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SIMULTANEOUS BINARY COLLISIONS IN THE EQUAL-MASS COLLINEAR FOUR-BODY PROBLEM

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ABSTRACT. In the four-body problem, it is not clear what initial conditions can lead to simultaneous binary collision (SBC), even in the collinear case. In this paper, we study SBC in the equal-mass collinear four-body problem and have a partial answer for initial conditions leading to SBC. After introducing a Levi-Civita type transformation, we analyze the new transformed differential system of SBC and solve for all possible solutions. The problem is studied in two cases: decoupled case and coupled case. In the decoupled case where SBC is treated as two separated binary collisions, the initial conditions leading to SBC satisfy several simple algebraic identities. This result gives insights to the coupled case, which is SBC in the equal-mass collinear four-body problem. Furthermore, we show from a different perspective that solutions passing through SBC must be analytic in the transformed system and the initial condition set leading to SBC has a measure 0.

1. Introduction and main results

The N-body problem in celestial mechanics considers the motion of N point masses m_1, $m_2, \dots, m_N$ governed by a Newtonian gravitational force. The equations of motion are

$$m_i \ddot{q}_i = -\sum_{j \neq i} \frac{m_i m_j (q_i - q_j)}{|q_i - q_j|^3}, \quad i = 1, \dots N, \tag{1.1}$$

where q_i denotes the position of the i-th body with mass m_i ($i = 1, 2, \dots, N$) and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$.

Definition 1.1. A solution $q(t) = (q_1(t), \dots, q_N(t))$ of differential equations (1.1) has a *singularity* at time $t^* < \infty$ if it cannot be extended beyond t^* under the variables $\{q_1, \dots, q_N, t\}$.

A collision occurs if $q_i = q_j$ for some $i \neq j$. It is clear that a collision is a singularity. The collision set is defined by

$$\triangle \equiv \cup \{q = (q_1, q_2, \dots, q_N) \in (\mathbb{R}^d)^N : q_i = q_j \text{ for some } i \neq j\}.$$

Definition 1.2. A singularity at time $t = t^*$ is a *collision singularity* if there exists $q^* \in \triangle$, such that $q(t) \to q^*$ as $t \to t^*$.

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The collision set $\triangle$ contains several types of collision singularities: binary collision, total collision, triple collision, simultaneous binary collision, etc. A *binary collision* occurs when only two masses collide. If all the masses coincide at the same point, we have a *total collision*. Between these two extreme cases, other types of collisions can also occur. A *triple collision* occurs if three of the point masses coincide while the rest have distinct positions. If two or more different pairs of binary collisions coincide at different locations, we have a *simultaneous binary collision* (SBC for short).

The simplest collision singularity is binary collision. "It is known that in the two-body problem, one can change both space and time variables so that a binary collision transforms to a regular point of equations. The solution can then be extended through the collision singularity in the new variables [12]." In the three-body problem, binary collision was successfully studied by Sundman [22, 23]. He found a convergent series solution in a new time variable $\tau = t^{1/3}$ for the three-body problem which ends in a binary collision at $t = 0$. This is commonly referred to as regularization of binary collision. About a decade later, Levi-Civita [8] proposed another approach and regularized the binary collision. The transformation introduced in his work is now known as the Levi-Civita transformation, which is an important tool in the regularization theory.

SBC is another type of collision singularity. In this case, k $(2 \leq k < \infty)$ pairs of binary collisions occur simultaneously at time $t = 0$. To be more precise, we assume that there exists a neighborhood $(-\delta, \delta)$ of t, such that SBC at $t = 0$ is the only singularity in $(-\delta, \delta)$. And at $t = 0$, the distances between the centers of each collision pairs are bounded below by a constant $\alpha_0 > 0$. The regularization of SBC has been widely studied. Sperling [21], Saari [18] and Belbruno [1] proved the existence of convergent series solution of SBC in the new time variable $\tau = t^{1/3}$ independently. Later, Lacomba and Simó [20] showed that SBC is time continuation regularizable. However, there are still some open questions. For example, it is not clear that if there exists a transformation such that SBC becomes a regular point in the transformed differential system. After changing of variables, SBC may still be a singular point of the new system. Can one solve this transformed differential system of SBC? Our work provides a positive answer for this question.

For the purpose of clarification, here we only study SBC in the equal-mass collinear four-body problem. After introducing a new time variable, the standard Hamiltonian system generated by Newtonian equations (1.1) becomes a new differential system, which is not a Hamiltonian. We use a device proposed by Poincaré [19] and transfer this differential system to a new Hamiltonian system. Then the study of SBC becomes to analyze the new Hamiltonian system. Since this new Hamiltonian has a complicated form, we introduce a technique of Siegel and Moser [19] and study it in two cases: decoupled case and coupled case. In the *decoupled case*, we ignore the interaction between the two collision pairs. In other words, SBC in the decoupled case is considered as two separated binary collisions happening at the same time. An analytic argument helps us solve the differential system in the decoupled case. The solutions of SBC in the decoupled case are all analytic and they actually form a one-parameter set. Similar results also hold for the *coupled case*, which is SBC in the equal-mass collinear four-body problem.

Before explaining the main results in detail, we introduce some notations. Without loss of generality, we assume that four masses locate on the x-axis. we denote the coordinate of mass m_k by $q_k \in \mathbb{R}$, and the linear momentum by p_k respectively, where $p_k = m_k \dot{q_k}$ $(k = 1, 2, 3, 4)$. Then the standard Hamiltonian system is

$$\dot{q_k} = E_{p_k}, \quad \dot{p_k} = -E_{q_k}, \quad E = T - U \quad (k = 1, 2, 3, 4),$$

where

$$T = \frac{1}{2} \sum_{k=1}^{4} \frac{p_k^2}{m_k}, \quad U = \sum_{1 \le j < i \le 4} \frac{m_i m_j}{\mid q_i - q_j \mid}.$$

Let $t = 0$ be the time of SBC. Choose $\delta > 0$ to be small enough, such that $t = 0$ is the *only* singular point in $(-\delta, \delta)$. As in Figure 1, we assume the coordinates and momenta of four masses satisfy $q_1 < q_2 < q_3 < q_4$, $p_1 > 0$, $p_3 > 0$, $p_2 < 0$ and $p_4 < 0$ for $t \in (-\delta, 0)$.

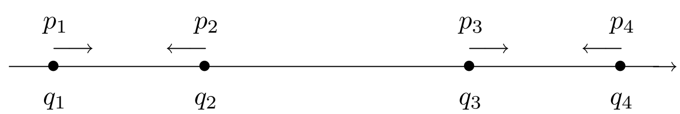

FIGURE 1. Mass configuration

The center of mass and the total linear momentum are set to be 0, i.e.

$$m_1 q_1 + m_2 q_2 + m_3 q_3 + m_4 q_4 = 0, \quad (m_1 = m_2 = m_3 = m_4 = 1),$$
$$m_1 \dot{q_1} + m_2 \dot{q_2} + m_3 \dot{q_3} + m_4 \dot{q_4} = 0, \quad \text{or } p_1 + p_2 + p_3 + p_4 = 0.$$

A standard canonical transformation involving mutual distances is introduced as follows:

$$x_1 = q_2 - q_1, \quad x_2 = q_4 - q_3, \quad x_3 = q_3 - q_2;$$
$$y_1 = -p_1, \quad y_2 = p_4, \quad y_3 = -p_1 - p_2.$$

The new time variable is defined as

$$s = \int_{\tau}^{t} (\frac{1}{x_1(t)} + \frac{1}{x_2(t)}) dt, \quad (\tau \le t < 0),$$

where $t = 0$ is the time of SBC, i.e. $x_1(0) = x_2(0) = 0$. Note that s is a regular function of t in the interval $\tau \le t < 0$ and

$$s_1 = \int_{\tau}^{0} (\frac{1}{x_1(t)} + \frac{1}{x_2(t)}) dt$$

is the time of SBC in the new coordinate. It has been proved that s_1 is finite [19]. without loss of generality, we can assume $s_1 = 0$.

A Levi-Civita type canonical transformation is defined as follows

$$\xi_1 = -x_1 y_1^2, \quad \xi_2 = -x_2 y_2^2, \quad \xi_3 = x_3,$$
$$\eta_1 = \frac{1}{y_1}, \quad \eta_2 = \frac{1}{y_2}, \quad \eta_3 = y_3.$$

The study of transformed differential system of SBC is accomplished in two cases: the decoupled case and the coupled case. The definitions of these two cases are as follows:

Definition 1.3. Assume $m_1 = m_2 = m_3 = m_4 = 1$. If $x_3 \equiv \infty$, $y_3 \equiv 0$, we say the system is the decoupled case; and we call SBC in the collinear four-body problem with equal masses the coupled case.

The following theorems are our main results.

Theorem 1.4. *Let $E = h$ be the total Hamiltonian energy of the decoupled system. In the decoupled case with total energy $h = 0$, the solutions $(\xi_1, \xi_2, \eta_1, \eta_2)$ of the transformed differential system of SBC are all analytic in a small neighborhood of $s = 0$ and they form a one-parameter set, where the parameter C satisfies*

$$C = \frac{\xi_1 - \xi_2}{\xi_1\eta_1^2 + \xi_2\eta_2^2} = \frac{x_1y_1^2 - x_2y_2^2 + x_2h}{x_1 + x_2} = \big(y_1^2 - \frac{1}{x_1}\big).$$

Note that the formula of C in Theorem 1.4 has a physical meaning. It is actually the total energy of the left collision pair (q_1, q_2) in Figure 1. It is clear that it is a first integral. without loss of generality, we may assume that $C > 0$. It is shown in Section 3.4 that $s = 0$ is a much weaker singular point of the transformed differential system in the decoupled case with total energy $E = h = 0$, which can be handled analytically. The differential system in the decoupled system with $E = h = 0$ eventually becomes the equations of N_1 and N_2:

$$N_1' = (1 - N_1^2)^2 \cdot \frac{N_2^2}{N_1^2 + N_2^2}, \tag{1.2}$$

$$N_2' = (1 + N_2^2)^2 \cdot \frac{N_1^2}{N_1^2 + N_2^2}, \tag{1.3}$$

with the initial condition

$$N_1(0) = N_2(0) = 0. \tag{1.4}$$

where

$$N_1(s) = C^{1/2}\eta_1\big(\frac{s}{C^{1/2}}\big), \quad N_2(s) = C^{1/2}\eta_2\big(\frac{s}{C^{1/2}}\big),$$

and C is an arbitrary positive constant. It is shown that the differential system (1.2)–(1.3), and (1.4) has a unique solution (N_1, N_2) in a small neighborhood of $s = 0$. Consequently, the unique solution (N_1, N_2) generates a one-parameter family of solutions (η_1, η_2) with C as the parameter. As a consequence, the following proposition holds.

Proposition 1.5. *For fixed total energy $E = h = 0$, the variables $(\xi_1, \xi_2, \eta_1, \eta_2)$ are on a 3-dimensional hypersurface. In a small neighborhood of 0 on the energy surface $E = h = 0$, the set of initial conditions leading to SBC is 2-dimensional. Actually, for any given small $(\eta_{10}, \eta_{20}) = (\epsilon_1, \epsilon_2)$ leading to SBC, there exists unique s_0, C_0, ξ_{10} and ξ_{20}, such that*

$$\tanh^{-1}(C_0^{1/2}\eta_1) + \tan^{-1}(C_0^{1/2}\eta_2) = \frac{C_0^{1/2}\eta_2}{1 + C_0\eta_2^2} + \frac{C_0^{1/2}\eta_1}{1 - C_0\eta_1^2} = C_0^{1/2}s, \tag{1.5}$$

$$\epsilon_1 = \eta_1(s_0, C_0), \quad \epsilon_2 = \eta_2(s_0, C_0),$$

$$\xi_{10} = \frac{1}{C_0\epsilon_1^2 - 1}, \quad \xi_{20} = \frac{-1}{C_0\epsilon_2^2 + 1}.$$

Here $(\eta_1(s_0, C_0), \eta_2(s_0, C_0))$ is the solution of (1.6) to (1.8):

$$\eta_1' = (-1 + C\eta_1^2)^2 \frac{\eta_2^2}{\eta_1^2 + \eta_2^2}, \tag{1.6}$$

$$\eta_2' = (1 + C\eta_2^2)^2 \frac{\eta_1^2}{\eta_1^2 + \eta_2^2}, \tag{1.7}$$

$$\eta_1(0) = \eta_2(0) = 0. \tag{1.8}$$

which is equivalent to solve the algebraic equations (1.5).

For the decoupled case with general total energy $E = h$, similar argument shows that:

Theorem 1.6. *In the decoupled case with total energy $E = h$, the solutions $(\xi_1, \xi_2, \eta_1, \eta_2)$ of the transformed differential system of SBC are all analytic in a small neighborhood of $s = 0$ and they form a one-parameter set, where the parameter D satisfies*

$$\begin{aligned} D &= \frac{\xi_1 - \xi_2}{\xi_1\eta_1^2 + \xi_2\eta_2^2} + \frac{(\xi_2\eta_2^2 - \xi_1\eta_1^2)h}{2(\xi_1\eta_1^2 + \xi_2\eta_2^2)} \\ &= C + \frac{(\xi_2\eta_2^2 - \xi_1\eta_1^2)h}{2(\xi_1\eta_1^2 + \xi_2\eta_2^2)} \\ &= \frac{x_1y_1^2 - x_2y_2^2 + x_2h}{x_1 + x_2} - \frac{h}{2} \\ &= C + \frac{h(x_2 - x_1)}{2(x_1 + x_2)}. \end{aligned}$$

Proposition 1.7. *For fixed total energy $E = h$, the variables $(\xi_1, \xi_2, \eta_1, \eta_2)$ are on a 3-dimensional hypersurface. In a small neighborhood of* 0 *on the energy surface $E = h$, the set of initial conditions leading to SBC is 2-dimensional. Actually, for any given small $(\eta_{10}, \eta_{20}) = (\epsilon_1, \epsilon_2)$ leading to SBC, there exists some s_0, D_0, ξ_{10} and ξ_{20}, such that*

$$\epsilon_1 = \eta_1(s_0, D_0), \quad \epsilon_2 = \eta_2(s_0, D_0),$$
$$\xi_{10} = \frac{1}{(D_0 + \frac{h}{2})\epsilon_1^2 - 1}, \quad \xi_{20} = \frac{1}{(-D_0 + \frac{h}{2})\epsilon_2^2 - 1},$$

where $(\eta_1(s_0, D_0), \eta_2(s_0, D_0))$ is the solution of the system

$$\eta_1' = [(D + \frac{1}{2}h)\eta_1^2 - 1]^2 \frac{\eta_2^2}{\eta_1^2 + \eta_2^2 - h\eta_1^2\eta_2^2}, \tag{1.9}$$

$$\eta_2' = [(-D + \frac{1}{2}h)\eta_2^2 - 1]^2 \frac{\eta_1^2}{\eta_1^2 + \eta_2^2 - h\eta_1^2\eta_2^2}. \tag{1.10}$$

with initial conditions

$$\eta_1(0) = \eta_2(0) = 0, \tag{1.11}$$

Remark 1.8. To solve the initial value problem (1.9) to (1.11), we can apply the separation of variables and integrate it. It is not hard to see that the solution satisfies algebraic identities like (1.5).

To understand the differential system in the coupled case, we need to study the connection between the coupled case and the decoupled case.

Theorem 1.9. *Let $E = h$ be the total Hamiltonian energy of the system in the coupled case. In the differential system* (5.3) *to* (5.9) *of the coupled case, there are infinitely many solutions $(\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3)$. All of the solutions are analytic in a* *small neighborhood of $s = 0$ and they form a one-parameter set, where D in (1.12) is the parameter.*

$$\begin{aligned} D &= \lim_{s\to 0}\big[\frac{\xi_1-\xi_2}{\xi_1\eta_1^2+\xi_2\eta_2^2}+\frac{(\xi_2\eta_2^2-\xi_1\eta_1^2)h}{2(\xi_1\eta_1^2+\xi_2\eta_2^2)}\big] \\ &= C+\lim_{s\to 0}\frac{(\xi_2\eta_2^2-\xi_1\eta_1^2)h}{2(\xi_1\eta_1^2+\xi_2\eta_2^2)} \\ &= \lim_{t\to 0}\frac{x_1y_1^2-x_2y_2^2+x_2h}{x_1+x_2}-\frac{h}{2}. \end{aligned} \tag{1.12}$$

Furthermore, for any given initial condition

$$\xi_1(0)=\xi_2(0)=-1,\quad \eta_1(0)=\eta_2(0)=0,\quad \xi_3(0)=\widehat{\xi}_3,\quad \eta_3(0)=\widehat{\eta}_3$$

at SBC and fixed total energy $E = h$, there is a one-to-one correspondence between solutions $(\xi_1, \xi_2, \eta_1, \eta_2)$ in the coupled system and the decoupled system.

In addition, we provide a variational way to understand the regularization of the collinear binary collision and collinear SBC in the decoupled case.

Theorem 1.10. *The collinear binary collision in the two-body problem and the collinear SBC in the decoupled case are regularizable.*

After introducing the time variable s, we can apply the variational argument to understand the regularization of binary collision in the collinear two-body problem and collinear SBC in the decoupled case. The proof of this theorem can be found in Sections 4.1 and 4.2.

Remark 1.11. Besides the regularization mentioned above, there is another type of regularization: block regularization, which is first introduced by Easton [2], and then by Martínez and Simó [10, 11] and ElBialy [3, 4, 5, 6, 7]. Very rich and interesting results can be found there and the references within [9, 12, 13, 14, 15, 16]. In this paper, we don't discuss this kind of regularization.

The paper is organized as follows. In Section 2, we simplify the Hamiltonian form and show that a Levi-Civita type canonical transformation is well-defined at SBC. In Section 3, The decoupled case is studied and all possible solutions of SBC in this case are found. In section 4, a variational argument is provided to understand the regularization of collinear binary collision in two-body problem and the decoupled of collinear SBC. In Section 5, the transformed system of the coupled case is analyzed. In the Appendix, power series solutions are calculated for each cases as a numerical evidence.

2. Preliminaries

2.1. **Simplified Hamiltonian form.** Let $t = 0$ be the time of SBC. As shown in Figure 1, we assume the positions of the four masses satisfy $q_4 > q_3 > q_2 > q_1$ for $t \in (-\delta, 0)$. We denote the linear momentum of mass m_i by $p_i = m_i\dot{q}_i$ $(i = 1, 2, 3, 4)$. Let the center of mass rest at the origin and the total linear momenta be 0. The Hamiltonian for this system is $E = T - U$, where

$$T=\frac{1}{2}\sum_{k=1}^{4}\frac{p_k^2}{m_k},\quad U=\sum_{1\le j<i\le 4}\frac{m_im_j}{\mid q_i-q_j\mid}.$$

In this section, we apply the standard first integrals: the center of mass and the total linear momenta to eliminate a pair of variables. The canonical transformation is defined as follows

$$\begin{gathered} x_1 = q_2 - q_1, \quad x_2 = q_4 - q_3, \quad x_3 = q_3 - q_2, \quad x_4 = q_4, \quad y_1 = -p_1, \\ y_2 = -p_1 - p_2 - p_3, \quad y_3 = -p_1 - p_2, \quad y_4 = p_1 + p_2 + p_3 + p_4. \end{gathered} \tag{2.1}$$

Note that the total linear momenta $p_1 + p_2 + p_3 + p_4 = 0$, then $y_2 = p_4$, $y_4 = 0$. Also the center of mass is assumed to be 0, thus

$$\begin{aligned} 0 &= m_1 q_1 + m_2 q_2 + m_3 q_3 + m_4 q_4 \\ &= m_1(x_4 - x_3 - x_2 - x_1) + m_2(x_4 - x_3 - x_2) + m_3(x_4 - x_2) + m_4 x_4. \end{aligned}$$

It follows that

$$x_4 = \frac{x_1 m_1 + x_3(m_1 + m_2) + x_2(m_1 + m_2 + m_3)}{m_1 + m_2 + m_3 + m_4}.$$

Under the canonical transformation (2.1), we only have to consider the following Hamiltonian system with 6 variables x_i, y_i $(i = 1, 2, 3)$:

$$\dot{x_k} = E_{y_k}, \quad \dot{y_k} = -E_{x_k}, \quad (k = 1, 2, 3) \tag{2.2}$$

where $E = T - U$ and

$$T = \frac{1}{2}\Big[\frac{y_1^2}{m_1} + \frac{(y_1 - y_3)^2}{m_2} + \frac{(y_3 - y_2)^2}{m_3} + \frac{y_2^2}{m_4}\Big], \tag{2.3}$$

$$U = \frac{m_1 m_2}{x_1} + \frac{m_1 m_3}{x_1 + x_3} + \frac{m_1 m_4}{x_1 + x_2 + x_3} + \frac{m_2 m_3}{x_3} + \frac{m_2 m_4}{x_2 + x_3} + \frac{m_3 m_4}{x_2}. \tag{2.4}$$

2.2. **Limits at SBC.** By our assumption, $q_1 < q_2 < q_3 < q_4$, $p_1 > 0$, $p_3 > 0$, $p_2 < 0$ and $p_4 < 0$ for any $t \in (-\delta, 0)$. It implies that

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0, \quad x_4 \ge 0, \quad y_1 < 0, \quad y_2 < 0, \quad \text{for any } t \in (-\delta, 0).$$

Since SBC happens at $t = 0$, it follows that $\lim_{t\to 0} x_1(t) = \lim_{t\to 0} x_2(t) = 0$. From the Newtonian equation (1.1), it is clear that

$$\lim_{t\to 0} x_1^2 \ddot{x}_1 = \lim_{t\to 0}(q_2 - q_1)^2(\ddot{q}_2 - \ddot{q}_1) = -(m_1 + m_2), \tag{2.5}$$

$$\lim_{t\to 0} x_2^2 \ddot{x}_2 = \lim_{t\to 0}(q_4 - q_3)^2(\ddot{q}_4 - \ddot{q}_3) = -(m_3 + m_4). \tag{2.6}$$

By identities (2.5) and (2.6), one can show the following result [1].

Lemma 2.1.

$$\lim_{t\to 0} \frac{x_1}{x_2} = \alpha, \quad \textit{where } \alpha = \Big(\frac{m_1 + m_2}{m_3 + m_4}\Big)^{1/3}, \tag{2.7}$$

and

$$\lim_{t\to 0}(q_2 - q_1)(\dot{q_2} - \dot{q_1})^2 = 2(m_1 + m_2), \tag{2.8}$$

$$\lim_{t\to 0}(q_4 - q_3)(\dot{q_4} - \dot{q_3})^2 = 2(m_3 + m_4). \tag{2.9}$$

Proof. By identity (2.5), it is clear that $x_1 = q_2 - q_1 = O(t^{2/3})$. It follows that

$$\lim_{t\to 0} \frac{(q_2 - q_1)(\dot{q_2} - \dot{q_1})^2}{(q_2 - q_1)^2(\ddot{q}_2 - \ddot{q}_1)} = -2.$$

Hence,

$$\lim_{t\to 0}(q_2 - q_1)(\dot{q_2} - \dot{q_1})^2 = 2(m_1 + m_2).$$

Similarly,

$$\lim_{t\to 0}(q_4-q_3)(\dot q_4-\dot q_3)^2 = 2(m_3+m_4).$$

The ratio of idetities (2.8) and (2.9) implies the limit (2.7). The proof is complete. □

To introduce a Levi-Civita type canonical transformation, we first need to show that $x_1y_1^2$ and $x_2y_2^2$ are well-defined at SBC. Similar results can be found in [17].

Lemma 2.2. $\lim_{t\to 0} x_1y_1^2$ *and* $\lim_{t\to 0} x_2y_2^2$ *exist, and*

$$\lim_{t\to 0} x_1y_1^2 = \lim_{t\to 0} x_1p_1^2 = \frac{2(m_1m_2)^2}{m_1+m_2},$$
$$\lim_{t\to 0} x_2y_2^2 = \lim_{t\to 0} x_2p_4^2 = \frac{2(m_3m_4)^2}{m_3+m_4}.$$

Proof. First, we show that both $x_1y_1^2$ and $x_2y_2^2$ are bounded when t approaches 0. By equation (2.4),

$$x_1U = m_1m_2 + x_1\frac{m_1m_3}{x_1+x_3} + x_1\frac{m_1m_4}{x_1+x_2+x_3} + x_1\frac{m_2m_3}{x_3} + x_1\frac{m_2m_4}{x_2+x_3} + x_1\frac{m_3m_4}{x_2}.$$

Note that $\lim_{t\to 0} x_1(t) = \lim_{t\to 0} x_2(t) = 0$ and $\lim_{t\to 0} x_3(t) > 0$. Then by Lemma 2.1,

$$\lim_{t\to 0} x_1U = \lim_{t\to 0}[m_1m_2 + x_1\frac{m_3m_4}{x_2}] = m_1m_2 + \alpha m_3m_4.$$

Let $E = T - U = h$ be the Hamiltonian constant. It follows that

$$\lim_{t\to 0} x_1T = \lim_{t\to 0} x_1(U+h) = m_1m_2 + \alpha m_3m_4.$$

By equation (2.3),

$$\lim_{t\to 0} x_1T = \lim_{t\to 0}\frac{1}{2}x_1[\frac{y_1^2}{m_1} + \frac{(y_1-y_3)^2}{m_2} + \frac{(y_3-y_2)^2}{m_3} + \frac{y_2^2}{m_4}] = m_1m_2+\alpha m_3m_4. \tag{2.10}$$

In particular,

$$0 \le x_1y_1^2 \le 2m_1(m_1m_2+\alpha m_3m_4) \quad\text{and}\quad 0 \le x_1y_2^2 \le 2m_4(m_1m_2+\alpha m_3m_4).$$

Therefore, both $x_1y_1^2$ and $x_2y_2^2 = \frac{x_2}{x_1}x_1y_2^2$ are bounded at SBC.

Next, we apply the boundedness of $x_1y_1^2$ and $x_2y_2^2$ and Lemma 2.1 to show the existence of their limits. Note that $y_1 = -p_1$, $y_2 = p_4$, $y_3 = -p_1-p_2$, then $x_1y_1^2 = x_1p_1^2$ and $x_2y_2^2 = x_2p_4^2$. By (2.8), (2.9), (2.10):

$$\lim_{t\to 0} x_1(\frac{p_1}{m_1} - \frac{p_2}{m_2})^2 = 2(m_1+m_2), \tag{2.11}$$
$$\lim_{t\to 0} x_2(\frac{p_3}{m_3} - \frac{p_4}{m_4})^2 = 2(m_3+m_4), \tag{2.12}$$
$$p_1+p_2+p_3+p_4 = 0, \tag{2.13}$$
$$\lim_{t\to 0} x_1[\frac{p_1^2}{m_1} + \frac{p_2^2}{m_2} + \frac{p_3^2}{m_3} + \frac{p_4^2}{m_4}] = 2(m_1m_2+\alpha m_3m_4). \tag{2.14}$$

By (2.14), we know that $x_1p_1^2$, $x_1p_2^2$, $x_1p_3^2$, $x_1p_4^2$ are all bounded when t approaches 0.

Consider the equation for y_3:

$$-\dot y_3 = E_{x_3} = \frac{m_1m_3}{(x_1+x_3)^2} + \frac{m_1m_4}{(x_1+x_2+x_3)^2} + \frac{m_2m_3}{(x_3)^2} + \frac{m_2m_4}{(x_2+x_3)^2}. \tag{2.15}$$

Since $x_3 = q_3 - q_2$ is strictly positive for $-\delta < t \le 0$, there exists a positive constant B, such that $x_3 > B > 0$. Integrating the above identity (2.15) from $-\delta$ to 0 implies that

$$0 \le -y_3(0) + y_3(-\delta) \le \frac{1}{B^2}\delta \cdot (m_1m_3 + m_1m_4 + m_2m_3 + m_2m_4).$$

It follows that $p_1(0) + p_2(0) = -y_3(0)$ is bounded. Hence,

$$\lim_{t\to 0} x_1(p_1+p_2)^2 = 0, \quad \lim_{t\to 0} x_1p_1(p_1+p_2) = \lim_{t\to 0} \frac{1}{p_1} x_1p_1^2(p_1+p_2) = 0.$$

By (2.11),

$$\begin{aligned}
&2(m_1+m_2)\\
&= \lim_{t\to 0} x_1\big(\frac{p_1}{m_1} - \frac{p_2}{m_2}\big)^2 = \lim_{t\to 0} x_1\big(\frac{p_1}{m_1} + \frac{p_1}{m_2} - \frac{p_1}{m_2} - \frac{p_2}{m_2}\big)^2\\
&= \lim_{t\to 0} x_1p_1^2\big(\frac{1}{m_1} + \frac{1}{m_2}\big)^2 + \frac{1}{m_2^2}\lim_{t\to 0} x_1(p_1+p_2)^2 - \frac{2(m_1+m_2)}{m_1m_2^2}\lim_{t\to 0} x_1p_1(p_1+p_2)\\
&= \lim_{t\to 0} x_1p_1^2\big(\frac{1}{m_1} + \frac{1}{m_2}\big)^2.
\end{aligned}$$

Therefore,

$$\lim_{t\to 0} x_1y_1^2 = \lim_{t\to 0} x_1p_1^2 = \frac{2(m_1m_2)^2}{m_1+m_2}.$$

Similarly, by considering equation (2.12), one can show that

$$\lim_{t\to 0} x_2y_2^2 = \lim_{t\to 0} x_2p_4^2 = \frac{2(m_3m_4)^2}{m_3+m_4}.$$

□

2.3. **New Hamiltonian F.** The transformation in this subsection is inspired by the book of Siegel and Moser [19]. To consider equations (1.1) at SBC, we introduce a new independent time variable

$$s = \int_\tau^t \big(\frac{m_1m_2}{x_1} + \frac{m_3m_4}{x_2}\big)dt, \quad (\tau \le t < 0),$$

where $t = 0$ is the time of SBC. Let $s = s_1$ be the corresponding collision time in the new time variable. Siegel and Moser [19] showed that $\int_\tau^0 U dt$ is finite, so $s_1 = \int_\tau^0 (\frac{m_1m_2}{x_1} + \frac{m_3m_4}{x_2})dt$ is finite. Without loss of generality, we can assume $s_1 = 0$.

Denote dx_k/ds by x_k' and dy_k/ds by y_k'. Then the Hamiltonian system (2.2) becomes

$$x_k' = \frac{1}{\frac{m_1m_2}{x_1} + \frac{m_3m_4}{x_2}} E_{y_k}, \quad y_k' = -\frac{1}{\frac{m_1m_2}{x_1} + \frac{m_3m_4}{x_2}} E_{x_k}, \quad (k = 1, 2, 3), \tag{2.16}$$

where $E = T - U$, and

$$T = \frac{1}{2}\Big[\frac{y_1^2}{m_1} + \frac{(y_1-y_3)^2}{m_2} + \frac{(y_3-y_2)^2}{m_3} + \frac{y_2^2}{m_4}\Big], \tag{2.17}$$

$$U = \frac{m_1m_2}{x_1} + \frac{m_1m_3}{x_1+x_3} + \frac{m_1m_4}{x_1+x_2+x_3} + \frac{m_2m_3}{x_3} + \frac{m_2m_4}{x_2+x_3} + \frac{m_3m_4}{x_2}. \tag{2.18}$$

To restore (2.16) to a Hamiltonian form, we apply a device introduced by Poincaré [19]. Let

$$F = \frac{1}{\frac{m_1m_2}{x_1} + \frac{m_3m_4}{x_2}}(E-h) = \frac{1}{\frac{m_1m_2}{x_1} + \frac{m_3m_4}{x_2}}(T-U-h),$$

where $E = T - U = h$. For the solution of Hamiltonian system (2.2) on the energy surface $E = h$, we have

$$F_{x_k} = \frac{1}{\frac{m_1m_2}{x_1} + \frac{m_3m_4}{x_2}}E_{x_k}, \quad F_{y_k} = \frac{1}{\frac{m_1m_2}{x_1} + \frac{m_3m_4}{x_2}}E_{y_k}.$$

Consequently, (2.16) can be written as

$$x_k' = F_{y_k}, \quad y_k' = -F_{x_k}, \quad (k = 1, 2, 3), \tag{2.19}$$

where

$$F = \frac{1}{\frac{m_1m_2}{x_1} + \frac{m_3m_4}{x_2}}(E-h).$$

The following result is due to Siegel and Moser [19].

Lemma 2.3. *If* (x_k, y_k) $(k = 1, 2, 3)$ *is a solution of the differential system* (2.16) *on the energy surface* $E = h$*, then it is also a solution of the Hamiltonian system* (2.19) *on the energy surface* $F = 0$*, where*

$$F = \frac{1}{\frac{m_1m_2}{x_1} + \frac{m_3m_4}{x_2}}(E-h).$$

Similarly, if (x_k, y_k) $(k = 1, 2, 3)$ *is a solution of the Hamiltonian system* F *on the energy surface* $F = 0$ *and* $x_k \neq 0$ $(k = 1, 2)$*, it is also a solution of the differential system* (2.16) *on the energy surface* $E = h$*.*

Proof. If (x_k, y_k) $(k = 1, 2, 3)$ is a solution of the differential system (2.16) on the energy surface $E = h$, then the function

$$F = \frac{1}{\frac{m_1m_2}{x_1} + \frac{m_3m_4}{x_2}}(E-h) = \frac{1}{\frac{m_1m_2}{x_1} + \frac{m_3m_4}{x_2}}(T-U-h)$$

of x_k, y_k $(k = 1, 2, 3)$ satisfies

$$F_{x_k} = \frac{1}{\frac{m_1m_2}{x_1} + \frac{m_3m_4}{x_2}}E_{x_k}, \quad F_{y_k} = \frac{1}{\frac{m_1m_2}{x_1} + \frac{m_3m_4}{x_2}}E_{y_k}.$$

It follows that (2.16) can be expressed as the Hamiltonian system

$$x_k' = F_{y_k}, \quad y_k' = -F_{x_k}, \quad (k = 1, 2, 3),$$

which is indeed satisfied by all solutions of the original equations of motion with energy $E = h$, hence $F = 0$.

Conversely, if $F = 0$ and $x_k \neq 0$ $(k = 1, 2)$, then (2.16) follows from (2.19). $\square$

Remark 2.4. To study the solutions of (2.2) with Hamiltonian $E = h$, it is equivalent to find the solutions of the Hamiltonian $F = \frac{1}{\frac{m_1m_2}{x_1} + \frac{m_3m_4}{x_2}}(E-h)$ on the energy surface $F = 0$.

3. Decoupled case with total energy $E = h = 0$

Let $m_1 = m_2 = m_3 = m_4 = 1$. In this section, we study a simple decoupled case, in which we assume $x_3 = \infty$, $y_3 = 0$, $h = 0$. Under these assumptions, the formulas (2.17) (2.18) become $T = y_1^2 + y_2^2$, $U = \frac{1}{x_1} + \frac{1}{x_2}$, and

$$F = \frac{y_1^2 + y_2^2}{\frac{1}{x_1} + \frac{1}{x_2}} - 1. \tag{3.1}$$

And the Hamiltonian system becomes

$$x_k' = F_{y_k}, \quad y_k' = -F_{x_k}, \quad (k = 1, 2) \tag{3.2}$$

with Hamiltonian

$$F = \frac{y_1^2 + y_2^2}{\frac{1}{x_1} + \frac{1}{x_2}} - 1 = 0. \tag{3.3}$$

3.1. **Equations for x_k and y_k $(k = 1, 2)$.** We write the Hamiltonian system (3.2) in explicit forms:

$$x_1' = \frac{2y_1}{\frac{1}{x_1} + \frac{1}{x_2}} = \frac{2y_1 x_1 x_2}{x_1 + x_2}, \tag{3.4}$$

$$y_1' = -\frac{y_1^2 + y_2^2}{(\frac{1}{x_1} + \frac{1}{x_2})^2} \frac{1}{x_1^2} = -\frac{x_2}{x_1(x_1 + x_2)}, \tag{3.5}$$

$$x_2' = \frac{2y_2}{\frac{1}{x_1} + \frac{1}{x_2}} = \frac{2y_2 x_1 x_2}{x_1 + x_2}, \tag{3.6}$$

$$y_2' = -\frac{y_1^2 + y_2^2}{(\frac{1}{x_1} + \frac{1}{x_2})^2} \frac{1}{x_2^2} = -\frac{x_1}{x_2(x_1 + x_2)}. \tag{3.7}$$

Lemma 3.1. *If $\{x_1, x_2, y_1, y_2\}$ is a solution of the Hamiltonian system* (3.2) *on the energy surface $F = 0$, then there exists a constant C such that*

$$y_1^2 = \frac{1}{x_1} + C, \quad y_2^2 = \frac{1}{x_2} - C, \quad C = \frac{x_1 y_1^2 - x_2 y_2^2}{x_1 + x_2}.$$

Proof. From equations (3.4) and (3.5),

$$\frac{dy_1}{dx_1} = \frac{y_1'}{x_1'} = -\frac{1}{2y_1 x_1^2}.$$

We separate the variables and integrate both sides:

$$\int 2y_1 dy_1 = \int \frac{-1}{x_1^2} dx_1.$$

Then

$$y_1^2 = \frac{1}{x_1} + C, \quad \text{i.e. } x_1 y_1^2 - C x_1 = 1, \tag{3.8}$$

where C is a constant, which only depends on the initial conditions. Similarly, we have

$$y_2^2 = \frac{1}{x_2} + \tilde{C}, \quad \text{i.e. } x_2 y_2^2 - \tilde{C} x_2 = 1, \tag{3.9}$$

where $\tilde{C}$ is another constant, which depends on the initial conditions.

Adding up equations (3.8) and (3.9) and applying the formula of F (3.3) in the Hamiltonian system (3.2), we obtain

$$y_1^2 + y_2^2 = \frac{1}{x_1} + \frac{1}{x_2} + C + \tilde{C} = y_1^2 + y_2^2 + C + \tilde{C}.$$

It follows that $\tilde{C} = -C$.

Then equation (3.9) becomes

$$y_2^2 = \frac{1}{x_2} - C, \quad \text{or} \quad x_2 y_2^2 + C x_2 = 1. \tag{3.10}$$

The difference of equation (3.8) and equation (3.10) implies

$$x_1 y_1^2 - x_2 y_2^2 - C(x_1 + x_2) = 0 \quad \text{thus} \quad C = \frac{x_1 y_1^2 - x_2 y_2^2}{x_1 + x_2}.$$

□

Remark 3.2. By Lemma 3.1, the first integral $C = y_1^2 - \frac{1}{x_1}$ is actually the total energy of the collision pair: m_1 and m_2.

Although we assume $m_1 = m_2 = m_3 = m_4 = 1$, the system is not necessary to be symmetric since the positions and velocities of the four masses are arbitrary.

3.2. **Levi-Civita type canonical transformation.** Let $s = 0$ be the time of SBC. In this subsection, we define a new Levi-Civita type canonical transformation, which is a generalization of Siegel and Moser 's work [19]. The transformation is defined as follows:

$$\xi_1 = -x_1 y_1^2, \quad \xi_2 = -x_2 y_2^2, \quad \eta_1 = \frac{1}{y_1}, \quad \eta_2 = \frac{1}{y_2}.$$

Then x_i, y_i can be solved in terms of ξ_i and η_i $(i = 1, 2)$:

$$x_1 = -\xi_1 \eta_1^2, \quad x_2 = -\xi_2 \eta_2^2, \quad y_1 = \frac{1}{\eta_1}, \quad y_2 = \frac{1}{\eta_2}.$$

Then the new Hamiltonian system becomes

$$\xi_k' = F_{\eta_k}, \quad \eta_k' = -F_{\xi_k}, \quad (k = 1, 2) \tag{3.11}$$

with

$$F = -\frac{\xi_1 \xi_2 (\eta_1^2 + \eta_2^2)}{\xi_1 \eta_1^2 + \xi_2 \eta_2^2} - 1. \tag{3.12}$$

By Lemma 2.3, we only need to consider the solution on the energy surface $F = 0$.

3.3. **First integral C in the transformed Hamiltonian system.** The following equations are the explicit forms of the Hamiltonian system (3.11):

$$\xi_1' = \frac{2\xi_1 \xi_2 \eta_1 \eta_2^2 (\xi_1 - \xi_2)}{(\xi_1 \eta_1^2 + \xi_2 \eta_2^2)^2}, \tag{3.13}$$

$$\xi_2' = -\frac{2\xi_1 \xi_2 \eta_2 \eta_1^2 (\xi_1 - \xi_2)}{(\xi_1 \eta_1^2 + \xi_2 \eta_2^2)^2}, \tag{3.14}$$

$$\eta_1' = \frac{(\eta_1^2 + \eta_2^2) \xi_2^2 \eta_2^2}{(\xi_1 \eta_1^2 + \xi_2 \eta_2^2)^2}, \tag{3.15}$$

$$\eta_2' = \frac{(\eta_1^2 + \eta_2^2) \xi_1^2 \eta_1^2}{(\xi_1 \eta_1^2 + \xi_2 \eta_2^2)^2}. \tag{3.16}$$

Actually, the Hamiltonian constant $F = -\frac{\xi_1\xi_2(\eta_1^2+\eta_2^2)}{\xi_1\eta_1^2+\xi_2\eta_2^2} - 1 = 0$ implies that

$$\xi_1\eta_1^2 + \xi_2\eta_2^2 = -\xi_1\xi_2(\eta_1^2+\eta_2^2).$$

Hence the differential equations of η_1 and η_2 can be simplified to:

$$\eta_1' = \frac{(\eta_1^2+\eta_2^2)\xi_2^2\eta_2^2}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2} = \frac{1}{\xi_1^2}\cdot\frac{\eta_2^2}{\eta_1^2+\eta_2^2}, \tag{3.17}$$

$$\eta_2' = \frac{(\eta_1^2+\eta_2^2)\xi_1^2\eta_1^2}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2} = \frac{1}{\xi_2^2}\cdot\frac{\eta_1^2}{\eta_1^2+\eta_2^2} \tag{3.18}$$

By Lemma 2.2, the initial condition at SBC is

$$\xi_1(0) = \xi_2(0) = -1, \quad \eta_1(0) = \eta_2(0) = 0. \tag{3.19}$$

Similar to Lemma 3.1, we define $f(s)$ to be the formula:

$$f(s) = \frac{\xi_1-\xi_2}{\xi_1\eta_1^2+\xi_2\eta_2^2}.$$

It is clear that $f(s)$ is not defined at $s=0$. We want to show that $f(s)$ is a first integral of the Hamiltonian system (3.11) for any $s \neq 0$.

Lemma 3.3. *If $\{\xi_1,\xi_2,\eta_1,\eta_2\}$ is a solution of the Hamiltonian system* (3.11) *with the initial condition* (3.19)*, then $\frac{df}{ds} = 0$ for any $s \neq 0$.*

Proof. Note that

$$\begin{aligned}
&\frac{df}{ds}\\
&= \frac{\partial f}{\partial \xi_1}\cdot\xi_1' + \frac{\partial f}{\partial \xi_2}\cdot\xi_2' + \frac{\partial f}{\partial \eta_1}\cdot\eta_1' + \frac{\partial f}{\partial \eta_2}\cdot\eta_2'\\
&= \frac{\xi_2(\eta_1^2+\eta_2^2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}\cdot\frac{2\xi_1\xi_2\eta_1\eta_2^2(\xi_1-\xi_2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2} + \frac{-\xi_1(\eta_1^2+\eta_2^2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}\cdot\frac{-2\xi_1\xi_2\eta_2\eta_1^2(\xi_1-\xi_2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}\\
&\quad + \frac{-2\xi_1\eta_1(\xi_1-\xi_2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}\cdot\frac{\xi_2^2\eta_2^2(\eta_1^2+\eta_2^2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2} + \frac{-2\xi_2\eta_2(\xi_1-\xi_2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}\cdot\frac{\xi_1^2\eta_1^2(\eta_1^2+\eta_2^2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}\\
&= 0.
\end{aligned}$$

□

We denote the first integral $f(s)$ by C, so

$$C = \frac{\xi_1-\xi_2}{\xi_1\eta_1^2+\xi_2\eta_2^2}.$$

Lemma 3.4. *By the two first integral F and C:*

$$0 = F = -\frac{\xi_1\xi_2(\eta_1^2+\eta_2^2)}{\xi_1\eta_1^2+\xi_2\eta_2^2} - 1, \tag{3.20}$$

$$C = \frac{\xi_1-\xi_2}{\xi_1\eta_1^2+\xi_2\eta_2^2}, \tag{3.21}$$

ξ_i can be solved in terms of η_i $(i=1,2)$:

$$\xi_1 = \frac{1}{-1+C\eta_1^2}, \quad \xi_2 = \frac{1}{-1-C\eta_2^2}. \tag{3.22}$$

Proof. From identity (3.20),

$$-\xi_1\xi_2(\eta_1^2+\eta_2^2)=\xi_1\eta_1^2+\xi_2\eta_2^2. \tag{3.23}$$

Dividing both sides by $\xi_1\xi_2$, we have

$$-(\eta_1^2+\eta_2^2)=\frac{\eta_1^2}{\xi_2}+\frac{\eta_2^2}{\xi_1}. \tag{3.24}$$

From identities (3.21) and (3.23), it follows that

$$\xi_1-\xi_2=C(\xi_1\eta_1^2+\xi_2\eta_2^2)=-C\xi_1\xi_2(\eta_1^2+\eta_2^2).$$

Dividing both sides by $(-\xi_1\xi_2)$ implies that

$$\frac{1}{\xi_1}-\frac{1}{\xi_2}=C(\eta_1^2+\eta_2^2). \tag{3.25}$$

Then we can solve for $\frac{1}{\xi_1}$ and $\frac{1}{\xi_2}$ from equations (3.24) and (3.25):

$$\frac{1}{\xi_1}=-1+C\eta_1^2,\quad \frac{1}{\xi_2}=-1-C\eta_2^2.$$

Then (3.22) follows. □

Remark 3.5. Note that $\xi_1=1/(-1+C\eta_1^2)$ in Lemma 3.4, then C can be solved in terms of ξ_1 and η_1:

$$C=\frac{1+\xi_1}{\xi_1\eta_1^2}.$$

Since $x_1=-\xi_1\eta_1^2$ and $\xi_1=-x_1y_1^2$, it follows that

$$C=\frac{1+\xi_1}{\xi_1\eta_1^2}=\frac{1-x_1y_1^2}{-x_1}=y_1^2-\frac{1}{x_1}.$$

Therefore, the first integral C in the differential system (3.13)–(3.18) is the total energy of the left collision pair: m_1 and m_2, which is the same constant as in Lemma 3.1.

Since ξ_1 and ξ_2 can be represented by η_1 and η_2, we only need to solve the equations of η_1 and η_2:

$$\eta_1'=(-1+C\eta_1^2)^2\cdot\frac{\eta_2^2}{\eta_1^2+\eta_2^2}, \tag{3.26}$$

$$\eta_2'=(-1-C\eta_2^2)^2\cdot\frac{\eta_1^2}{\eta_1^2+\eta_2^2}, \tag{3.27}$$

with initial condition $\eta_1(0)=\eta_2(0)=0$. By considering differential equations (3.13) to (3.18), we find an identity between η_1 and η_2.

Lemma 3.6. *η_1 and η_2 satisfy*

$$\frac{\eta_1}{1-C\eta_1^2}+\frac{\eta_2}{1+C\eta_2^2}=s. \tag{3.28}$$

Proof. From the differential equations (3.17) and (3.18), we have

$$\eta_1'\xi_1+\eta_2'\xi_2=\frac{1}{\xi_1}\frac{\eta_2^2}{\eta_1^2+\eta_2^2}+\frac{1}{\xi_2}\frac{\eta_1^2}{\eta_1^2+\eta_2^2}=\frac{\xi_1\eta_1^2+\xi_2\eta_2^2}{\xi_1\xi_2(\eta_1^2+\eta_2^2)}.$$

By the first integral of F (3.20),

$$\frac{\xi_1\eta_1^2+\xi_2\eta_2^2}{\xi_1\xi_2(\eta_1^2+\eta_2^2)}=-1.$$

Then $\eta_1'\xi_1+\eta_2'\xi_2=-1$.

Similarly, by differential equations (3.13) and (3.14),

$$\eta_1\xi_1'+\eta_2\xi_2'=\frac{2\xi_1\xi_2\eta_1^2\eta_2^2(\xi_1-\xi_2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}-\frac{2\xi_1\xi_2\eta_1^2\eta_2^2(\xi_1-\xi_2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}=0.$$

Therefore,

$$(\eta_1\xi_1+\eta_2\xi_2)'=\eta_1'\xi_1+\eta_2'\xi_2+\eta_1\xi_1'+\eta_2\xi_2'=-1+0=-1.$$

Since $\xi_1(0)=\xi_2(0)=-1$ and $\eta_1(0)=\eta_2(0)=0$, it follows that

$$\eta_1\xi_1+\eta_2\xi_2=-s.$$

Applying formula (3.22) of ξ_1 and ξ_2 in Lemma 3.4, we have (3.28). □

When $C=0$, we have $\xi_1=\xi_2=-1$ and $\eta_1=\eta_2=s/2$. Consequently, $x_1=x_2$ and $y_1=y_2$, which is the symmetric case. The two collision pairs have exactly the same motion.

When $C<0$, the equations (3.26) and (3.27) can be written as

$$\eta_1'=(-1-|C|\eta_1^2)^2\cdot\frac{\eta_2^2}{\eta_1^2+\eta_2^2},$$
$$\eta_2'=(-1+|C|\eta_2^2)^2\cdot\frac{\eta_1^2}{\eta_1^2+\eta_2^2}.$$

Note that the solutions $\{\eta_1,\eta_2\}$ of the above two equations are exactly the solutions $\{\eta_2,\eta_1\}$ of (3.26) and (3.27) with $C_1=\mid C\mid$. Without loss of generality, we assume $C>0$ for the rest of this article.

Lemma 3.7. *Let $\{\eta_1,\eta_2\}$ be the solution of equations* (3.26) *and* (3.27) *with initial condition $\eta_1(0)=\eta_2(0)=0$. Define $N_1(s)=C^{1/2}\eta_1(\frac{s}{C^{1/2}})$ and $N_2(s)=C^{1/2}\eta_2(\frac{s}{C^{1/2}})$. Then*

$$\tanh^{-1}(N_1)+\tan^{-1}(N_2)=s, \tag{3.29}$$
$$\frac{N_1}{1-N_1^2}+\frac{N_2}{1+N_2^2}=s. \tag{3.30}$$

Proof. Consider the ratio between the two equations (3.26) and (3.27):

$$\frac{\eta_1'}{\eta_2'}=\frac{(-1+C\eta_1^2)^2}{(-1-C\eta_2^2)^2}\cdot\frac{\eta_2^2}{\eta_1^2}.$$

Separate the variables and integrate both sides:

$$\frac{\eta_1^2}{(-1+C\eta_1^2)^2}d\eta_1=\frac{\eta_2^2}{(1+C\eta_2^2)^2}d\eta_2,$$
$$-\frac{1}{2C}\frac{\eta_1}{-1+C\eta_1^2}-\frac{1}{2}\frac{\tanh^{-1}(C^{1/2}\eta_1)}{C^{\frac{3}{2}}}=-\frac{1}{2C}\frac{\eta_2}{1+C\eta_2^2}+\frac{1}{2}\frac{\tan^{-1}(C^{1/2}\eta_2)}{C^{\frac{3}{2}}}+\overline{C}, \tag{3.31}$$

where $\overline{C}$ is a constant. By the initial condition $\eta_1(0) = \eta_2(0) = 0$, $\overline{C} = 0$. Simplifying the identity (3.31),

$$\frac{C^{1/2}\eta_1}{-1 + C\eta_1^2} + \tanh^{-1}(C^{1/2}\eta_1) = \frac{C^{1/2}\eta_2}{1 + C\eta_2^2} - \tan^{-1}(C^{1/2}\eta_2). \tag{3.32}$$

Note that from (3.28) in Lemma 3.6, it follows that

$$\tanh^{-1}(C^{1/2}\eta_1) + \tan^{-1}(C^{1/2}\eta_2) = \frac{C^{1/2}\eta_2}{1 + C\eta_2^2} + \frac{C^{1/2}\eta_1}{1 - C\eta_1^2} = C^{1/2}s. \tag{3.33}$$

Let $N_1(s) = C^{1/2}\eta_1(\frac{s}{C^{1/2}})$, $N_2(s) = C^{1/2}\eta_2(\frac{s}{C^{1/2}})$. Then (3.33) becomes (3.29), (3.30). □

In fact, the differential equations of η_1 and η_2 become the equations of N_1 and N_2:

$$N_1' = (1 - N_1^2)^2 \cdot \frac{N_2^2}{N_1^2 + N_2^2}, \tag{3.34}$$

$$N_2' = (1 + N_2^2)^2 \cdot \frac{N_1^2}{N_1^2 + N_2^2}, \tag{3.35}$$

with the initial condition

$$N_1(0) = N_2(0) = 0. \tag{3.36}$$

3.4. **Existence and uniqueness of the solution $\{N_1(s), N_2(s)\}$.** In this section, we show the existence and uniqueness of the above initial-value problem (3.34) and (3.35) with initial condition (3.36) in a small neighborhood of 0.

Theorem 3.8. *The differential system* (3.34) *and* (3.35) *with initial condition* (3.36) *has a unique solution $(N_1(s), N_2(s))$ which is analytic for small enough s.*

By Lemma 3.7, the following identity holds for solution $(N_1(s), N_2(s))$:

$$\tanh^{-1}(N_1) + \tan^{-1}(N_2) = \frac{N_1}{1 - N_1^2} + \frac{N_2}{1 + N_2^2},$$

or

$$-\tanh^{-1}(N_1) - \tan^{-1}(N_2) + \frac{N_1}{1 - N_1^2} + \frac{N_2}{1 + N_2^2} = 0. \tag{3.37}$$

To prove Theorem 3.8, we first introduce an extended implicit function theorem and apply it to show that N_1 is analytic with respect to N_2 is a small neighborhood of 0.

Proposition 3.9 (Extended implicit function theorem)**.** *Assume $(N_1(s), N_2(s))$ satisfies* (3.37) *and the initial condition* (3.36)*. Then there exist intervals $I = (-\delta_1, \delta_1)$ and $J = (-\delta_2, \delta_2)$ and a unique function g, such that*

$$g : J \to I, \quad N_2 \mapsto N_1 = g(N_2).$$

Proof. Let

$$G(N_1, N_2) = -\tanh^{-1}(N_1) - \tan^{-1}(N_2) + \frac{N_1}{1 - N_1^2} + \frac{N_2}{1 + N_2^2}.$$

By assumption, the identity (3.37) holds, i.e. $G(N_1, N_2) = 0$. Denote $\frac{\partial G}{\partial N_1}$ by G'_{N_1}, the second partial derivative $\frac{\partial^2 G}{\partial N_1^2}$ by G''_{N_1}, and the third partial derivative $\frac{\partial^3 G}{\partial N_1^3}$ by

G'''_{N_1}. Differentiating $G(N_1, N_2)$ with respect to N_1 three times, we can find some properties about the partial derivatives of $G(N_1, N_2)$:

$$G(0,0) = 0, \quad G'_{N_1}(0, N_2) = 0, \quad G''_{N_1}(0, N_2) = 0, \quad G'''_{N_1}(0, N_2) = 4 \neq 0.$$

Because $G'''_{N_1}(0, N_2) = 4 > 0$ and G'''_{N_1} is continuous, there exists a rectangular area R:

$$R = \{(N_1, N_2) : |\, N_1 \,| < \delta_1, \quad |N_2| < \delta_2'\},$$

such that

$$m = \min_{(N_1,N_2)\in R} G'''_{N_1}(N_1, N_2) > 1 > 0.$$

Since in the rectangular area R, $G''_{N_1}(0, N_2) = 0$, $G'''_{N_1}(0, N_2) > 0$ and $G''_{N_1}(N_1, N_2)$ is continuous and strictly increasing with respect to N_1,

$$G''_{N_1}(N_1, N_2) > 0 \quad \text{for } 0 < N_1 < \delta_1,$$
$$G''_{N_1}(N_1, N_2) < 0 \quad \text{for } -\delta_1 < N_1 < 0.$$

It follows that $G'_{N_1}(N_1, N_2)$ is strictly increasing with respect to N_1 when $0 < N_1 < \delta_1$ and $G'_{N_1}(N_1, N_2)$ is strictly decreasing with respect to N_1 when $N_1 \in (-\delta_1, 0)$. Note that $G'_{N_1}(0, N_2) = 0$,

$$G'_{N_1}(N_1, N_2) > 0 \quad \text{for } 0 < N_1 < \delta_1,$$
$$G'_{N_1}(N_1, N_2) > 0 \quad \text{for } -\delta_1 < N_1 < 0.$$

That is,

$$G'_{N_1}(N_1, N_2) > 0 \quad \text{for } -\delta_1 < N_1 < \delta_1,\ N_1 \neq 0.$$

Also note that $G(0,0) = 0$, then

$$G(-\delta_1, 0) < 0, \quad G(\delta_1, 0) > 0.$$

By continuity of $G(N_1, N_2)$, there exists $0 < \delta_2 < \delta_2'$, such that when $|N_2| < \delta_2$,

$$G(-\delta_1, N_2) < 0, \quad G(\delta_1, N_2) > 0.$$

Consider the intervals $I = (-\delta_1, \delta_1)$ and $J = (-\delta_2, \delta_2)$. For any N_2 in J, the function $G(N_1, N_2)$ is strictly increasing with respect to N_1 in I, then by the intermediate value theorem for continuous function, there exists exactly one $N_1 \in I$ such that $G(N_1, N_2) = 0$. By the definition of function, there exist a unique function g such that

$$g : J \to I, \quad N_2 \mapsto N_1 = g(N_2).$$

Hence, so far we have proved the existence and uniqueness of N_1 as a function of N_2 which satisfy $G(N_1, N_2) = 0$. $\square$

It is necessary to point out that this extended implicit function theorem works for any function G which satisfies the identity

$$G(0,0) = 0, \quad G'_{N_1}(0, N_2) = 0, \quad G''_{N_1}(0, N_2) = 0, \quad G'''_{N_1}(0, N_2) \neq 0,$$

where N_2 is in a small neighborhood of 0.

Proposition 3.10. *Assume that* $(N_1(s), N_2(s))$ *satisfies the identity* (3.37) *and the initial condition* (3.36)*, then* N_1 *is a real analytic function of* N_2 *in a small neighborhood of* $N_2 = 0$.

Proof. From the definition of $G(N_1, N_2)$, there exists a small neighborhood V of (0,0), such that $G(N_1, N_2)$ is analytic in V with respect to (N_1, N_2). The Taylor series expansion of $G(N_1, N_2)$ at $(0, 0)$ is

$$\sum_{n=1}^{\infty} \frac{2n}{2n+1} N_1^{2n+1} + \sum_{n=1}^{\infty} \frac{2n(-1)^n}{2n+1} N_2^{2n+1},$$

and N_1 and N_2 satisfy $G(N_1, N_2) = 0$. Therefore, in the small neighborhood V,

$$\sum_{n=1}^{\infty} \frac{2n}{2n+1} N_1^{2n+1} + \sum_{n=1}^{\infty} \frac{2n(-1)^n}{2n+1} N_2^{2n+1} = 0;$$

that is,

$$\sum_{n=1}^{\infty} \frac{2n}{2n+1} N_1^{2n+1} = \sum_{n=1}^{\infty} \frac{2n(-1)^{n+1}}{2n+1} N_2^{2n+1},$$

$$N_1{}^3\Big(\frac{2}{3} + \sum_{n=2}^{\infty} \frac{2n}{2n+1} N_1^{2n-2}\Big) = N_2{}^3\Big(\frac{2}{3} + \sum_{n=2}^{\infty} \frac{2n(-1)^{n+1}}{2n+1} N_2^{2n-2}\Big). \tag{3.38}$$

For simplicity, let

$$h_1(N_1) = \frac{2}{3} + \sum_{n=2}^{\infty} \frac{2n}{2n+1} N_1^{2n-2},$$

$$h_2(N_2) = \frac{2}{3} + \sum_{n=2}^{\infty} \frac{2n(-1)^{n+1}}{2n+1} N_2^{2n-2}.$$

By the ratio test, we see that $h_1(N_1)$ and $h_2(N_2)$ both are analytic in a neighborhood of 0 and the radius of convergence is 1. Note that when $r \neq 0$, $(1+x)^r$ is analytic for $x \in (-1, 1)$ and the Taylor series at 0 is

$$(1+x)^r = \sum_{k=0}^{\infty} \frac{r[r-1][r-2]\dots[r-(k-1)]}{k!} x^k.$$

Let

$$u_1(N_1) = \frac{3}{2} \sum_{n=2}^{\infty} \frac{2n}{2n+1} N_1^{2n-2}, \quad u_2(N_2) = \frac{3}{2} \sum_{n=2}^{\infty} \frac{2n(-1)^{n+1}}{2n+1} N_1^{2n-2},$$

then

$$\big[\frac{3}{2} h_1(N_1)\big]^{1/3} = [1 + u_1]^{1/3}$$

is an analytic function of u_1 and it is obvious that $u_1(N_1)$ is also analytic with respect to N_1. It follows that $[\frac{3}{2} h_1(N_1)]^{1/3}$ is analytic for N_1 in a small neighborhood V_1 of 0, and so is $[h_1(N_1)]^{1/3}$. Similarly, we can show that $[h_2(N_2)]^{1/3}$ is analytic for N_2 in a small neighborhood V_2 of 0.

By equation (3.38),

$$N_1^3 \cdot h_1(N_1) = N_2^3 \cdot h_2(N_2).$$

Taking the cube roots on both sides,

$$N_1 \cdot [h_1(N_1)]^{1/3} = N_2 \cdot [h_2(N_2)]^{1/3}.$$

Let

$$\Gamma(N_1, N_2) = N_1 \cdot [h_1(N_1)]^{1/3} - N_2 \cdot [h_2(N_2)]^{1/3},$$

then by the above argument, $\Gamma(N_1, N_2)$ is analytic with respect to (N_1, N_2) in a small neighborhood of $(0,0)$. It is clear that $\Gamma(0,0)=0$. To apply the analytic implicit function theorem, we need to check the condition

$$\begin{aligned}\frac{\partial\Gamma}{\partial N_1}(0,0) &= [h_1(N_1)]^{1/3} + N_1\cdot\frac{1}{3}[h_1(N_1)]^{-\frac{2}{3}}\cdot h_1'(N_1)\mid_{N_1=0}\\ &= (\frac{2}{3})^{1/3} + 0\cdot\frac{1}{3}\cdot(\frac{2}{3})^{-\frac{2}{3}}\cdot 0\\ &= (\frac{2}{3})^{1/3}\neq 0,\end{aligned}$$

By Cauchy's analytic implicit function theorem, there exists $r_0>0$, and a power series

$$N_1 = N_1(N_2) = \sum_{i=0}^{\infty} a_i N_2^i$$

such that $N_1(N_2)=\sum_{i=0}^{\infty} a_i N_2^i$ is absolutely convergent for $|N_2|<r_0$ and that $\Gamma(N_1(N_2),N_2)=0$. That is, N_1 is an analytic function of N_2 when $|N_2|<r_0$. $\square$

Proof of Theorem 3.8. Since (3.29) and (3.30) hold if N_1 and N_2 satisfy the differential equations, That is

$$\begin{gathered}\tanh^{-1}(N_1)+\tan^{-1}(N_2)=s,\\ \frac{N_1}{1-N_1^2}+\frac{N_2}{1+N_2^2}=s.\end{gathered}$$

By Proposition 3.10 and 3.9, N_1 is an analytic function of N_2 when N_2 close to 0. By the setting,

$$N_1 = N_1(N_2) = \sum_{i=0}^{\infty} a_i N_2^i = a_0 + a_1N_2 + a_2N_2^2 + \dots$$

$a_0=0$ since $N_1(0)=0$. First, we show that $a_1=1$. Because

$$\begin{gathered}\sum_{n=1}^{\infty}\frac{2n}{2n+1}N_1^{2n+1} = \sum_{n=1}^{\infty}\frac{2n(-1)^{n+1}}{2n+1}N_2^{2n+1},\\ \frac{2}{3}N_1^3+\frac{4}{5}N_1^5+\dots = \frac{2}{3}N_2^3-\frac{4}{5}N_2^5+\dots\end{gathered}$$

Substituting N_1 by $\sum_{i=1}^{\infty} a_i N_2^i$ and comparing the coefficients of N_2^3 on both sides, we have $\frac{2}{3}a_1^3=\frac{2}{3}$, it follows that $a_1=1$. Therefore, N_1/N_2 is also a real analytic function of N_2 and

$$\lim_{s\to 0}\frac{N_1}{N_2} = \lim_{N_2\to 0}\frac{N_1}{N_2} = 1.$$

Next, we show that both N_1 and N_2 are analytic functions of s in a small enough neighborhood of 0. Rewrite the differential equation of N_2 as

$$N_2' = (1+N_2^2)^2\cdot\frac{N_1^2}{N_1^2+N_2^2} = (1+N_2^2)^2\big[1-\frac{1}{1+(\frac{N_1}{N_2})^2}\big].$$

When $s\to 0$, N_2 also approaches 0, and

$$\frac{N_1}{N_2} = 1 + \sum_{n=2}^{\infty} a_n N_2^{n-1}.$$

Let

$$(\frac{N_1}{N_2})^2 = 1 + \sum_{n=1}^{\infty} d_n N_2^n \equiv 1 + \phi(N_2),$$

where $\phi(N_2) = \sum_{n=1}^{\infty} d_n N_2^n$ is an analytic function of N_2 in a small neighborhood of 0.

$$\frac{1}{1+(\frac{N_1}{N_2})^2} = \frac{1}{2+\phi(N_2)} = \frac{1}{2}\frac{1}{1+\frac{\phi(N_2)}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(\frac{\phi(N_2)}{2})^n$$

which is an analytic function of N_2 in a neighborhood of 0 with radius of convergence 1. Therefore,

$$(1+N_2^2)^2\frac{N_1^2(N_2)}{N_1^2(N_2)+N_2^2} = (1+N_2^2)^2\big[1 - \frac{1}{1+(\frac{N_1}{N_2})^2}\big]$$

is also analytic with respect to N_2 in a small neighborhood of 0.

Note that $N_2'(0) = 1$, by Cauchy's theorem, $N_2' = (1+N_2^2)^2\frac{N_1^2(N_2)}{N_1^2(N_2)+N_2^2}$, $N_2(0) = 0$ has a unique analytic solution $N_2 = N_2(s)$ in a small neighborhood of 0. Hence, $N_1(s) = N_1(N_2(s))$ is also analytic in a small neighborhood of 0. □

So far, we know that for any given constant C, the solution $(\eta_1(s,C), \eta_2(s,C))$ of the initial value problem (3.39)–(3.41) is unique.

$$\eta_1' = (-1 + C\eta_1^2)^2\frac{\eta_2^2}{\eta_1^2+\eta_2^2}, \tag{3.39}$$

$$\eta_2' = (1 + C\eta_2^2)^2\frac{\eta_1^2}{\eta_1^2+\eta_2^2}, \tag{3.40}$$

$$\eta_1(0) = \eta_2(0) = 0. \tag{3.41}$$

Note that $C = y_1^2 - \frac{1}{x_1} = \frac{1+\xi_1}{\xi_1\eta_1^2}$ is the energy of one collision pair (m_1 and m_2). Therefore, in the decoupled system with total energy $E = h = 0$, the following theorems hold.

Theorem 3.11. *In the decoupled case with total energy $h = 0$, the solutions $(\xi_1, \xi_2, \eta_1, \eta_2)$ of the transformed differential system of SBC are all analytic in a small neighborhood of $s = 0$ and they form a one-parameter set, where the parameter C satisfies*

$$C = \frac{\xi_1 - \xi_2}{\xi_1\eta_1^2 + \xi_2\eta_2^2} = \frac{x_1y_1^2 - x_2y_2^2 + x_2h}{x_1 + x_2} = (y_1^2 - \frac{1}{x_1}).$$

Proposition 3.12. *For fixed total energy $E = h = 0$, the variables $(\xi_1, \xi_2, \eta_1, \eta_2)$ are on a 3-dimensional hypersurface. In a small neighborhood of 0 on the energy surface $E = h = 0$, the initial conditions leading to SBC is 2-dimensional. Actually, for any given small $(\eta_{10}, \eta_{20}) = (\epsilon_1, \epsilon_2)$ leading to SBC, there exists unique s_0, C_0, ξ_{10} and ξ_{20}, such that*

$$\epsilon_1 = \eta_1(s_0, C_0), \quad \epsilon_2 = \eta_2(s_0, C_0),$$
$$\xi_{10} = \frac{1}{C_0\epsilon_1^2 - 1}, \quad \xi_{20} = \frac{-1}{C_0\epsilon_2^2 + 1},$$

where $(\eta_1(s_0, C_0), \eta_2(s_0, C_0))$ is the solution of the system (3.39)–(3.41).

3.5. **Decoupled system with total energy $E = h$.** The goal in this section is to show that Theorem 3.11 and Proposition 3.12 hold for general total energy $E = h$. To consider the solution of the decoupled system on the energy surface $E = h$, we define a new Hamiltonian

$$F = \frac{y_1^2 + y_2^2 - h}{\frac{1}{x_1} + \frac{1}{x_2}}.$$

Note that $F = 1$. The differential equations are:

$$x_1' = \frac{2y_1}{\frac{1}{x_1} + \frac{1}{x_2}}, \quad x_2' = \frac{2y_2}{\frac{1}{x_1} + \frac{1}{x_2}},$$

$$y_1' = -\frac{y_1^2 + y_2^2 - h}{(\frac{1}{x_1} + \frac{1}{x_2})^2 x_1^2} = -\frac{x_2}{x_1(x_1 + x_2)},$$

$$y_2' = -\frac{y_1^2 + y_2^2 - h}{(\frac{1}{x_1} + \frac{1}{x_2})^2 x_2^2} = -\frac{x_1}{x_2(x_1 + x_2)}.$$

By a similar argument, we have

$$y_1^2 = \frac{1}{x_1} + \bar{C}, \quad y_2^2 = \frac{1}{x_2} + \bar{C}_1.$$

Then $\bar{C} + \bar{C}_1 = h$ and

$$\bar{C} = \frac{x_1 y_1^2 - x_2 y_2^2 + x_2 h}{x_1 + x_2}, \quad \bar{C}_1 = \frac{x_2 y_2^2 - x_1 y_1^2 + x_1 h}{x_1 + x_2}.$$

By the same canonical transformation, the new Hamiltonian becomes

$$F = -\frac{\xi_1 \xi_2 (\eta_1^2 + \eta_2^2) - h \xi_1 \xi_2 \eta_1^2 \eta_2^2}{\xi_1 \eta_1^2 + \xi_2 \eta_2^2}$$

and the equations become:

$$\eta_1' = (\bar{C}\eta_1^2 - 1)^2 \frac{\eta_2^2}{\eta_1^2 + \eta_2^2 - h\eta_1^2\eta_2^2},$$

$$\eta_2' = [(h - \bar{C})\eta_2^2 - 1]^2 \frac{\eta_1^2}{\eta_1^2 + \eta_2^2 - h\eta_1^2\eta_2^2}.$$

Let $D = \bar{C} - \frac{1}{2}h$, then the equations become

$$\eta_1' = [(D + \frac{1}{2}h)\eta_1^2 - 1]^2 \frac{\eta_2^2}{\eta_1^2 + \eta_2^2 - h\eta_1^2\eta_2^2}, \tag{3.42}$$

$$\eta_2' = [(-D + \frac{1}{2}h)\eta_2^2 - 1]^2 \frac{\eta_1^2}{\eta_1^2 + \eta_2^2 - h\eta_1^2\eta_2^2}. \tag{3.43}$$

with initial conditions

$$\eta_1(0) = \eta_2(0) = 0. \tag{3.44}$$

For any given constant D, a similar argument shows that η_1 and η_2 have unique solutions $(\eta_1(s, D), \eta_2(s, D))$ in a small interval of 0. Note that

$$\xi_1 = \frac{1}{(D + \frac{h}{2})\eta_1^2 - 1}, \quad \xi_2 = \frac{1}{(-D + \frac{h}{2})\eta_2^2 - 1}.$$

Theorem 3.13. *In the decoupled case with total energy $E = h$, the solutions $(\xi_1, \xi_2, \eta_1, \eta_2)$ of the transformed differential system of SBC are all analytic in a small neighborhood of $s = 0$ and they form a one-parameter set, where the parameter D satisfies*

$$\begin{aligned} D &= \frac{\xi_1 - \xi_2}{\xi_1\eta_1^2 + \xi_2\eta_2^2} + \frac{(\xi_2\eta_2^2 - \xi_1\eta_1^2)h}{2(\xi_1\eta_1^2 + \xi_2\eta_2^2)} \\ &= C + \frac{(\xi_2\eta_2^2 - \xi_1\eta_1^2)h}{2(\xi_1\eta_1^2 + \xi_2\eta_2^2)} \\ &= \frac{x_1y_1^2 - x_2y_2^2 + x_2h}{x_1 + x_2} - \frac{h}{2} \\ &= C + \frac{h(x_2 - x_1)}{2(x_1 + x_2)}. \end{aligned}$$

As a consequence, the following proposition holds.

Proposition 3.14. *For fixed total energy $E = h$, the variables $(\xi_1, \xi_2, \eta_1, \eta_2)$ are on a 3-dimensional hypersurface. In a small neighborhood of 0 on the energy surface $E = h$, the initial conditions leading to SBC is 2-dimensional. Actually, for any given small $(\eta_{10}, \eta_{20}) = (\epsilon_1, \epsilon_2)$ leading to SBC, there exists some s_0, D_0, ξ_{10} and ξ_{20}, such that*

$$\begin{gathered} \epsilon_1 = \eta_1(s_0, D_0), \quad \epsilon_2 = \eta_2(s_0, D_0), \\ \xi_{10} = \frac{1}{(D_0 + \frac{h}{2})\epsilon_1^2 - 1}, \quad \xi_{20} = \frac{1}{(-D_0 + \frac{h}{2})\epsilon_2^2 - 1}, \end{gathered}$$

where $(\eta_1(s_0, D_0), \eta_2(s_0, D_0))$ is the solution of (3.42)–(3.44).

4. Understanding collision from the variational perspective

4.1. Regularization of binary collision in two-body problem. In this section, we study the regularization of one-dimensional binary collision from the variational perspective. We show that the solution of binary collision can be realized by the limit of non-collision solutions. This argument will help us understand the regularization of the decoupled case of SBC.

Let $m_1 = m_2 = 1$, $x = q_2 - q_1$ and $x(0) = 0$ be a binary collision point. Assume the center of mass and the total momentum to be 0, it follows that $q_1 = -q_2$ and $\dot{q}_1 = -\dot{q}_2$. In the one-dimensional two-body problem, the kinetic energy is $\frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) = \frac{1}{4}\dot{x}^2$. The action functional has the form

$$\mathcal{A}(x) = \int_0^1 \Big(\frac{1}{4}|\dot{x}|^2 + \frac{1}{|x|}\Big)dt, \tag{4.1}$$

where

$$x \in S = \{x \in W^{1,2}([0,1]) : x(0) = 0,\ x(1) = \alpha\},$$

and $\alpha > 0$ is some fixed number. without loss of generality, we assume $x(t) > 0$, for $t \in (0, 1]$. The binary collision happens at $t = 0$. As we know, the stationary points of the action functional $\mathcal{A}(x)$ are trajectories that satisfy the equations of motions, i.e., Newton's law of gravity. A new time variable s is defined by

$$s(t) = \int_0^t \frac{1}{x}\,dt.$$

It is known that $s(t)$ is finite for $0 \le t \le 1$. Under the new variable s, let $u(s) = x(t)$, and $u'(s) = \frac{du}{ds}$. Then $\dot{x}(t) = u'(s)\frac{ds}{dt} = \frac{u'(s)}{u(s)}$. Note that the first variation of (4.1) is

$$\frac{d}{d\tau}\mathcal{A}(x+\tau\phi)\big|_{\tau=0} = \frac{d}{d\tau}\Big(\int_0^1 (\frac{1}{4}((x+\tau\phi)')^2 + \frac{1}{x+\tau\phi})dt\Big)\Big|_{\tau=0} = 0, \tag{4.2}$$

for any $\phi \in C_0^1([0,1])$. Let $\beta = s(1) = \int_0^1 \frac{1}{x}dt$. It follows that the corresponding first variational form is

$$\begin{aligned}
\frac{d}{d\tau}\mathcal{A}(u+\tau\phi)\big|_{\tau=0} &= \frac{d}{d\tau}\Big(\int_0^\beta \Big(\frac{1}{4}\frac{((u+\tau\phi)')^2}{u^2} + \frac{1}{u+\tau\phi}\Big)\frac{dt}{ds}ds\Big)\Big|_{\tau=0} \\
&= \Big(\int_0^\beta \Big(\frac{(u+\tau\phi)'\phi'}{2u^2} + \frac{1}{u+\tau\phi}\Big)uds\Big)\Big|_{\tau=0} \\
&= \int_0^\beta (\frac{u'\phi'}{2u} - \frac{\phi}{u})ds = 0,
\end{aligned} \tag{4.3}$$

for any $\phi \in C_0^1([0,\beta])$. Hence, the Euler-Lagrange equation of (4.1) with variable s is

$$-\frac{u''}{2u} + \frac{(u')^2}{2u^2} - \frac{1}{u} = 0.$$

Since $u(s) > 0$, for $0 < s \le \beta$, multiplying $-2u$ on both sides of the above equation implies

$$u(s)'' - \frac{(u(s)')^2}{u(s)} + 2 = 0, \quad 0 < s \le \beta, \tag{4.4}$$

with boundary conditions

$$u(0) = 0, \quad u(\beta) = \alpha, \quad \text{where } \beta = \int_0^1 \frac{1}{x}dt.$$

Next, we solve this boundary value problem. Both sides of (4.4) multiplying by $\frac{u'}{2u^2}$, it becomes

$$(\frac{(u')^2}{4u^2})' + \frac{u'}{u^2} = 0, \quad 0 < s \le \beta.$$

Then

$$\Big(\frac{(u')^2}{4u^2}\Big) - \frac{1}{u} = C_0, \tag{4.5}$$

where C_0 is a constant, and $0 < s \le \beta$. From (4.5), we have

$$|\frac{(u')^2}{u}| = 4C_0u + 4 < \infty, \quad \text{as } s \to 0.$$

Therefore, (4.4) can be extended to the domain $s \in [-\beta, \beta]$ and any solution $u(s)$ of equation (4.4) for $0 < s < \beta$ can be extended to

$$u(s) = \begin{cases} u(s), & 0 < s \le \beta, \\ 0, & s = 0, \\ u(-s), & -\beta \le s < 0. \end{cases}$$

Note that the Hamiltonian is constant

$$-h = \frac{(u'(s))^2}{4u(s)^2} - \frac{1}{u(s)}, \quad 0 < s \le \beta; \tag{4.6}$$

it follows that $C_0 = -h$, where $h \geq 0$. Actually, one can solve for u in equation (4.6):

$$\frac{du}{\sqrt{-hu^2+u}} = 2ds, \quad 0 < s \leq \beta.$$

Integrating the above implies

$$\frac{1}{\sqrt{h}} \arcsin(2hu-1) = 2s + \widehat{C}_0, \quad 0 < s \leq \beta,$$

then

$$u = \frac{\sin(2\sqrt{h}s + \sqrt{h}\widehat{C}_0) + 1}{2h}, \quad 0 < s \leq \beta.$$

By the boundary condition $u(0) = 0$, it follows that $\widehat{C}_0 = -\frac{\pi}{2\sqrt{h}}$, and

$$u(s) = \frac{1-\cos(2\sqrt{h}s)}{2h} = s^2 - \frac{hs^4}{3} + \dots \quad 0 < s \leq \beta. \tag{4.7}$$

Note that $\frac{dt}{ds} = u(s)$, it follows that

$$t(s) = \frac{1}{2h}(s - \frac{\sin(2\sqrt{h}s)}{2\sqrt{h}}) = \frac{s^3}{3} - \frac{hs^5}{15} + \dots \quad 0 < s \leq \beta. \tag{4.8}$$

4.2. **Regularization of SBC in the decoupled case.** Let $x_1 = q_2 - q_1$, $x_2 = q_4 - q_3$, and $x_1(0) = x_2(0) = 0$ be the collinear SBC in the decoupled case. Similar to the case of binary collision, the action functional can be defined as

$$\mathcal{A}(x) = \int_0^1 (\frac{|\dot{x}_1|^2}{4} + \frac{|\dot{x}_2|^2}{4} + \frac{1}{|x_1|} + \frac{1}{|x_2|})dt, \tag{4.9}$$

where

$$(x_1, x_2) \in S = \{x \in W^{1,2}([0,1]) : x_1(0) = 0, x_2(0) = 0, x_1(1) = \alpha_1,\ x_2(1) = \alpha_2\},$$

and $\alpha_1, \alpha_2 > 0$ are fixed. A new variable s is defined by

$$s(t) = \int_0^t (\frac{1}{x_1} + \frac{1}{x_2})dt.$$

It is known that $s(t)$ is finite for $0 \leq t \leq 1$. Under the new variable s, let $u_1(s) = x_1(t)$, $u_2(s) = x_2(t)$ and $u_i'(s) = \frac{du_i}{ds}$ $(i = 1, 2)$. Then $\dot{x}_i(t) = u_i'(s)\frac{ds}{dt} = u_i'(s)(\frac{1}{u_1} + \frac{1}{u_2})$. Let $\beta_0 = \int_0^1 (\frac{1}{x_1} + \frac{1}{x_2})dt$. Note that the first variation of action form (4.9) is

$$\frac{d}{d\tau_i}\Big(\int_0^1 \frac{1}{4}[(x_1+\tau_1\phi_1)']^2 + \frac{1}{4}[(x_2+\tau_2\phi_2)']^2 + \frac{1}{x_1+\tau_1\phi_1} + \frac{1}{x_2+\tau_2\phi_2}\Big)dt\Big|_{\tau_i=0},$$

for any $\phi_1,\ \phi_2 \in C_0^1([0,1])$ and $i = 1, 2$. Then the corresponding first variational form in variable s is $\frac{d}{d\tau_i}\Big(\int_0^1 \widehat{L}\,ds\Big)\Big|_{\tau_i=0}$ $(i = 1, 2)$, where

$$\begin{aligned} \widehat{L} = &\frac{1}{4}\left[[(u_1+\tau_1\phi_1)']^2 + [(u_2+\tau_2\phi_2)']^2\right](\frac{1}{u_1} + \frac{1}{u_2}) \\ &+ (\frac{1}{u_1+\tau_1\phi_1} + \frac{1}{u_2+\tau_2\phi_2})\frac{1}{\frac{1}{u_1}+\frac{1}{u_2}}. \end{aligned}$$

Therefore, the corresponding Euler-Lagrange equations in the variable s are

$$-\frac{d}{ds}\Big(\frac{u_1'}{2}(\frac{1}{u_1} + \frac{1}{u_2})\Big) - \frac{1}{u_1^2}\frac{1}{\frac{1}{u_1}+\frac{1}{u_2}} = 0, \quad 0 < s \leq \beta_0, \tag{4.10}$$

$$-\frac{d}{ds}\Big(\frac{u_2'}{2}(\frac{1}{u_1}+\frac{1}{u_2})\Big)-\frac{1}{u_2^2}\frac{1}{\frac{1}{u_1}+\frac{1}{u_2}}=0,\quad 0<s\le\beta_0 \tag{4.11}$$

with boundaries

$$u_1(0)=u_2(0)=0,\quad u_1(1)=\alpha_1,\quad u_2(1)=\alpha_2,\quad \text{where } \beta_0=\int_0^1(\frac{1}{x_1}+\frac{1}{x_2})dt. \tag{4.12}$$

Next we try to solve the above boundary value problem. Multiplying $-u_1'(\frac{1}{u_1}+\frac{1}{u_2})$ on both sides of (4.10), it follows that

$$\frac{1}{2}u_1''u_1'(\frac{1}{u_1}+\frac{1}{u_2})^2+\frac{1}{2}(u_1')^2\big(-\frac{u_1'}{u_1^2}-\frac{u_2'}{u_2^2}\big)\big(\frac{1}{u_1}+\frac{1}{u_2}\big)+\frac{u_1'}{u_1^2}=0,\quad 0<s\le\beta_0.$$

Then

$$\frac{1}{4}(u_1')^2(\frac{1}{u_1}+\frac{1}{u_2})^2-\frac{1}{u_1}=\widehat{C}_{10},\quad 0<s\le\beta_0. \tag{4.13}$$

where $\widehat{C}_{10}$ is a constant. Similarly, equation (4.11) implies

$$\frac{1}{4}(u_2')^2(\frac{1}{u_1}+\frac{1}{u_2})^2-\frac{1}{u_2}=\widehat{C}_{20},\quad 0<s\le\beta_0. \tag{4.14}$$

where $\widehat{C}_{20}$ is a constant.

On the other hand, the total energies of collision pairs (q_1,q_2) and (q_3,q_4) are

$$\frac{1}{4}(u_1')^2(\frac{1}{u_1}+\frac{1}{u_2})^2-\frac{1}{u_1}=h_1, \tag{4.15}$$

$$\frac{1}{4}(u_2')^2(\frac{1}{u_1}+\frac{1}{u_2})^2-\frac{1}{u_2}=h_2, \tag{4.16}$$

where $h_1+h_2=h$ is the total energy of the decoupled SBC system. It follows that

$$\widehat{C}_{10}=h_1,\quad \widehat{C}_{20}=h_2.$$

Hence, the solution (u_1,u_2) of the boundary value problem (4.10), (4.11) and (4.12) for $s\in(0,\beta_0]$ is equivalent to the solution of (4.15)–(4.16). Then the solutions $u_1(s),u_2(s)$ $s\in(0,\beta_0]$ can be extended to $s\in[-\beta_0,\beta_0]$ in the following way:

$$u_i(s)=\begin{cases}u_i(s) & 0<s\le\beta_0;\\ 0 & s=0;\\ u_i(-s) & -\beta_0\le s<0,\end{cases},\quad i=1,2.$$

5. Coupled case

In this section, we consider the coupled case, which studies SBC in the collinear four-body problem. The Hamiltonian F in the coupled case is

$$\begin{aligned}F&=\frac{1}{\frac{m_1m_2}{x_1}+\frac{m_3m_4}{x_2}}\cdot(T-U-h)\\&=\frac{1}{\frac{m_1m_2}{x_1}+\frac{m_3m_4}{x_2}}\big[\frac{y_1^2}{2m_1}+\frac{(y_1-y_3)^2}{2m_2}+\frac{(y_3-y_2)^2}{2m_3}+\frac{y_2^2}{2m_4}\big]\\&\quad-\frac{1}{\frac{m_1m_2}{x_1}+\frac{m_3m_4}{x_2}}\big[\frac{m_1m_2}{x_1}+\frac{m_1m_3}{x_1+x_3}+\frac{m_1m_4}{x_1+x_2+x_3}+\frac{m_2m_3}{x_3}\\&\quad+\frac{m_2m_4}{x_2+x_3}+\frac{m_3m_4}{x_2}+h\big].\end{aligned}$$

Let $m_1 = m_2 = m_3 = m_4 = 1$, then

$$\begin{aligned}
F &= \frac{1}{\frac{1}{x_1}+\frac{1}{x_2}}(y_1^2+y_2^2+y_3^2-y_1y_3-y_2y_3)\\
&\quad - \frac{1}{\frac{1}{x_1}+\frac{1}{x_2}}\Big(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_1+x_3}+\frac{1}{x_2+x_3}+\frac{1}{x_1+x_2+x_3}+h\Big)\\
&= \frac{y_1^2+y_2^2}{\frac{1}{x_1}+\frac{1}{x_2}} - \frac{(y_1+y_2)y_3}{\frac{1}{x_1}+\frac{1}{x_2}} - \frac{1}{\frac{1}{x_1}+\frac{1}{x_2}}\Big[\frac{1}{x_3}+\frac{1}{x_1+x_3}+\frac{1}{x_2+x_3}\\
&\quad + \frac{1}{x_1+x_2+x_3}+h-y_3^2\Big]-1.
\end{aligned}$$

We introduce a canonical transformation to simplify the Hamiltonian form of F. Set

$$\begin{gathered}
Y_1 = y_1 - \frac{1}{2}y_3, \quad Y_2 = y_2 - \frac{1}{2}y_3, \quad Y_3 = y_3,\\
X_1 = x_1, \quad X_2 = x_2, \quad X_3 = \frac{1}{2}x_1 + \frac{1}{2}x_2 + x_3.
\end{gathered}$$

Under this canonical transformation, the new hamiltonian becomes

$$\begin{aligned}
F &= \frac{Y_1^2+Y_2^2}{\frac{1}{X_1}+\frac{1}{X_2}} - 1 - \frac{h-\frac{1}{2}Y_3^2}{\frac{1}{X_1}+\frac{1}{X_2}} - \frac{1}{\frac{1}{X_1}+\frac{1}{X_2}}\Big[\frac{1}{X_3-\frac{1}{2}X_1-\frac{1}{2}X_2}\\
&\quad + \frac{1}{X_3+\frac{1}{2}X_1-\frac{1}{2}X_2} + \frac{1}{X_3+\frac{1}{2}X_2-\frac{1}{2}X_1} + \frac{1}{X_3+\frac{1}{2}X_1+\frac{1}{2}X_2}\Big].
\end{aligned}$$

Let

$$\begin{aligned}
A &= A(X_i, Y_3)\\
&= \frac{1}{X_3-\frac{1}{2}X_1-\frac{1}{2}X_2} + \frac{1}{X_3+\frac{1}{2}X_1-\frac{1}{2}X_2} + \frac{1}{X_3+\frac{1}{2}X_2-\frac{1}{2}X_1}\\
&\quad + \frac{1}{X_3+\frac{1}{2}X_1+\frac{1}{2}X_2} + h - \frac{1}{2}Y_3^2,
\end{aligned} \tag{5.1}$$

then

$$F = \frac{Y_1^2+Y_2^2}{\frac{1}{X_1}+\frac{1}{X_2}} - \frac{1}{\frac{1}{X_1}+\frac{1}{X_2}}A(X_i, Y_3) - 1. \tag{5.2}$$

Note that F is a Hamiltonian if and only if $E = T - U = h$. And $F = 0$ holds for any solution on the energy surface $E = h$. So we only consider the case $F = 0$.

5.1. **New transformation.** We introduce a canonical transformation similar to the one defined in the decoupled case:

$$\xi_1 = -X_1Y_1^2, \quad \xi_2 = -X_2Y_2^2, \quad \xi_3 = X_3, \quad \eta_1 = \frac{1}{Y_1}, \quad \eta_2 = \frac{1}{Y_2}, \quad \eta_3 = Y_3.$$

And X_i, Y_i $(i = 1, 2, 3)$ can be solved in terms of ξ_i and η_i $(i = 1, 2, 3)$:

$$X_1 = -\xi_1\eta_1^2, \quad X_2 = -\xi_2\eta_2^2, \quad X_3 = \xi_3, \quad Y_1 = \frac{1}{\eta_1}, \quad Y_2 = \frac{1}{\eta_2}, \quad Y_3 = \eta_3.$$

Then the Hamiltonian F becomes

$$F = -\frac{\xi_1\xi_2(\eta_1^2+\eta_2^2)}{\xi_1\eta_1^2+\xi_2\eta_2^2} - 1 + \frac{\xi_1\xi_2\eta_1^2\eta_2^2}{\xi_1\eta_1^2+\xi_2\eta_2^2}\Big[h - \frac{1}{2}\eta_3^2$$

$$+\frac{1}{-\frac{1}{2}\xi_1\eta_1^2-\frac{1}{2}\xi_2\eta_2^2+\xi_3}+\frac{1}{\frac{1}{2}\xi_1\eta_1^2-\frac{1}{2}\xi_2\eta_2^2+\xi_3}\Big]$$
$$+\frac{\xi_1\xi_2\eta_1^2\eta_2^2}{\xi_1\eta_1^2+\xi_2\eta_2^2}\Big[\frac{1}{-\frac{1}{2}\xi_1\eta_1^2+\frac{1}{2}\xi_2\eta_2^2+\xi_3}+\frac{1}{\frac{1}{2}\xi_1\eta_1^2+\frac{1}{2}\xi_2\eta_2^2+\xi_3}\Big],$$

and the corresponding differential equations are

$$\xi_1'=\frac{2\xi_1\xi_2\eta_1\eta_2^2(\xi_1-\xi_2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}+M_1, \tag{5.3}$$

$$\xi_2'=\frac{-2\xi_1\xi_2\eta_2\eta_1^2(\xi_1-\xi_2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}+M_2, \tag{5.4}$$

$$\eta_1'=-F_{\xi_1}=\frac{\xi_2^2\eta_2^2(\eta_1^2+\eta_2^2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}+G_1, \tag{5.5}$$

$$\eta_2'=-F_{\xi_2}=\frac{\xi_1^2\eta_1^2(\eta_1^2+\eta_2^2)}{(\xi_1\eta_1^2+\xi_2\eta_2^2)^2}+G_2, \tag{5.6}$$

$$\xi_3'=K_1, \tag{5.7}$$

$$\eta_3'=K_2, \tag{5.8}$$

$$\xi_1(0)=\xi_2(0)=-1,\quad \eta_1(0)=\eta_2(0)=0,\quad \xi_3(0)=\widehat{\xi}_3,\quad \eta_3(0)=\widehat{\eta}_3, \tag{5.9}$$

where M_1, M_2 , K_1 and K_2 are $O(s)$; G_1, G_2 are $O(s^2)$.

Different from the decoupled case, the differential equations of ξ_i and η_i ($i=1,2,3$) are much more complicated in this coupled case. Note that if $\widehat{\xi}_3=\infty$ and $\widehat{\eta}_3=0$, the above system is exactly the system in the decoupled case. Actually, the solutions in the decoupled case and the coupled case are closely related. The result is shown in Lemma 5.2 in the next subsection.

5.2. **Limits of u_i and v_i at SBC ($i=1,2$).** To study the solution of SBC, we introduce another transformation as follows

$$\frac{\xi_i+1}{s}=u_i,\quad \frac{\eta_i}{s}-\frac{1}{2}=v_i,\quad i=1,2;$$
$$\xi_3=\widehat{\xi}_3+u_3,\quad \eta_3=\widehat{\eta}_3+v_3,$$

where $\widehat{\xi}_3=\lim_{s\to0}\xi_3$ and $\widehat{\eta}_3=\lim_{s\to0}\eta_3$ are the limits at SBC. By the definition of u_3 and v_3, it is clear that $\lim_{s\to0}u_3=\lim_{s\to0}v_3=0$. We first show that u_1, u_2, v_1 and v_2 all have limit 0 at $s=0$.

Lemma 5.1.
$$\lim_{s\to0}u_1=\lim_{s\to0}u_2=\lim_{s\to0}v_1=\lim_{s\to0}v_2=0.$$

Proof. We first show the limits of v_i ($i=1,2$) is 0. By Lemma 2.2, we have

$$\lim_{s\to0}\frac{\eta_2^2}{\eta_1^2}=\lim_{t\to0}\frac{Y_1^2}{Y_2^2}=\lim_{t\to0}\frac{y_1^2}{y_2^2}=\lim_{t\to0}\frac{x_1p_1^2}{x_1p_4^2}=\frac{2(m_1m_2)^2}{(m_1+m_2)}\cdot\frac{(m_3+m_4)}{2\alpha(m_3m_4)^2},$$

where $\alpha=(\frac{m_1+m_2}{m_3+m_4})^{1/3}$.

Since $m_1=m_2=m_3=m_4=1$, it implies that $\alpha=1$ and

$$\lim_{s\to0}\frac{\eta_2^2}{\eta_1^2}=1.$$

Note that when $t \in (-\delta, 0)$, both y_1 and y_2 are negative. And when $t \in (0, \delta)$, they are both positive. So $\lim_{s\to 0} \frac{\eta_2}{\eta_1}$ is positive. Therefore,

$$\lim_{s\to 0} \frac{\eta_2}{\eta_1} = 1.$$

By L'Hospital rule,

$$\lim_{s\to 0} \frac{\eta_1}{s} = \lim_{s\to 0} \eta_1' = \lim_{s\to 0} \frac{\xi_2^2 \eta_2^2 (\eta_1^2 + \eta_2^2)}{(\xi_1 \eta_1^2 + \xi_2 \eta_2^2)^2} + \lim_{s\to 0} G_1.$$

If the limit on the right hand side is finite, then the limit on the left hand side also exists and equals to the same value. According to section 2.2, we have

$$\lim_{s\to 0} \eta_1 = \lim_{s\to 0} \eta_2 = 0, \quad \text{and} \quad \lim_{s\to 0} \xi_1 = \lim_{s\to 0} \xi_2 = -1.$$

So

$$\lim_{s\to 0} \frac{\xi_2^2 \eta_2^2 (\eta_1^2 + \eta_2^2)}{(\xi_1 \eta_1^2 + \xi_2 \eta_2^2)^2} = \lim_{s\to 0} \frac{\xi_2^2 \frac{\eta_1^2}{\eta_2^2} (1 + \frac{\eta_1^2}{\eta_2^2})}{(\xi_1 + \xi_2 \frac{\eta_1^2}{\eta_2^2})^2} = \frac{1 \cdot 1 \cdot 2}{(-1-1)^2} = \frac{1}{2}.$$

Because $\lim_{s\to 0} G_1 = 0$, it follows that

$$\lim_{s\to 0} \frac{\eta_1}{s} = \frac{1}{2}, \quad \text{and} \quad \lim_{s\to 0} \frac{\eta_2}{s} = \lim_{s\to 0} \frac{\eta_2}{\eta_1} \cdot \lim_{s\to 0} \frac{\eta_1}{s} = \frac{1}{2}.$$

Therefore, $\lim_{s\to 0} v_1 = \lim_{s\to 0} v_2 = 0$. To find the limit of u_i $(i = 1, 2)$, we consider the differential equations of ξ_i $(i = 1, 2)$. Since $\lim_{s\to 0} \xi_1 = \lim_{s\to 0} \xi_2 = -1$, $\lim_{s\to 0} \frac{\eta_1}{s} = \lim_{s\to 0} \frac{\eta_2}{s} = \frac{1}{2}$ and $\lim_{s\to 0} M_1 = \lim_{s\to 0} M_2 = 0$, it follows that

$$\lim_{s\to 0} (\xi_1' - \xi_2') = \lim_{s\to 0} \frac{2\xi_1 \xi_2 \eta_1 \eta_2 (\eta_1 + \eta_2)(\xi_1 - \xi_2)}{(\xi_1 \eta_1^2 + \xi_2 \eta_2^2)^2} = 2 \lim_{s\to 0} \frac{\xi_1 - \xi_2}{s}.$$

By the definition of derivative, $\lim_{s\to 0} \frac{\xi_1 - \xi_2}{s} = \lim_{s\to 0} (\xi_1' - \xi_2')$. Therefore,

$$\lim_{s\to 0} (\xi_1' - \xi_2') = 2 \lim_{s\to 0} (\xi_1' - \xi_2') = 0. \tag{5.10}$$

Similarly,

$$\begin{aligned} \lim_{s\to 0} (\xi_1' + \xi_2') &= \lim_{s\to 0} \frac{2\xi_1 \xi_2 \eta_1 \eta_2^2 (\xi_1 - \xi_2)}{(\xi_1 \eta_1^2 + \xi_2 \eta_2^2)^2} + \lim_{s\to 0} \frac{-2\xi_1 \xi_2 \eta_2 \eta_1^2 (\xi_1 - \xi_2)}{(\xi_1 \eta_1^2 + \xi_2 \eta_2^2)^2} \\ &= \lim_{s\to 0} \frac{2\xi_1 \xi_2 \eta_1 \eta_2 (\xi_1 - \xi_2)(\eta_2 - \eta_1)}{(\xi_1 \eta_1^2 + \xi_2 \eta_2^2)^2}. \end{aligned}$$

Note that

$$\lim_{s\to 0} \xi_1 = \lim_{s\to 0} \xi_2 = -1, \quad \lim_{s\to 0} \frac{\xi_1 - \xi_2}{s} = 0, \quad \lim_{s\to 0} \frac{\eta_1}{s} = \lim_{s\to 0} \frac{\eta_2}{s} = \frac{1}{2}.$$

Then,

$$\lim_{s\to 0} (\xi_1' + \xi_2') = 0. \tag{5.11}$$

From equations (5.11) and (5.10), it follows that $\lim_{s\to 0} \xi_1' = \lim_{s\to 0} \xi_2' = 0$. Then by L'Hospital rule,

$$\lim_{s\to 0} u_1 = \lim_{s\to 0} \frac{\xi_1 + 1}{s} = \lim_{s\to 0} \xi_1' = 0.$$

Similarly, we have $\lim_{s\to 0} u_2 = 0$. □

Lemma 5.2. *For any solution of* (5.3)-(5.9)*, its limit by letting* $\widehat{\xi}_3 \to \infty$ *and* $\widehat{\eta}_3 \to 0$ *is a solution in the decoupled system.*

Proof. By Lemma 5.1, it implies that the solution in the coupled system (5.3)–(5.9) is C^1 with respect to s for s small enough. And the solution is continuous with respect to $\widehat{\xi}_3$ and $\widehat{\eta}_3$. If $\widehat{\xi}_3 \to \infty$ and $\widehat{\eta}_3 \to 0$, the limit of the solution satisfies the decoupled system with initial condition

$$\xi_1(0) = \xi_2(0) = -1, \quad \eta_1(0) = \eta_2(0) = 0, \quad \xi_3(0) = \infty, \quad \eta_3(0) = 0.$$

The hamiltonian F converges to

$$-\frac{\xi_1\xi_2(\eta_1^2+\eta_2^2)}{\xi_1\eta_1^2+\xi_2\eta_2^2} - 1 + \frac{h\xi_1\xi_2\eta_1^2\eta_2^2}{\xi_1\eta_1^2+\xi_2\eta_2^2}$$

uniformly when $\widehat{\xi}_3 \to \infty$ and $\widehat{\eta}_3 \to 0$. It follows that this limit is a solution of the decoupled system. ☐

5.3. **Analytic Solutions of u_i and v_i $(i = 1, 2, 3)$ at $s = 0$.** By definition,

$$\begin{gathered}
\frac{\xi_i+1}{s} = u_i, \quad \frac{\eta_i}{s} - \frac{1}{2} = v_i, \quad i = 1, 2, \\
\xi_3 = \widehat{\xi}_3 + u_3, \quad \eta_3 = \widehat{\eta}_3 + v_3.
\end{gathered}$$

The new differential system becomes

$$\begin{gathered}
s\frac{du_1}{ds} = F_{\eta_1} - u_1, \quad s\frac{dv_1}{ds} = -F_{\xi_1} - v_1 - \frac{1}{2}, \\
s\frac{du_2}{ds} = F_{\eta_2} - u_2, \quad s\frac{dv_2}{ds} = -F_{\xi_2} - v_2 - \frac{1}{2}, \\
\frac{du_3}{ds} = F_{\eta_3}, \quad \frac{dv_3}{ds} = -F_{\xi_3},
\end{gathered}$$

with initial conditions $u_i(0) = v_i(0) = 0$ $(i = 1, 2, 3)$.

We only consider the ejection solution, that is $s > 0$. Let $s = e^{-\tau} > 0$, this system can be rewritten as a system with seven variables u_i, v_i and s:

$$\begin{gathered}
\frac{du_1}{d\tau} = -F_{\eta_1} + u_1, \quad \frac{dv_1}{d\tau} = F_{\xi_1} + v_1 + \frac{1}{2}, \\
\frac{du_2}{d\tau} = -F_{\eta_2} + u_2, \quad \frac{dv_2}{d\tau} = F_{\xi_2} + v_2 + \frac{1}{2}, \\
\frac{du_3}{d\tau} = -sF_{\eta_3}, \quad \frac{dv_3}{d\tau} = sF_{\xi_3}, \\
\frac{ds}{d\tau} = -s.
\end{gathered} \tag{5.12}$$

For simplicity, we use different notation:

$$\frac{d\sigma_k}{d\tau} = \Sigma_{l=1}^{7} b_{kl}\sigma_l + \varphi_k, \quad (k = 1, \dots, 7)$$

where $\sigma = (\sigma_1, \dots, \sigma_7)^T = (u_1, u_2, v_1, v_2, u_3, v_3, s)^T$. The initial value is $\sigma_k = 0$ (k=1,…,7) and φ_k are power series in σ_1, …,σ_7 beginning with quadratic terms, and b_{kl} are real constants. From the differential system (5.12), we can calculate the

7×7 linearized matrix (b_{kl}) at $s = 0$:

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \omega \\ 1 & 0 & 0 & 0 & 0 & 0 & \omega \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \tag{5.13}$$

where $\omega = \frac{1}{4}h - \frac{1}{8}\widehat{\eta}_3^2 + \frac{1}{\widehat{\xi}_3}$.

Theorem 5.3. *The differential system*

$$-s\frac{d\sigma}{ds} = B\sigma + \varphi, \quad \varphi = (\varphi_1, \dots, \varphi_7)^T,$$

has the initial condition $\sigma = (\sigma_1, \dots, \sigma_7)^T = 0$ and B is define by (5.13)*, where $\omega = \frac{1}{4}h - \frac{1}{8}\widehat{\eta}_3^2 + \frac{1}{\widehat{\xi}_3}$. Also, φ_k (k=1,2. . . ,7) are power series in σ_1, . . . ,σ_7 beginning with quadratic terms. Then this system has an analytic solution σ for s in a sufficiently small neighborhood of* 0.

Proof. Note that the eigenvalues of B are -1, -1, 0, 0, 1, 1, 3 and B is similar to a diagonal matrix. The standard technique to prove this theorem is the method of majorants, which can be found in Saari's work [18]. The details are omitted here. □

Theorem 5.3 shows that any formal power series solution of the coupled system is actually convergent in a small neighborhood of 0. That is, the coupled system has analytic solutions passing through SBC. On the other hand, we know that the solutions in the decoupled system form a one-parameter set, and all the solutions are analytic in a small neighborhood of $s = 0$. Let

$$\begin{aligned} D &= \lim_{s\to 0}[\frac{1+\xi_1}{\xi_1\eta_1^2} - \frac{h}{2}] = \lim_{s\to 0}[-\frac{1+\xi_2}{\xi_2\eta_2^2} + \frac{h}{2}] \\ &= \lim_{s\to 0}\big[\frac{\xi_1 - \xi_2}{\xi_1\eta_1^2 + \xi_2\eta_2^2} + \frac{(\xi_2\eta_2^2 - \xi_1\eta_1^2)h}{2(\xi_1\eta_1^2 + \xi_2\eta_2^2)}\big]. \end{aligned} \tag{5.14}$$

Similar to the decoupled case, the physical meaning of D is $\bar{C} - \frac{h}{2}$, where $\bar{C}$ is the total energy of the left collision pair (m_1 and m_2) at SBC. Next, we show that there is a one-to-one correspondence between solutions in the coupled system and solutions in the decoupled system.

Theorem 5.4. *Let $E = h$ be the total Hamiltonian energy of the system in the coupled case. In the differential system* (5.3) *to* (5.9) *of the coupled case, there are infinitely many solutions $(\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3)$. All of the solutions are analytic in a small neighborhood of $s = 0$ and they form a one-parameter set, where D in* (5.14) *is the parameter. Furthermore, for any given initial condition*

$$\xi_1(0) = \xi_2(0) = -1, \quad \eta_1(0) = \eta_2(0) = 0, \quad \xi_3(0) = \widehat{\xi}_3, \quad \eta_3(0) = \widehat{\eta}_3$$

at SBC and fixed total energy $E = h$, there is a one-to-one correspondence between solutions $(\xi_1, \xi_2, \eta_1, \eta_2)$ in the coupled system and the decoupled system.

Proof. Let $\{\xi_i(s,D),\eta_i(s,D)\}$ $(i=1,2,3)$ be a formal series solution of the coupled system (5.3) to (5.9), where D as in formula (5.14). By Theorem 5.3, this series solution is convergent and it is a real solution of the coupled system. Note that

$$D=\lim_{s\to 0}[\frac{1+\xi_1}{\xi_1\eta_1^2}-\frac{h}{2}]=\bar{C}-\frac{h}{2},$$

where $\bar{C}$ is the total energy of the left collision pair (m_1 and m_2) at SBC. When SBC happening, the total energy of the left collision pair (m_1 and m_2) is arbitrary. Hence, for any given constant D_1, one can construct such a formal series solution $\{\xi_i(s,D_1),\eta_i(s,D_1)\}$ $(i=1,2,3)$, which is convergent. (The power series form can be found in Appendices.) It follows that there exists a one-parameter set of analytic solutions of the coupled system (5.3) to (5.9).

On the other hand, we claim that all the solutions of the coupled system are analytic. If there exists a non-analytic solution in the coupled system, By Lemma 5.2, its limit in the decoupled case is also non-analytic and this limit is a solution in the decoupled case. However, theorem 3.14 implies that the solutions of the decoupled system are all analytic and they form a one-parameter set with D as the parameter. Contradiction! Therefore, for any given $\xi_3(0)=\widehat{\xi}_3$, $\eta_3(0)=\widehat{\eta}_3$ at SBC and fixed total energy $E=h$, there is a one-to-one correspondence between solutions in the coupled system and solutions in the decoupled system. $\square$

Appendix

By calculation, the series solution of N_1 and N_2 are as follows

$$\begin{aligned}N_1(s)&=\frac{1}{2}s-\frac{1}{20}s^3+\frac{1}{160}s^5-\frac{29}{36000}s^7+\dots,\\ N_2(s)&=\frac{1}{2}s+\frac{1}{20}s^3+\frac{1}{160}s^5+\frac{29}{36000}s^7+\dots\end{aligned}$$

Then the solutions $\{\xi_1,\xi_2,\eta_1,\eta_2\}$ for the decoupled system on the energy surface $E=h=0$ are

$$\begin{aligned}\xi_1(s,C)&=-1-\frac{1}{4}Cs^2-\frac{1}{80}C^2s^4+\frac{1}{1600}C^3s^6+\frac{7}{288000}C^4s^8+\dots,\\ \xi_2(s,C)&=-1+\frac{1}{4}Cs^2-\frac{1}{80}C^2s^4-\frac{1}{1600}C^3s^6+\frac{7}{288000}C^4s^8+\dots,\\ \eta_1(s,C)&=\frac{1}{2}s-\frac{C}{20}s^3+\frac{C^2}{160}s^5-\frac{29C^3}{36000}s^7+\dots,\\ \eta_2(s,C)&=\frac{1}{2}s+\frac{C}{20}s^3+\frac{C^2}{160}s^5+\frac{29C^3}{36000}s^7+\dots,\end{aligned}$$

where

$$C=\lim_{t\to 0}(y_1^2-\frac{1}{x_1})=\lim_{t\to 0}\frac{x_1y_1^2-x_2y_2^2}{x_1+x_2}=\lim_{s\to 0}\frac{1+\xi_1}{\xi_1\eta_1^2}=\lim_{s\to 0}\frac{\xi_1-\xi_2}{\xi_1\eta_1^2+\xi_2\eta_2^2}.$$

In the decoupled case with total energy $E=h$, the solutions are

$$\begin{aligned}\xi_1(s,D)&=-1+\big(-\frac{1}{8}h-\frac{1}{4}D\big)s^2+\big(-\frac{1}{192}h^2-\frac{1}{60}hD-\frac{1}{80}D^2\big)s^4\\ &\quad+\big(-\frac{1}{11520}h^3-\frac{1}{4032}Dh^2+\frac{11}{67200}D^2h+\frac{1}{1600}D^3\big)s^6+O(s^8),\\ \xi_2(s,D)&=-1+\big(-\frac{1}{8}h+\frac{1}{4}D\big)s^2+\big(-\frac{1}{192}h^2+\frac{1}{60}hD-\frac{1}{80}D^2\big)s^4\end{aligned}$$

$$
\begin{aligned}
&\quad + \big(- \frac{1}{11520} h^3 + \frac{1}{4032} D h^2 + \frac{11}{67200} D^2 h - \frac{1}{1600} D^3 \big) s^6 + O(s^8), \\
\eta_1(s,D) &= \frac{1}{2} s + \big(- \frac{1}{48} h - \frac{1}{20} D \big) s^3 + \big(\frac{1}{960} h^2 + \frac{1}{160} D^2 + \frac{3}{560} h D \big) s^5 \\
&\quad + \big(- \frac{17}{322560} h^3 - \frac{19}{44800} h^2 D - \frac{139}{134400} h D^2 - \frac{29}{36000} D^3 \big) s^7 + O(s^9), \\
\eta_2(s,D) &= \frac{1}{2} s + \big(- \frac{1}{48} h + \frac{1}{20} D \big) s^3 + \big(\frac{1}{960} h^2 + \frac{1}{160} D^2 - \frac{3}{560} h D \big) s^5 \\
&\quad + \big(- \frac{17}{322560} h^3 + \frac{19}{44800} h^2 D - \frac{139}{134400} h D^2 + \frac{29}{36000} D^3 \big) s^7 + O(s^9),
\end{aligned}
$$

where

$$
D = \frac{x_1 y_1^2 - x_2 y_2^2 + x_2 h}{x_1 + x_2} - \frac{h}{2} = C + \frac{h(x_2 - x_1)}{2(x_1 + x_2)} = C + \frac{(\xi_2 \eta_2^2 - \xi_1 \eta_1^2) h}{2(\xi_1 \eta_1^2 + \xi_2 \eta_2^2)}.
$$

If $h = 0$, $D = C$, the above solutions $\{\xi_i(s,D), \eta_i(s,D)\}$ $(i = 1,2)$ match the solutions $\{\xi_i(s,C), \eta_i(s,C)\}$ $(i = 1,2)$ in the decoupled case with $h = 0$.

For the coupled system with $m_1 = m_2 = m_3 = m_4 = 1$, the power series solutions are

$$
\begin{aligned}
\xi_1 &= -1 + \big(- \frac{1}{8} W - \frac{1}{4} D \big) s^2 + \big(- \frac{1}{192} W^2 - \frac{1}{60} W D - \frac{1}{80} D^2 \big) s^4 \\
&\quad + \big(- \frac{1}{11520} W^3 - \frac{1}{4032} D W^2 + \frac{11}{67200} D^2 W + \frac{1}{1600} D^3 \big) s^6 + O(s^8), \\
\xi_2 &= -1 + \big(- \frac{1}{8} W + \frac{1}{4} D \big) s^2 + \big(- \frac{1}{192} W^2 + \frac{1}{60} W D - \frac{1}{80} D^2 \big) s^4 \\
&\quad + \big(- \frac{1}{11520} W^3 + \frac{1}{4032} D W^2 + \frac{11}{67200} D^2 W - \frac{1}{1600} D^3 \big) s^6 + O(s^8), \\
\eta_1 &= \frac{1}{2} s + \big(- \frac{1}{48} W - \frac{1}{20} D \big) s^3 + \big(\frac{1}{960} W^2 + \frac{1}{160} D^2 + \frac{3}{560} W D \big) s^5 \\
&\quad + \big(- \frac{17}{322560} W^3 - \frac{19}{44800} W^2 D - \frac{139}{134400} W D^2 - \frac{29}{36000} D^3 \big) s^7 + O(s^9), \\
\eta_2 &= \frac{1}{2} s + \big(- \frac{1}{48} W + \frac{1}{20} D \big) s^3 + \big(\frac{1}{960} W^2 + \frac{1}{160} D^2 - \frac{3}{560} W D \big) s^5 \\
&\quad + (- \frac{17}{322560} W^3 + \frac{19}{44800} W^2 D - \frac{139}{134400} W D^2 + \frac{29}{36000} D^3) s^7 + O(s^9), \\
\xi_3 &= \widehat{\xi}_3 + u_3 = \widehat{\xi}_3 + \frac{1}{24} \widehat{\eta}_3 s^3 + \frac{1}{960} W \widehat{\eta}_3 s^5 \\
&\quad - \frac{1}{288 \widehat{\xi}_3^2} s^6 + \frac{\widehat{\eta}_3}{7} \big(\frac{1}{11520} W^2 - \frac{1}{1600} D^2 \big) s^7 + O(s^8), \\
\eta_3 &= \widehat{\eta}_3 + v_3 = \widehat{\eta}_3 - \frac{1}{6 \widehat{\xi}_3^2} s^3 - \frac{W}{240 \widehat{\xi}_3^2} s^5 \\
&\quad + \frac{\widehat{\eta}_3}{144 \widehat{\xi}_3^3} s^6 + \frac{1}{7 \widehat{\xi}_3^2} \big(- \frac{151}{46080} W^2 + \frac{1}{400} D^2 \big) s^7 + O(s^8),
\end{aligned}
$$

where

$$
\begin{aligned}
D &= \lim_{s \to 0} [\frac{1 + \xi_1}{\xi_1 \eta_1^2} - \frac{h}{2}] = \lim_{s \to 0} [- \frac{1 + \xi_2}{\xi_2 \eta_2^2} + \frac{h}{2}] \\
&= \lim_{s \to 0} \big[\frac{\xi_1 - \xi_2}{\xi_1 \eta_1^2 + \xi_2 \eta_2^2} + \frac{(\xi_2 \eta_2^2 - \xi_1 \eta_1^2) h}{2(\xi_1 \eta_1^2 + \xi_2 \eta_2^2)} \big],
\end{aligned}
$$

and

$$W = 4\omega = 4\Big[\frac{1}{4}h - \frac{1}{8}\widehat{\eta}_3^2 + \frac{1}{\widehat{\xi}_3}\Big] = \lim_{s\to 0} A,$$

with A defined in equation (5.1). By comparing the series forms in the coupled system and the decoupled system, it is clear that $\{\xi_i, \eta_i\}$ $(i = 1, 2)$ in the coupled system becomes the series solution $\{\xi_i(s, D), \eta_i(s, D)\}$ $(i = 1, 2)$ in the decoupled case if we set $\widehat{\eta} = 1/\widehat{\xi}_3 = 0$.

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