to (1.1). To state the main result, let us first adapt to our problem the definition of entropy subsolutions and entropy supersolutions, which were originally defined in [26, 24]. Denote by f^+, f^- the positive and negative parts of a function f with $f = f^+ - f^-$.

Definition 3.1. A function $\underline{v}(x)$ is an entropy subsolution of (2.2) if, for all $k > 0$, we have $\underline{v} \in L^q(\Omega), T_k(\underline{v}) \in W_0^{1,p(\cdot)}(\Omega)$, and it holds that

$$\int_\Omega |\nabla \underline{v}|^{p(x)-2}\nabla \underline{v} \nabla T_k(\underline{v}-\varphi)^+ dx + \int_\Omega |\underline{v}|^{q-1}\underline{v} T_k(\underline{v}-\varphi)^+ dx \le \int_\Omega g T_k(\underline{v}-\varphi)^+ dx, \tag{3.1}$$

for any $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

On the other hand, a function $\overline{v}(x)$ is an entropy supersolution of problem (2.2) if, for all $k > 0$, we have $\overline{v} \in L^q(\Omega), T_k(\overline{v}) \in W_0^{1,p(\cdot)}(\Omega)$, and it holds that

$$\int_\Omega |\nabla \overline{v}|^{p(x)-2}\nabla \overline{v} \nabla T_k(\overline{v}-\varphi)^- dx + \int_\Omega |\overline{v}|^{q-1}\overline{v} T_k(\overline{v}-\varphi)^- dx \ge \int_\Omega g T_k(\overline{v}-\varphi)^- dx, \tag{3.2}$$

for any $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Definition 3.2. A function $\underline{u}(x,t)$ is an entropy subsolution of (1.1) if, for all $T, k > 0$, we have $\underline{u}(x,t) \in C([0,T]; L^1(\Omega)) \cap L^q(Q_T), T_k(\underline{u}) \in L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega))$, $\nabla T_k(\underline{u}) \in (L^{p(\cdot)}(Q_T))^N$, and it holds that

$$\begin{aligned}
&\int_\Omega \Phi_k((\underline{u}-\varphi)^+)(T)dx - \int_\Omega \Phi_k((\underline{u}_0-\varphi(0))^+)dx \\
&+ \int_{Q_T} |\nabla \underline{u}|^{p(x)-2}\nabla \underline{u}\nabla T_k(\underline{u}-\varphi)^+ \,dx\,dt \\
&+ \int_0^T \langle \varphi_t, T_k(\underline{u}-\varphi)^+\rangle dt + \int_{Q_T} |\underline{u}|^{q-1}\underline{u} T_k(\underline{u}-\varphi)^+ \,dx\,dt \\
&\le \int_{Q_T} g T_k(\underline{u}-\varphi)^+ \,dx\,dt,
\end{aligned} \tag{3.3}$$

for any $\varphi \in C^1(\overline{Q}_T)$ with $\varphi = 0$ on $\partial\Omega \times (0,T)$ and $\underline{u}(x,0) \equiv \underline{u}_0(x) \le u_0(x)$ a.e. in Ω with $\underline{u}_0 \in L^1(\Omega)$.

On the other hand, a function $\overline{u}(x,t)$ is an entropy supersolution of problem (1.1) if, for all $T, k > 0$, we have $\overline{u}(x,t) \in C([0,T]; L^1(\Omega)) \cap L^q(Q_T), T_k(\overline{u}) \in L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega))$, $\nabla T_k(\overline{u}) \in (L^{p(\cdot)}(Q_T))^N$, and it holds that

$$\begin{aligned}
&\int_\Omega \Phi_k((\overline{u}-\varphi)^-)(T)dx - \int_\Omega \Phi_k((\overline{u}_0-\varphi(0))^-)dx \\
&+ \int_{Q_T} |\nabla \overline{u}|^{p(x)-2}\nabla \overline{u}\nabla T_k(\overline{u}-\varphi)^- \,dx\,dt \\
&+ \int_0^T \langle \varphi_t, T_k(\overline{u}-\varphi)^-\rangle dt + \int_{Q_T} |\overline{u}|^{q-1}\overline{u} T_k(\overline{u}-\varphi)^- \,dx\,dt \\
&\ge \int_{Q_T} g T_k(\overline{u}-\varphi)^- \,dx\,dt,
\end{aligned} \tag{3.4}$$

for any $\varphi \in C^1(\overline{Q}_T)$ with $\varphi = 0$ on $\partial\Omega \times (0,T)$ and $\overline{u}(x,0) \equiv \overline{u}_0(x) \ge u_0(x)$ a.e. in Ω with $\overline{u}_0 \in L^1(\Omega)$.

Remark 3.3. Taking $T_k(u^n-\varphi)^+$, $T_k(u^n-\varphi)^-$ as test functions in (2.5) and passing to the limits, we obtain that an entropy solution to problem (1.1) is both an entropy subsolution and an entropy supersolution of the same problem. In the same way, an entropy solution to the elliptic problem (2.2) also turns out to be an entropy subsolution and an entropy supersolution to the problem.

Remark 3.4. Similar to the observation in Remark 2.6, we may find that an entropy subsolution (entropy supersolution) $\underline{v}$ (respectively $\overline{v}$) of the elliptic problem (2.2) is automatically an entropy subsolution (entropy supersolution) of (1.1) with itself as initial data.

Lemma 3.5. *Let $u_0, g \in L^1(\Omega)$, and $\overline{u}, \underline{u}$ be an entropy supersolution and an entropy subsolution to problem* (1.1) *respectively. Let u be the unique entropy solution to the same problem. Then for any $t > 0$, we have $\underline{u} \le u \le \overline{u}$ almost everywhere in Ω.*

The proof of the above lemma is almost the same as that of [26, Lemma 3.3], we omit it.

Theorem 3.6. *Let $\overline{v}$ and $\underline{v}$ be, respectively, an entropy supersolution and an entropy subsolution to problem* (2.2) *respectively. Assume* (2.1) *holds, $g, u_0 \in L^1(\Omega)$ and $\underline{v} \le u_0 \le \overline{v}$. If*

$$\theta(x) \doteq \max\{\frac{p(x)q}{(q+1)}, \quad p(x) - \frac{N}{N+1}\} > 1 \text{ in } \overline{\Omega},$$

then the unique entropy solution u of problem (1.1) *converges in $L^q(\Omega)$ to the unique entropy solution v of problem* (2.2) *as t tends to infinity.*

Corollary 3.7. *Assume* (2.1) *holds, $g \in L^1(\Omega), u_0 \equiv 0$. If $\theta(x) > 1$ in $\overline{\Omega}$, then the unique entropy solution $u(t)$ of problem* (1.1) *converges to the unique entropy solution v of problem* (2.2) *in $L^q(\Omega)$ as t tends to infinity.*

Proof of Theorem 3.6. Consider the nonlinear problem

$$\begin{gathered}
(u_m)_t - \operatorname{div}(|\nabla u_m|^{p(x)-2}\nabla u_m) + |u_m|^{q-1}u_m = g \quad \text{in } \Omega \times (0,1), \\
u_m = 0 \quad \text{on } \partial\Omega \times (0,1), \\
u_m(x,0) = u(x,m) \quad \text{in } \Omega,
\end{gathered} \tag{3.5}$$

where $m \in \mathbb{N} \cup \{0\}$, and $u(x,0) = \overline{v}$. Let $u(t)$ be the entropy solution for problem (1.1) with $\overline{v}$ as initial data. Thanks to the uniqueness of entropy solutions and the independency of t for the data g, $u_m(t)$ is just the restriction of $u(t)$ on the interval $[m, m+1)$. Note that v and $\overline{v}$ are the entropy subsolution and the entropy supersolution respectively for problem (1.1) with initial data $\overline{v}$. Thanks to Lemma 3.5, $v \le u(t) \le \overline{v}$ for any $t > 0$. Similarly, let $u(s+t)$ be the solution with $u(s)$ as initial data, then we have $u(s+t) \le u(t)$ for any $t, s > 0$, which implies that $u(t)$ is decreasing in t. Thus for $0 < t < 1$,

$$\overline{v}(x) \ge u_m(x,0) \ge u_m(x,t) = u(x, m+t) \ge u_{m+1}(x,0) \ge v(x). \tag{3.6}$$

Hence there must be a function $w(x) \ge v(x)$ such that $u(x,t)$ converges to $w(x)$ almost everywhere in Ω as t tends to infinity. And then by the dominated convergence theorem, we have

$$u(x,t) \to w(x) \text{ in } L^1(\Omega) \quad \text{as } t \to +\infty. \tag{3.7}$$

Next, following the ideas of [26] (see also [27]), we can perform some estimates for the sequence $\{u_m\}$, to prove that $w(x)$ is actually the entropy solution v to the elliptic problem (2.2).

Thanks to (3.6), we have

$$\|u_m(t)\|_{L^1(\Omega)} = \|u(m+t)\|_{L^1(\Omega)} \le \|\overline{v}\|_{L^1(\Omega)} + \|v\|_{L^1(\Omega)} \le C, \quad 0 < t \le 1, \tag{3.8}$$

where C is obviously independent of m. Thus the sequence $\{u_m\}$ is bounded in $L^\infty(0,1;L^1(\Omega))$. Taking $u_0 = u(x,m)$, $T = 1$ and $\varphi = 0$ in Definition 2.4, we obtain

$$\begin{aligned}
&\int_\Omega \Phi_k(u_m)(1)dx + \int_0^1 \int_\Omega |\nabla T_k(u_m)|^{p(x)}\,dx\,d\tau + \int_0^1 \int_\Omega |u_m|^q |T_k(u_m)|\,dx\,d\tau \\
&\le k\|g\|_{L^1(\Omega)} + \int_\Omega \Phi_k(u(x,m))dx.
\end{aligned} \tag{3.9}$$

Note that

$$0 \le \Phi_k(s) \le k|s| \le \Phi_k(s) + \frac{k^2}{2}. \tag{3.10}$$

Noticing (3.8), we deduce from (3.9) that

$$\int_0^1 \int_\Omega |\nabla T_k(u_m)|^{p(x)}\,dx\,d\tau \le Ck, \tag{3.11}$$

$$\begin{aligned}
\int_0^1 \int_\Omega |u_m|^q\,dx\,d\tau &\le \int_0^1 \int_\Omega |u_m|^q |T_k(u_m)|\,dx\,d\tau + |\Omega| \\
&\le \int_\Omega \Phi_1(u(x,m))dx + |\Omega| + \|g\|_{L^1(\Omega)} \le C.
\end{aligned} \tag{3.12}$$

For a given function $f(x,t)$ defined on Q_T, we set

$$\{f \ge k\} = \{(x,t) \in Q_T : f(x,t) \ge k\}, \{f \le k\} = \{(x,t) \in Q_T : f(x,t) \le k\}.$$

Then setting $\alpha(\cdot) = p(\cdot)/(q+1)$ in $\overline{\Omega}$ and using (3.12), we deduce that

$$\begin{aligned}
&\int_{\{|\nabla u_m|^{\alpha(x)} > k\}} k^q\,dx\,dt \\
&\le \int_{\{|\nabla u_m|^{\alpha(x)} > k\} \cap \{|u_m| \le k\}} k^q\,dx\,dt + \int_{\{|u_m| > k\}} k^q\,dx\,dt \\
&\le \int_{\{|u_m| \le k\}} k^q \Big(\frac{|\nabla u_m|^{\alpha(x)}}{k}\Big)^{\frac{p(x)}{\alpha(x)}}\,dx\,dt + \int_{Q_1} |u_m|^q\,dx\,dt \\
&\le \frac{1}{k} \int_{Q_1} |\nabla T_k(u_m)|^{p(x)}\,dx\,dt + C \le C,
\end{aligned} \tag{3.13}$$

which implies that $|\nabla u_m|^{p(\cdot)/(q+1)}$ is bounded in $M^q(Q_1)$, and hence we conclude from Lemma 2.1 that

$$|\nabla u_m|^{\beta(\cdot)} \text{ is bounded in } L^1(Q_1) \text{ for } \beta \in C(\overline{\Omega}) \text{ satisfying } \beta(\cdot) < p(\cdot)q/(q+1) \text{ in } \overline{\Omega}. \tag{3.14}$$

On the other hand, let $s \in C(\overline{\Omega})$ such that $1 < s(\cdot) < (N+1)p(\cdot)/N$ in $\overline{\Omega}$. From the continuity of s and p, for any $x \in \Omega$, there exists a ball $B_\delta(x)$ of x, such that $s^+(B_\delta(x) \cap \Omega) < ((N+1)p(\cdot)/N)^-\,(B_\delta(x) \cap \Omega)$, where

$$s^+(B_\delta(x) \cap \Omega) = \max\{s(y) : y \in \overline{B_\delta(x) \cap \Omega}\},$$

$$((N+1)p(\cdot)/N)^-\,(B_\delta(x) \cap \Omega) = \min\{(N+1)p(y)/N : y \in \overline{B_\delta(x) \cap \Omega}\}.$$

It is obvious that $\cup_{x \in \Omega} B_\delta(x)$ is an open covering of $\overline{\Omega}$. Since $\overline{\Omega}$ is compact, there is a finite sub-covering $B_{\delta_i}(x_i), i = 1, 2, \dots, l$. For convenience, we denote the set

$B_{\delta_i}(x_i)\cap\Omega$ by U_i hereafter. Assume that $\operatorname{meas}(U_i) > c > 0, i = 1, 2, \dots, l$. Denoting ${s_i}^+ = s^+(U_i), {p_i}^- = p^-(U_i)$, we have

$$((N+1)p(\cdot)/N)^-(U_i) = (N+1){p_i}^-/N.$$

Setting $U_{i,1} = U_i \times (0, 1)$, we deduce that

$$\begin{aligned}
&\int_{U_{i,1}\cap\{|u_m|>k\}} k^{\frac{(N+1){p_i}^-}{N}}\,dx\,dt \\
&\le \int_0^1\int_{U_i} |T_k(u_m)|^{\frac{(N+1){p_i}^-}{N}}\,dx\,dt \\
&\le 2^{\frac{(N+1){p_i}^-}{N}}\int_0^1\int_{U_i} |T_k(u_m) - (T_k(u_m))_i|^{\frac{(N+1){p_i}^-}{N}}\,dx\,dt \\
&\quad + 2^{\frac{(N+1){p_i}^-}{N}}\int_0^1\int_{U_i} |(T_k(u_m))_i|^{\frac{(N+1){p_i}^-}{N}}\,dx\,dt,
\end{aligned} \tag{3.15}$$

where

$$(T_k(u_m))_i = \frac{1}{\operatorname{meas}(U_i)}\int_{U_i} T_k(u_m)dx \text{ for almost all } t \in (0, 1).$$

Thanks to (3.8), we have

$$|(T_k(u_m))_i| \le \frac{1}{\operatorname{meas}(U_i)}\int_{U_i} |T_k(u_m)|dx \le C,$$

where c is independent of k, m. By the well-known Gagliardo-Nirenberg inequality, we have, for almost all $t \in (0, 1)$,

$$\begin{aligned}
&\int_{U_i} |T_k(u_m) - (T_k(u_m))_i|^{\frac{(N+1){p_i}^-}{N}}\,dx \\
&\le C\int_{U_i} |\nabla T_k(u_m)|^{{p_i}^-}\,dx\Big(\int_{U_i} |T_k(u_m) - (T_k(u_m))_i|dx\Big)^{\frac{{p_i}^-}{N}}.
\end{aligned}$$

Integrating the above inequality over $(0, 1)$, we deduce that

$$\begin{aligned}
&\int_0^1\int_{U_i} |T_k(u_m) - (T_k(u_m))_i|^{\frac{(N+1){p_i}^-}{N}}\,dx\,dt \\
&\le C\left(\|u_m\|_{L^\infty(0,1;L^1(\Omega))} + C|\Omega|\right)^{\frac{{p_i}^-}{N}}\int_0^1\int_{U_i} |\nabla T_k(u_m)|^{{p_i}^-}\,dx\,dt.
\end{aligned}$$

Taking the last inequality and (3.8), (3.11) into (3.15), we obtain that

$$\int_{U_{i,1}\cap\{|u_m|>k\}} k^{\frac{(N+1){p_i}^- - N}{N}}\,dx\,dt \le C, \tag{3.16}$$

where C may depend on $\|g\|_{L^1(\Omega)}, \|\overline{v}\|_{L^1(\Omega)}, |\Omega|$, but it is independent of k, m. Since ${s_i}^+ < \frac{(N+1){p_i}^-}{N}$ in U_i, (3.16) implies that $(k \ge 1)$

$$\begin{aligned}
\int_{U_{i,1}\cap\{|u_m|>k\}} k^{s(x)-1}\,dx\,dt &\le \int_{U_{i,1}\cap\{|u_m|>k\}} k^{{s_i}^+-1}\,dx\,dt \\
&\le \int_{U_{i,1}\cap\{|u_m|>k\}} k^{\frac{(N+1){p_i}^- - N}{N}}\,dx\,dt \le C.
\end{aligned}$$

Then

$$\int_{\{|u_m|>k\}} k^{s(x)-1}dx \le \sum_{i=1}^{l} \int_{U_i\cap\{|u_m|>k\}} k^{s(x)-1}dx \le lC. \tag{3.17}$$

Hence $\{u_m\}$ is bounded in $M^{s(x)-1}(Q_1)$. Similar calculations as (3.13) help us to obtain that $\{|\nabla u_m|^{\frac{p(x)}{s(x)}}\}$ is bounded in $M^{s(x)-1}(Q_1)$. By Lemma 2.1, we have that

$$\text{the set } \{|\nabla u_m|^{\beta_1(x)}\} \text{ is bounded in } L^1(Q_1) \text{ for } \beta_1 \in C(\overline{\Omega}) \text{ satisfying } \beta_1(\cdot) < p(\cdot) - N/(N+1). \tag{3.18}$$

Combining (3.14) and (3.18), we obtain that $\{|\nabla u_m|^{\gamma(x)}\}$ is bounded in $L^1(Q_1)$ for any $\gamma \in C(\overline{\Omega})$ satisfying

$$0 < \gamma(x) < \theta(x) \doteq \max\{p(x)q/(q+1),\ p(x) - N/(N+1)\} \quad \text{in } \overline{\Omega}.$$

Thus if $\theta(x) > 1$ in $\overline{\Omega}$ (for example, in the case $p(x) > 2 - 1/(N+1)$ or $q > 1/(p(x)-1)$ in $\overline{\Omega}$), we can choose a constant $1 < r_0 < \theta(x)$ such that

$$\{u_m(t)\} \text{ is bounded in } L^{r_0}(0,1;W_0^{1,r_0}(\Omega)).$$

On the other hand, note that

$$p(x) - N/(N+1) > p(x) - 1 \text{ in } \overline{\Omega}.$$

The definition of $\theta(x)$ allows us to choose a positive function γ_0 in $\overline{\Omega}$, such that

$$p(x) - 1 < \gamma_0(x) < \theta(x) \quad \text{in } \overline{\Omega}.$$

Hence $\{(|\nabla u_m|^{(p(x)-1)})^{\gamma_0(x)/(p(x)-1)}\}$ $(=\{|\nabla u_m|^{\gamma_0(x)}\})$ is bounded in $L^1(Q_1)$. Therefore, $\{|\nabla u_m|^{p(x)-1}\}$ is bounded in $L^{s_0}(Q_1)$ for $1 < s_0 \le \gamma_0(x)/(p(x)-1)$ in $\overline{\Omega}$, which implies that $\operatorname{div}(|\nabla u_m|^{p(x)-2}\nabla u_m)$ is uniformly (with respect to m) bounded in $L^{s_0}(0,1;W^{-1,s_0}(\Omega))$. Then we deduce from the equation that

$$\{(u_m)_t\} \text{ is bounded in } L^{s_0}(0,1;W^{-1,s_0}(\Omega)) + L^1(0,1;L^1(\Omega)).$$

So, thanks to the well-known compactness result of Aubin's type, see for example [32], there is a subsequence of $\{u_m\}$, denoted by $\{u_{m_k}\}$, which converges to a function $\widetilde{u}$ in $L^1(Q_1)$. Since $u_{m_k}(x,t) = u(x, t+m_k)$, we may conclude from (3.7) that $\widetilde{u} = w(x)$ is independent of time.

Now, let us show that $\widetilde{u}$ is actually the entropy solution v of the elliptic problem (2.2). From (3.11), we have

$$\begin{gathered} T_k(u_m) \to T_k(\widetilde{u}) \quad \text{weakly in } L^{p^-}(0,1;W_0^{1,p(\cdot)}(\Omega)), \\ \nabla T_k(u_m) \to \nabla T_k(\widetilde{u}) \quad \text{weakly in } (L^{p(\cdot)}(Q_1))^N. \end{gathered}$$

From the estimate on u_m, we know that

$$u_m \to \widetilde{u} \quad \text{weakly in } L^{r_0}(0,1;W_0^{1,r_0}(\Omega)), \text{ for some } 1 < r_0 < \theta(x).$$

Furthermore, with very minor modifications on the proof of [11, Theorem 3.3], one can prove that

$$\nabla u_m \to \nabla \widetilde{u} \quad \text{a.e. in } Q_1.$$

Since u_m is the entropy solution of (3.5), we have

$$\int_\Omega \Phi_k(u_m - \varphi)(1)dx - \int_\Omega \Phi_k(u(x,m) - \varphi(0))dx + \int_0^1 \langle \varphi_t, T_k(u_m - \varphi)\rangle dt \tag{3.19}$$

$$+\int_{Q_1} |\nabla u_m|^{p(x)-2}\nabla u_m\cdot\nabla T_k(u_m-\varphi)\,dx\,dt+\int_{Q_1}|u_m|^{q-1}u_mT_k(u_m-\varphi)\,dx\,dt \tag{3.20}$$

$$\le\int_{Q_1} gT_k(u_m-\varphi)dx \tag{3.21}$$

for any $\varphi\in C^1(\overline{Q}_1)$ with $\varphi=0$ in $\partial\Omega\times(0,1)$. Using the convergence results above for u_m, and passing to the limit, we deduce that

$$(3.19)\to\int_\Omega \Phi_k(\widetilde{u}-\varphi(1))dx-\int_\Omega\Phi_k(\widetilde{u}-\varphi(0))dx+\int_0^1\langle\varphi_t,T_k(\widetilde{u}-\varphi)\rangle dt.$$

Since $\widetilde{u}=w(x)$ is independent of time, similar to Remark 2.6 we have

$$\int_\Omega \Phi_k(\widetilde{u}-\varphi(1))dx-\int_\Omega\Phi_k(\widetilde{u}-\varphi(0))dx+\int_0^1\langle\varphi_t,T_k(\widetilde{u}-\varphi)\rangle dt=0. \tag{3.22}$$

Passing to the limit in (3.21), we obtain that

$$(3.21)\to\int_{Q_1} gT_k(\widetilde{u}-\varphi)dx. \tag{3.23}$$

At last, passing to the limit in (3.20), we note that

$$\int_{Q_1} |\nabla u_m|^{p(x)-2}\nabla u_m\cdot\nabla T_k(u_m-\varphi)\,dx\,dt \tag{3.24}$$

$$=\int_{Q_1}(|\nabla u_m|^{p(x)-2}\nabla u_m-|\nabla\varphi|^{p(x)-2}\nabla\varphi)\nabla T_k(u_m-\varphi)\,dx\,dt \tag{3.25}$$

$$+\int_{Q_1}|\nabla\varphi|^{p(x)-2}\nabla\varphi\cdot\nabla T_k(u_m-\varphi)\,dx\,dt. \tag{3.26}$$

Using Fatou's lemma and the weak convergence of $\nabla T_k(u_m)$ we obtain

$$\lim_{m\to\infty}(3.24)\ge\int_{Q_1}|\nabla\widetilde{u}|^{p(x)-2}\nabla\widetilde{u}\nabla T_k(\widetilde{u}-\varphi)\,dx\,dt. \tag{3.27}$$

Similarly, since

$$\begin{aligned}\int_{Q_1}|u_m|^{q-1}u_mT_k(u_m-\varphi)\,dx\,dt&=\int_{Q_1}(|u_m|^{q-1}u_m-|\varphi|^{q-1}\varphi)T_k(u_m-\varphi)\,dx\,dt\\&\quad+\int_{Q_1}|\varphi|^{q-1}\varphi T_k(u_m-\varphi)\,dx\,dt,\end{aligned} \tag{3.28}$$

we deduce that

$$\lim_{m\to\infty}(3.28)\ge\int_{Q_1}|\widetilde{u}|^{q-1}\widetilde{u}\nabla T_k(\widetilde{u}-\varphi)\,dx\,dt. \tag{3.29}$$

We then conclude from (3.22), (3.23), (3.27) and (3.29) that

$$\begin{aligned}&\int_{Q_1}|\nabla\widetilde{u}|^{p(\cdot)-2}\nabla\widetilde{u}\nabla T_k(\widetilde{u}-\varphi)\,dx\,dt+\int_{Q_1}|\widetilde{u}|^{q-1}\widetilde{u}T_k(\widetilde{u}-\varphi)\,dx\,dt\\&\le\int_{Q_1}gT_k(\widetilde{u}-\varphi)\,dx\,dt.\end{aligned}$$

Especially, for $\phi \in C^1(\overline{\Omega}), \phi\,|_{\partial\Omega}= 0$, we have

$$\int_\Omega |\nabla\widetilde{u}|^{p(\cdot)-2}\nabla\widetilde{u}\nabla T_k(\widetilde{u}-\phi)\,dx + \int_\Omega |\widetilde{u}|^{q-1}\widetilde{u}T_k(\widetilde{u}-\phi)\,dx \le \int_\Omega T_k(\widetilde{u}-\phi)g\,dx.$$

Then from the density result, we conclude that $\widetilde{u}$ satisfies the entropy formulation of problem (2.2) and hence it coincides with the unique entropy solution v. Performing a similar argument, we can prove that the entropy solution $u^1(t)$ for problem (1.1) with $\underline{v}$ as initial data also converges in $L^1(\Omega)$ to the entropy solution v of problem (2.2).

For the entropy solution $u^2(t)$ of (1.1) corresponding to the initial data u_0 with $\underline{v} \le u_0 \le \overline{v}$, thanks to the comparison result, we have

$$\underline{v} \le u^1(t) \le u^2(t) \le u(t) \le \overline{v}.$$

Thus we obtain that $u^2(t)$ converges to v in $L^1(\Omega)$. Since $\underline{v}, \overline{v}$ all lie in $L^q(\Omega)$, we obtain the convergence result in $L^q(\Omega)$. $\square$

Proof of Corollary 3.7. Let v_1, v_2 be the entropy solutions to the following two problems:

$$\begin{gathered} -\operatorname{div}(|\nabla v|^{p(x)-2}\nabla v) + |v|^{q-1}v = g^+ \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.30}$$

and

$$\begin{gathered} -\operatorname{div}(|\nabla v|^{p(x)-2}\nabla v) + |v|^{q-1}v = -g^- \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.31}$$

Note that 0 is an entropy subsolution of (3.30), and it is an entropy supersolution of (3.31). Since the entropy solution can be obtained as the limit of the solutions for the approximate problems, similar to Lemma 3.5, it is not difficult to show the following comparison result, see [24, 27] for the constant exponents case,

$$v_2 \le 0 \le v_1 \ \textit{a.e.} \text{ in } \Omega.$$

On the other hand, thanks to Remark 3.3, v_1 is an entropy supersolution of (3.30). And hence, it is an entropy supersolution of (2.2). Similarly, v_2 is an entropy subsolution of (2.2). Thus, the result of the corollary follows immediately from Theorem 3.6. $\square$

Remark 3.8. Let $w(t) = u(t) - v$. We can prove that $w(t)$ converges to zero in $L^r(\Omega)$ for any $1 \le r < \infty$ as t tends to infinity. Indeed, consider the approximate problem for (2.2),

$$\begin{gathered} -\operatorname{div}(|\nabla v^n|^{p(x)-2}\nabla v^n) + |v^n|^{q-1}v^n = g^n \quad \text{in } \Omega, \\ v^n = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.32}$$

where $\{g^n\}$ is the same sequence as in (2.5). Problem (3.32) admits a unique solution v^n for each n, and up to subsequences, $\{v^n\}$ converges to the unique entropy solution v of (2.2) in $L^1(\Omega)$, see [13, 39]. Subtracting (3.32) from (2.5) and setting $w^n = u^n - v^n$, we have

$$\begin{gathered} w_t^n - \operatorname{div}(|\nabla u^n|^{p(x)-2}\nabla u^n - |\nabla v^n|^{p(x)-2}\nabla v^n) + |u^n|^{q-1}u^n - |v^n|^{q-1}v^n = 0 \\ \text{in } \Omega \times \mathbb{R}^+, \\ w^n = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ w^n(x,0) = u_0^n - v^n \quad \text{in } \Omega. \end{gathered} \tag{3.33}$$

Thanks to the convergence results for u^n and v^n, we know that, up to a subsequence, w^n converges to $w = u - v$ in $C([0,T]; L^1(\Omega))$ for any $T > 0$.

Taking $T_k(u^n)(k \geq 1)$ as a test function in (2.5), we deduce that

$$\frac{d}{dt}\int_\Omega \Phi_k(u^n)(t)dx + \int_\Omega |\nabla T_k(u^n)|^{p(x)}dx + \int_\Omega |u^n|^q|T_k(u^n)|dx \leq k\|g\|_{L^1(\Omega)}. \tag{3.34}$$

From the definition of $\Phi_k(\cdot)$ we obtain

$$\int_\Omega \Phi_k(u^n)(t)dx \leq k\|u^n(t)\|_{L^1(\Omega)} \leq \int_\Omega \Phi_k(u^n)(t)dx + \frac{k^2}{2}|\Omega|, \tag{3.35}$$

where $|\Omega|$ is the Lebesgue measure of Ω. Note that

$$\int_\Omega \Phi_1(u^n)(t)dx \leq \int_\Omega |u^n|^q|T_1(u^n)|dx + |\Omega|.$$

We deduce from (3.34) that

$$\frac{d}{dt}\int_\Omega \Phi_1(u^n)(t)dx + \int_\Omega \Phi_1(u^n)(t)dx \leq \|g\|_{L^1(\Omega)} + |\Omega|.$$

Standard Gronwall type inequality implies that

$$\int_\Omega \Phi_1(u^n)(t)dx \leq \|g\|_{L^1(\Omega)} + |\Omega| + e^{-t}\int_\Omega \Phi_1(u_0^n)dx, \quad t > 0.$$

Thanks to (3.10), we have

$$\|u^n(t)\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)} + \frac{3}{2}|\Omega| + \|g\|_{L^1(\Omega)}, \quad t > 0. \tag{3.36}$$

Integrating (3.34) on $[t, t+1]$, we obtain

$$\begin{aligned} \int_t^{t+1}\int_\Omega |u^n|^q\,dx\,d\tau &\leq \int_t^{t+1}\int_\Omega (|u^n|^q|T_1(u^n)| + 1)\,dx\,d\tau \\ &\leq \|g\|_{L^1(\Omega)} + \|u_0\|_{L^1(\Omega)} + |\Omega|. \end{aligned} \tag{3.37}$$

Multiplying (3.32) by $T_1(v^n)$, we deduce that

$$\int_\Omega |v^n|^q dx \leq \int_\Omega (|v^n|^q|T_1(v^n)| + 1)dx \leq \|g\|_{L^1(\Omega)} + |\Omega|. \tag{3.38}$$

Combining (3.37), (3.38), we have

$$\int_t^{t+1}\int_\Omega |w^n(\tau)|^q\,dx\,d\tau \leq 2\|g\|_{L^1(\Omega)} + 2|\Omega| + \|u_0\|_{L^1(\Omega)}, \quad \text{for any } t \geq 0. \tag{3.39}$$

Taking $|w^n|^{q-2}w^n$ as a test function in (3.33) (if $q < 2$, we can take $((|w^n|+\epsilon)^{q-1} - \epsilon^{q-1})sgn(w^n)$ as a test function and then let ϵ go to zero to justify this calculation.

Here for simplicity, we assume that $q \geq 2$.), we deduce that

$$\frac{1}{q}\frac{d}{dt}\int_\Omega |w^n(t)|^q\,dx + C\int_\Omega |w^n(t)|^{2q-1}dx \leq 0. \tag{3.40}$$

Integrating the above inequality from s to $t+1 (0 \leq t \leq s < t+1)$, yields

$$\int_\Omega |w^n(t+1)|^q\,dx \leq \int_\Omega |w^n(s)|^q\,dx.$$

Integrating this inequality with respect to s from t to $t+1$ and using (3.39), yields

$$\int_\Omega |w^n(t+1)|^q\,dx \leq \int_t^{t+1}\int_\Omega |w^n(\tau)|^q\,dx\,d\tau \leq C, \quad \text{for any } t \geq 0, \tag{3.41}$$

with C independent of n, t. Integrating (3.40) on $[t, t+1]$ for any $t \geq 1$, and using (3.41) we deduce that

$$\int_t^{t+1}\int_\Omega |w^n(\tau)|^{2q-1}\,dx\,d\tau \leq C\int_\Omega |w^n(t)|^q\,dx \leq C \tag{3.42}$$

Now setting $q_1 = 2q-1$, and using $|w^n|^{q_1-2}w^n$ as a test function in (3.33), we obtain

$$\frac{1}{q_1}\frac{d}{dt}\int_\Omega |w^n|^{q_1}dx + C\int_\Omega |w^n|^{q_1+q-1}dx \leq 0. \tag{3.43}$$

Integrating (3.43) from s to $t+1 (1 \leq t \leq s < t+1)$, yields

$$\int_\Omega |w^n(t+1)|^{q_1}dx \leq \int_\Omega |w^n(s)|^{q_1}dx.$$

Integrating the above inequality with respect to s from t to $t+1$ and using (3.42), we obtain

$$\int_\Omega |w^n(t+1)|^{q_1}dx \leq \int_t^{t+1}\int_\Omega |w^n(\tau)|^{q_1}\,dx\,d\tau \leq C, \quad \text{for any } t \geq 1, \tag{3.44}$$

with C independent of n, t. Now integrating (3.43) on $[t, t+1]$ for $t \geq 2$, and using (3.44) we have

$$\int_t^{t+1}\int_\Omega |w^n(\tau)|^{q_1+q-1}dx\tau \leq C\int_\Omega |w^n(t)|^{q_1}dx \leq C.$$

Bootstrapping the above processes, we can deduce that

$$\int_\Omega |w^n(t)|^{q_k}dx \leq C, \quad \text{for } t \geq T_k,$$

with $q_k = q_{k-1} + q - 1$, $q_0 = q$, C being independent of n. Passing to the limit, we obtain the same estimate for w. Combining this estimate with the convergence result obtained in Theorem 3.6, we obtain that $w(t)$ converges to zero in $L^r(\Omega)$ for any $1 \leq r < \infty$ as t tends to infinity.

Remark 3.9. Although we performed all the calculations under the assumption that $g \in L^1(\Omega)$, with very minor modifications, we can show that all the results can be extended to the case $g \in L^1(\Omega) + W^{-1,p'(x)}(\Omega)$.

If we replace the $p(x)$-Laplacian operator $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ by a more general Leray-Lions type operator involving variable exponent $-\operatorname{div}(a(x, \nabla v))$, where a :

$\Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (i.e. $a(x,\xi)$ is measurable on Ω for all $\xi \in \mathbb{R}^N$, and $a(x,\xi)$ is continuous on $\mathbb{R}^N$ for a.e. $x \in \Omega$) such that

$$
\begin{gathered}
a(x,\xi)\xi \geq \alpha|\xi|^{p(x)}, \\
|a(x,\xi)| \leq \beta[b(x) + |\xi|^{p(x)-1}], \\
(a(x,\xi) - a(x,\eta))(\xi - \eta) > 0,
\end{gathered}
$$

for almost every $x \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$, α, β being positive constants, $b(x)$ being a nonnegative function in $L^{p(\cdot)/p(\cdot)-1}(\Omega)$, then the results obtained above still hold.

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School of Mathematical Sciences, Anhui University, Hefei 230601, China

E-mail address, Xiaojuan Chai: `chaixj@ahu.edu.cn`

E-mail address, Haisheng Li: `squeensy@sina.com`

E-mail address, Weisheng Niu: `niuwsh@ahu.edu.cn`