Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 86, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A PRESCRIBED MEAN-CURVATURE PROBLEM WITH CRITICAL GROWTH

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Abstract. In this work we study an existence and multiplicity of solutions for the prescribed mean-curvature problem with critical growth,

$$
\begin{gathered}
-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big)= \lambda|u|^{q-2}u+|u|^{2^*-2}u \quad \text{in } \Omega \\
u=0 \quad \text{on } \partial\Omega,
\end{gathered}
$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N\geq 3$ and $1<q<2$. To employ variational arguments, we consider an auxiliary problem which is proved to have infinitely many solutions by genus theory. A clever estimate in the gradient of the solutions of the modified problem is necessary to recover solutions of the original problem.

1. Introduction

In this work we study the existence and multiplicity of solutions for quasilinear problems with nonlinearity of Brézis-Nirenberg type (see [4])

$$
\begin{gathered}
-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big)= \lambda|u|^{q-2}u+|u|^{2^*-2}u \quad \text{in } \Omega \\
u=0 \quad \text{on } \partial\Omega,
\end{gathered}
\tag{1.1}
$$

where $\Omega\subset\mathbb{R}^N$ is a bounded smooth domain, $\lambda>0$, $1<q<2$ and $2^*=\frac{2N}{N-2}$. This problem has applications not just to describe a surface given by $u(x)$, whose mean curvature is described by the right hand side of (1.1), but also in *capillarity theory* where when the nonlinearity is replaced by κu, the resultant equation describe the equilibrium of a liquid surface with constant surface tension in a uniform gravity field [18, p. 262].

Problems like (1.1) has been intensively studied over the previous decades. In [6], the authors studied a related subcritical problem in which they obtained positive solutions. In [11], the authors proved the existence of infinitely many solutions for a subcritical version of (1.1). In the recent work [3], Bonheure, Derlet and Valeriola studied a purely subcritical version of (1.1), where they proved the existence and

2010 *Mathematics Subject Classification.* 35J93, 35J62, 35J20.
Key words and phrases. Prescribed mean-curvature problem; critical exponent; variational methods.

Submitted March 24, 2014. Published April 7, 2015.

multiplicity of nodal $H_0^1(\Omega)$ solutions, to sufficiently large values of λ. They overcame the difficulty in working in the $BV(\Omega)$ space, which is the natural functional space to treat (1.1), by doing a truncation in the degenerate part of the mean-curvature operator in order to make possible construct a variational framework in the Sobolev space $H_0^1(\Omega)$. Nevertheless, this truncation requires sharp estimates on the gradient of the solutions, in order to prove that the solutions of the modified problem in fact are solutions of the original one.

When $\Omega = \mathbb{R}^N$ and the nonlinearity is substituted by u^q; i.e., the Gidas-Spruck analogue for the mean-curvature operator, Ni and Serr [12, 13] proved that if $1 < q < \frac{N}{N-2}$ no positive solution exist, while for $q \geq 2^* - 1$ there exist infinitely many solutions. In the range $\frac{N}{N-2} < q < 2^* - 1$ some contributions has been given by Clément et al [5] and by Del Pino and Guerra [7], where in the latter the authors prove that many positive solutions do exist if $q < 2^* - 1$ is sufficiently close to $2^* - 1$.

Still in the case $\Omega = \mathbb{R}^N$ but with nonlinearity given by $\lambda u + u^p$, Peletier and Serr [14] succeed in proving the existence of positive radial solutions when $\lambda < 0$ is small enough and p is subcritical. In the case $\lambda > 0$, they stated there is no regular solution to that problem no matter how much small or large p is.

In this work, because of the boundedness of Ω, we prove a result in a strike opposition of that [14], in which we obtain the existence of infinitely many regular solutions of (1.1), for small enough $\lambda > 0$. More specifically, we prove the following result.

Theorem 1.1. *If $1 < q < 2$, then there exists $\lambda^* > 0$ such that if $0 < \lambda < \lambda^*$,* (1.1) *has infinitely many solutions. Moreover, if u_λ is a solution of* (1.1)*, then $u_\lambda \in H_0^1(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1)$, and*

$$\lim_{\lambda \to 0} \|u_\lambda\| = \lim_{\lambda \to 0} \|u_\lambda\|_\infty = \lim_{\lambda \to 0} \|\nabla u_\lambda\|_\infty = 0,$$

where $\|\cdot\|$ is the Sobolev norm in $H_0^1(\Omega)$.

Our approach follows the main ideas of Bonheure et al [3], to make possible consider a related modified problem in $H_0^1(\Omega)$. Afterwards, to get solutions of the modified problem we apply Krasnoselskii genus theory in the same way that Azorero and Alonso [1]. Finally, we use the Moser iteration technique and a regularity result by Lewy and Stampacchia [17] to get decay in λ of the gradient of the solutions, which will imply that the solutions of the modified problem in fact are solutions of the original one.

It is worth pointing out that in fact, our result with minor modifications could be used to prove the existence of infinitely many solutions of a supercritical problem like

$$\begin{gathered} -\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = \lambda |u|^{q-2}u + |u|^{s-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where $s > 2^*$. If this were the case, its enough to proceed as in [8, 19, 16], by truncating the nonlinearity, substituting it by

$$f_K(t) = \begin{cases} \lambda |t|^{q-2}t + |t|^{s-2}t & \text{if } |t| \leq K \\ \lambda |t|^{q-2}t + K^{s-2^*}|t|^{2^*-2}t & \text{if } |t| > K \end{cases}$$

where $K > 0$. Since f_K is an odd continuous function and $f_K(t)| \leq \lambda|t|^{q-1} + K^{s-2^*}|t|^{2^*-1}$ for all $t \in \mathbb{R}$, just few modifications in some of our technical results allow one to obtain infinitely many solutions of the truncated problem for λ small enough. Hence, if u_λ is one of such solutions, as $\lim_{\lambda\to 0} \|\nabla u_\lambda\|_\infty = 0$, for λ small enough $\|\nabla u_\lambda\|_\infty < K$ and then u_λ would be a solution of $(\bar{P}_\lambda)$.

This article is organized as follows. In the second section we present the auxiliary problem and the variational framework. In the third one we make a brief review of Genus theory. In the fourth we prove some technical results which imply on the existence of infinitely many solutions of the auxiliary problem. The last one is dedicated to present the proof of the main result, which consists in estimates in $L^\infty(\Omega)$ norm of the gradient of solutions.

2. Auxiliary problem and variational framework

Let us consider $r \geq 0$, $\delta > 0$ and a function $\eta \in C^1([r, r+\delta])$ such that

$$\eta(r) = \frac{1}{\sqrt{1+r}}, \quad \eta(r+\delta) = \frac{1}{\sqrt{1+r+\delta}},$$
$$\eta'(r) = -\frac{1}{2\sqrt{(1+r)^3}}, \quad \eta'(r+\delta) = 0.$$

Now we define

$$a(t) := \begin{cases} \frac{1}{\sqrt{1+t}}, & \text{if } 0 \leq t \leq r, \\ \eta(t), & \text{if } r \leq t \leq r+\delta, \\ K_0 = \frac{1}{\sqrt{1+r+\delta}}, & \text{if } t \geq r+\delta. \end{cases}$$

Note that $a \in C^1([0,\infty))$ is decreasing and $K_0 \leq a(t) \leq 1$ for $t \in [0,\infty)$. Let us fix $r > 0$ such that

$$\frac{2}{2^*} < K_0 < 1. \tag{2.1}$$

The proof of the Theorem 1.1 is based on a careful study of solutions of the auxiliary problem

$$\begin{gathered} -\operatorname{div}(a(|\nabla u|^2)\nabla u) = \lambda|u|^{q-2}u + |u|^{2^*-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.2}$$

We say that $u \in H_0^1(\Omega)$ is a weak solution (2.2) if it satisfies

$$\int_\Omega a(|\nabla u|^2)\nabla u \nabla\phi\,dx = \lambda\int_\Omega |u|^{q-2}u\phi\,dx + \int_\Omega |u|^{2^*-2}u\phi\,dx,$$

for all $\phi \in H_0^1(\Omega)$. Let us consider $H_0^1(\Omega)$ with its usual norm $\|u\| = \big(\int_\Omega |\nabla u|^2\big)^{1/2}$ and define the C^1-functional $I_\lambda : H_0^1(\Omega) \to \mathbb{R}$ by

$$I_\lambda(u) = \frac{1}{2}\int_\Omega A(|\nabla u|^2)\,dx - \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}\,dx,$$

where $A(t) = \int_0^t a(s)\,ds$. Note that

$$I'_\lambda(u)\phi = \int_\Omega a(|\nabla u|^2)\nabla u\nabla\phi\,dx - \lambda\int_\Omega |u|^{q-2}u\phi\,dx - \int_\Omega |u|^{2^*-2}u\phi\,dx,$$

for all $\phi \in H_0^1(\Omega)$ and then, critical points of I_λ are weak solutions of (2.2).

To use variational methods, we first derive some results related to the Palais-Smale compactness condition.

We say that a sequence $(u_n) \subset H_0^1(\Omega)$ is a $(PS)_{c_\lambda}$ sequence for I_λ if

$$I_\lambda(u_n) \to c_\lambda \text{ and } \|I'_\lambda(u_n)\|_{H^{-1}(\Omega)} \to 0, \quad \text{as } n \to \infty, \tag{2.3}$$

where

$$c_\lambda = \inf_{\pi\in\Gamma} \max_{t\in[0,1]} I_\lambda(\pi(t)) > 0,$$

$$\Gamma := \{\pi \in C([0,1], H_0^1(\Omega)) : \pi(0) = 0,\ I_\lambda(\pi(1)) < 0\}.$$

If (2.3) implies the existence of a subsequence $(u_{n_j}) \subset (u_n)$ which converges in $H_0^1(\Omega)$, we say that I_λ satisfies the Palais-Smale condition on the level c_λ.

3. Genus theory

We start by considering some basic facts on the Krasnoselskii genus theory that we will use in the proof of Theorem 1.1.

Let E be a real Banach space. Let us denote by $\mathfrak{A}$ the class of all closed subsets $A \subset E \setminus \{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$.

Definition 3.1. Let $A \in \mathfrak{A}$. The Krasnoselskii genus $\gamma(A)$ of A is defined as being the least positive integer k such that there is an odd mapping $\phi \in C(A, \mathbb{R}^k)$ such that $\phi(x) \neq 0$ for all $x \in A$. When such number does not exist we set $\gamma(A) = \infty$. Furthermore, by definition, $\gamma(\emptyset) = 0$.

In the sequel we establish only the properties of the genus that will be used through this work. More informations on this subject may be found [10].

Theorem 3.2. *Let $E = \mathbb{R}^N$ and $\partial\Omega$ be the boundary of an open, symmetric and bounded subset $\Omega \subset \mathbb{R}^N$ such that $0 \in \Omega$. Then $\gamma(\partial\Omega) = N$.*

Corollary 3.3. $\gamma(S^{N-1}) = N$.

Proposition 3.4. *If $K \in \mathfrak{A}$, $0 \notin K$ and $\gamma(K) \geq 2$, then K has infinitely many points.*

4. Technical results

The genus theory requires that the functional I_λ is bounded from below. Since this is not the case, it is necessary to work with a related functional, which will be done employing some ideas contained [1].

In light of Proposition 3.4, it seems to be useful proving that the set of critical points of the related functional has genus greater than 2, to obtain infinitely many solutions of (2.2).

Let us present the way in which we truncate the function I_λ . From (2.1) and Sobolev's embedding, we obtain

$$I_\lambda(u) \geq \frac{K_0}{2}\|u\|^2 - \frac{\lambda}{qS_q^{q/2}}\|u\|^q - \frac{1}{2^*S^{2^*/2}}\|u\|^{2^*} = g(\|u\|^2),$$

S and S_q are, respectively, the best constants of the Sobolev's embeddings $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ and $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ and

$$g(t) = \frac{K_0}{2}t - \frac{\lambda}{qS_q^{q/2}}t^{q/2} - \frac{1}{2^*S^{2^*/2}}t^{2^*/2}. \tag{4.1}$$

Hence, there exists $\tau_1 > 0$ such that, if $\lambda \in (0, \tau_1)$, g attains its positive maximum.

Let $R_0 < R_1$ the roots of g. We have that $R_0 = R_0(\tau_1)$ and the following result holds.

Lemma 4.1.
$$R_0(\tau_1) \to 0 \quad as\ \lambda \to 0. \tag{4.2}$$

Proof. From $g(R_0(\tau_1)) = 0$ and $g'(R_0(\tau_1)) > 0$, we have
$$\frac{K_0}{2}R_0(\tau_1) = \frac{\lambda}{qS_q^{q/2}}R_0(\tau_1)^{q/2} + \frac{1}{2^*S^{2^*/2}}R_0(\tau_1)^{2^*/2}, \tag{4.3}$$
$$\frac{K_0}{2} > \frac{\lambda}{2qS_q^{q/2}}R_0(\tau_1)^{(q-2)/2} + \frac{1}{2S^{2^*/2}}R_0(\tau_1)^{(2^*-2)/2}, \tag{4.4}$$
for all $\lambda \in (0, \tau_1)$. From (4.3), we conclude that $R_0(\tau_1)$ is bounded. Suppose that $R_0(\tau_1) \to R_0 > 0$ as $\lambda \to 0$. Then
$$\frac{K_0}{2} = \frac{1}{2^*S^{2^*/2}}R_0(\tau_1)^{(2^*-2)/2}, \tag{4.5}$$
$$\tag{4.6}$$
$$\frac{K_0}{2} \geq \frac{1}{2S^{2^*/2}}R_0(\tau_1)^{(2^*-2)/2}, \tag{4.7}$$
which is a contradiction, because $2^* > 2$. Therefore $R_0 = 0$. $\square$

We consider τ_1 such that $R_0 \leq r$ and we modify the functional I_λ in the following way. Take $\phi \in C^\infty([0, +\infty))$, $0 \leq \phi \leq 1$ such that $\phi(t) = 1$ if $t \in [0, R_0]$ and $\phi(t) = 0$ if $t \in [R_1, +\infty)$. Now, we consider the truncated functional
$$J_\lambda(u) = \frac{1}{2}\int_\Omega A(|\nabla u|^2)\,dx - \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \phi(\|u\|^2)\frac{1}{2^*}\int_\Omega |u|^{2^*}\,dx.$$
Note that $J_\lambda \in C^1(H_0^1(\Omega), \mathbb{R})$ and, as in (4.1), $J_\lambda(u) \geq \overline{g}(\|u\|^2)$, where
$$\overline{g}(t) = \frac{K_0}{2}t - \frac{\lambda}{qS_q^{q/2}}t^{q/2} - \phi(t)\frac{1}{2^*S^{2^*/2}}t^{2^*/2}.$$

Let us remark that if $\|u\|^2 \leq R_0$, then $J_\lambda(u) = I_\lambda(u)$ and if $\|u\|^2 \geq R_1$, then
$$J_\lambda(u) = \frac{1}{2}\int_\Omega A(|\nabla u|^2)\,dx - \frac{\lambda}{q}\int_\Omega |u|^q\,dx,$$
which implies that J_λ is coercive and hence bounded from below.

Now we show that J_λ satisfy the local Palais-Smale condition. For this, we need the following technical result, which is analogous of [1, Lemma 4.2].

Lemma 4.2. *Let $(u_n) \subset H_0^1(\Omega)$ be a bounded sequence such that*
$$I_\lambda(u_n) \to c_\lambda \quad and \quad I'_\lambda(u_n) \to 0.$$
If
$$\begin{aligned} c_\lambda &< (\frac{K_0}{2} - \frac{1}{2^*})K_0^{(N-2)/2}S^{N/2} - \lambda(\frac{1}{q} - \frac{1}{2^*})|\Omega|^{\frac{(2^*-q)}{2}} \\ &\quad \times \Big[\frac{q}{2^*}\lambda\big(\frac{1}{q} - \frac{1}{2^*}\big)|\Omega|^{(2^*-q)/2^*}\Big(\big(\frac{K_0}{2} - \frac{1}{2^*}\big)\frac{1}{S^{2^*/2}}\Big)^{-1}\Big]^{\frac{q}{(2^*-q)}} \end{aligned}$$
hold, then, up to a subsequence, (u_n) is strongly convergent in $H_0^1(\Omega)$.

Proof. Taking a subsequence, we may suppose that

$$|\nabla u_n|^2 \rightharpoonup |\nabla u|^2 + \sigma \quad \text{and} \quad |u_n|^{2^*} \rightharpoonup |u|^{2^*} + \nu$$

in the weak* sense of measures.

Using the concentration compactness-principle due to Lions [9, Lemma 2.1], we obtain an at most countable index set Λ, sequences $(x_i) \subset \Omega$, $(\mu_i), (\sigma_i), (\nu_i), \subset [0, \infty)$, such that

$$\nu = \sum_{i\in\Lambda} \nu_i \delta_{x_i}, \quad \sigma \geq \sum_{i\in\Lambda} \sigma_i \delta_{x_i}, \quad S\nu_i^{2/2^*} \leq \sigma_i, \tag{4.8}$$

for all $i \in \Lambda$, where δ_{x_i} is the Dirac mass at $x_i \in \Omega$.

Now we claim that $\Lambda = \emptyset$. Arguing by contradiction, assume that $\Lambda \neq \emptyset$ and fix $i \in \Lambda$. Consider $\psi \in C_0^\infty(\Omega, [0, 1])$ such that $\psi \equiv 1$ on $B_1(0)$, $\psi \equiv 0$ on $\Omega \setminus B_2(0)$ and $|\nabla\psi|_\infty \leq 2$. Defining $\psi_\varrho(x) := \psi((x-x_i)/\varrho)$ where $\varrho > 0$, we have that $(\psi_\varrho u_n)$ is bounded. Thus $I'_\lambda(u_n)(\psi_\varrho u_n) \to 0$; that is,

$$\begin{aligned} &\int_\Omega a(|\nabla u_n|^2) u_n \nabla u_n \nabla \psi_\varrho \, dx + \int_\Omega a(|\nabla u_n|^2)\psi_\varrho |\nabla u_n|^2 \, dx \\ &= \lambda \int_\Omega |u_n|^q \psi_\varrho \, dx + \int_\Omega \psi_\varrho |u_n|^{2^*} \, dx + o_n(1). \end{aligned}$$

Since $\operatorname{supp}(\psi_\varrho) \subset B_{2\varrho}(x_i)$, we obtain

$$\Big|\int_\Omega u_n \nabla u_n \nabla \psi_\varrho \, dx\Big| \leq \int_{B_{2\rho}(x_i)} |\nabla u_n||u_n \nabla \psi_\varrho| \, dx.$$

By Hölder inequality and the fact that the sequence (u_n) is bounded in $H_0^1(\Omega)$ we have

$$\Big|\int_\Omega u_n \nabla u_n \nabla \psi_\varrho \, dx\Big| \leq C\Big(\int_{B_{2\varrho}(x_i)} |u_n \nabla \psi_\varrho|^2 \, dx\Big)^{1/2}.$$

By the Dominated Convergence Theorem $\int_{B_{2\varrho}(x_i)} |u_n \nabla\psi_\varrho|^2 \, dx \to 0$ as $n \to +\infty$ and $\varrho \to 0$. Thus, we obtain

$$\lim_{\varrho\to 0} \Big[\lim_{n\to\infty} \int_\Omega u_n \nabla u_n \nabla \psi_\varrho \, dx\Big] = 0.$$

Since $0 < K_0 \leq a(t) \leq 1$, for all $t \in \mathbb{R}$, we obtain

$$\lim_{\varrho\to 0} \lim_{n\to\infty} \Big[\int_\Omega a(|\nabla u_n|^2) u_n \nabla u_n \nabla \psi_\varrho \, dx\Big] = 0.$$

Moreover, similar arguments applies in order to obtain

$$\lim_{\varrho\to 0} \lim_{n\to\infty} \Big[\int_\Omega \psi_\varrho |u_n|^q \, dx\Big] = 0.$$

Thus, we have

$$K_0 \int_\Omega \psi_\varrho \mathrm{d}\sigma \leq \int_\Omega \psi_\varrho \mathrm{d}\nu + o_\varrho(1).$$

Letting $\varrho \to 0$ and using standard theory of Radon measures, we conclude that $K_0\sigma_i \leq \nu_i$. It follows from (4.8) that

$$\sigma_i \geq K_0^{(N-2)/2} S^{N/2}. \tag{4.9}$$

Now we shall prove that the above expression cannot occur, and therefore the set Λ is empty. Indeed, if for some $i \in \Lambda$ (4.9) hold, then

$$c_\lambda = I_\lambda(u_n) - \frac{1}{2^*} I'_\lambda(u_n)u_n + o_n(1)$$

which implies

$$c_\lambda \geq (\frac{K_0}{2} - \frac{1}{2^*}) \int_\Omega |\nabla u_n|^2 \,dx - \lambda(\frac{1}{q} - \frac{1}{2^*}) \int_\Omega |u_n|^q \,dx.$$

Since $\frac{2}{2^*} < K_0 < 1$ (see (2.1)), letting $n \to \infty$ we obtain

$$c_\lambda \geq (\frac{K_0}{2} - \frac{1}{2^*})\sigma_i + (\frac{K_0}{2} - \frac{1}{2^*}) \int_\Omega |\nabla u|^2 \,dx - \lambda(\frac{1}{q} - \frac{1}{2^*}) \int_\Omega |u|^q \,dx.$$

Hence,

$$c_\lambda \geq (\frac{K_0}{2} - \frac{1}{2^*})K_0^{(N-2)/2} S^{N/2} + (\frac{K_0}{2} - \frac{1}{2^*}) \int_\Omega |\nabla u|^2 \,dx - \lambda(\frac{1}{q} - \frac{1}{2^*}) \int_\Omega |u|^q \,dx.$$

By Hölder's inequality and Sobolev's embedding we obtain

$$\begin{aligned} c_\lambda &\geq (\frac{K_0}{2} - \frac{1}{2^*})K_0^{(N-2)/2} S^{N/2} + (\frac{K_0}{2} - \frac{1}{2^*}) \frac{1}{S^{2^*/2}} \int_\Omega |u|^{2^*} \,dx \\ &\quad - \lambda(\frac{1}{q} - \frac{1}{2^*})|\Omega|^{\frac{(2^*-q)}{2^*}} \Big(\int_\Omega |u|^{2^*} \,dx\Big)^{q/2^*}. \end{aligned}$$

Note that

$$f(t) = (\frac{K_0}{2} - \frac{1}{2^*}) \frac{1}{S^{2^*/2}} t^{2^*} - \lambda(\frac{1}{q} - \frac{1}{2^*})|\Omega|^{\frac{(2^*-q)}{2^*}} t^q$$

is a continuous function that attains its absolute minimum, for $t > 0$, at the point

$$\alpha_0 = \Big[\frac{q}{2^*}\lambda\big(\frac{1}{q} - \frac{1}{2^*}\big)|\Omega|^{(2^*-q)/2^*} \Big(\big(\frac{K_0}{2} - \frac{1}{2^*}\big)\frac{1}{S^{2^*/2}}\Big)^{-1}\Big]^{\frac{1}{(2^*-q)}}.$$

Then

$$c_\lambda \geq (\frac{K_0}{2} - \frac{1}{2^*})K_0^{(N-2)/2} S^{N/2} + (\frac{K_0}{2} - \frac{1}{2^*}) \frac{1}{S^{2^*/2}} \alpha_0^{2^*} - \lambda(\frac{1}{q} - \frac{1}{2^*})|\Omega|^{\frac{(2^*-q)}{2}} \alpha_0^q.$$

So

$$c_\lambda \geq (\frac{K_0}{2} - \frac{1}{2^*})K_0^{(N-2)/2} S^{N/2} - \lambda(\frac{1}{q} - \frac{1}{2^*})|\Omega|^{\frac{(2^*-q)}{2}} \alpha_0^q.$$

Thus, we conclude that

$$\begin{aligned} c_\lambda &\geq (\frac{K_0}{2} - \frac{1}{2^*})K_0^{(N-2)/2} S^{N/2} - \lambda(\frac{1}{q} - \frac{1}{2^*})|\Omega|^{\frac{(2^*-q)}{2}} \\ &\quad \times \Big[\frac{q}{2^*}\lambda\big(\frac{1}{q} - \frac{1}{2^*}\big)|\Omega|^{(2^*-q)/2^*} \Big(\big(\frac{K_0}{2} - \frac{1}{2^*}\big)\frac{1}{S^{2^*/2}}\Big)^{-1}\Big]^{\frac{q}{(2^*-q)}}, \end{aligned}$$

which is a contradiction. Thus Λ is empty and it follows that $u_n \to u$ in $L^{2^*}(\Omega)$. Thus, up to a subsequence,

$$\|u_n - u\|^2 \leq \frac{1}{K_0} \int_\Omega a(|\nabla u_n|^2)|\nabla u_n - \nabla u|^2 = I_\lambda(u_n)u_n - I_\lambda(u_n)u + o_n(1) = o_n(1).$$

$\square$

By Lemma 4.2 we conclude that, there exists $\tau_2 > 0$ such that, for all $\lambda \in (0, \tau_2)$ we obtain

$$\begin{aligned}&(\frac{K_0}{2} - \frac{1}{2^*})K_0^{(N-2)/2} S^{N/2} - \lambda(\frac{1}{q} - \frac{1}{2^*})|\Omega|^{\frac{(2^*-q)}{2}} \\ &\times \Big[\frac{q}{2^*}\lambda(\frac{1}{q} - \frac{1}{2^*})|\Omega|^{(2^*-q)/2^*}\Big((\frac{K_0}{2} - \frac{1}{2^*})\frac{1}{S^{2^*/2}}\Big)^{-1}\Big]^{\frac{q}{(2^*-q)}} > 0\end{aligned}$$

and, hence, if (u_n) is a bounded sequence such that $I_\lambda(u_n) \to c$, $I'_\lambda(u_n) \to 0$ with $c < 0$, then (u_n) has a convergent subsequence.

Lemma 4.3. *If $J_\lambda(u) < 0$, then $\|u\|^2 < R_0 \le r$ and $J_\lambda(v) = I_\lambda(v)$, for all v in a small neighborhood of u. Moreover, J_λ satisfies a local Palais-Smale condition for $c < 0$.*

Proof. Since $\overline{g}(\|u\|^2) \le J_\lambda(u) < 0$, then $\|u\|^2 < R_0 \le r$. By the choice of τ_1 in (4.2) we have that $J_\lambda(u) = I_\lambda(u)$. Moreover, since J_λ is continuous, we conclude that $J_\lambda(v) = I_\lambda(v)$, for all $v \in B_{R_0/2}(0)$. Besides, if (u_n) is a sequence such that $J_\lambda(u_n) \to c < 0$ and $J'_\lambda(u_n) \to 0$ as $n \to \infty$, then for n sufficiently large $I_\lambda(u_n) = J_\lambda(u_n) \to c < 0$ and $I'_\lambda(u_n) = J'_\lambda(u_n) \to 0$ as $n \to \infty$. Since J_λ is coercive, we obtain that (u_n) is bounded in $H_0^1(\Omega)$. From Lemma 4.2, for all $\lambda \in (0, \tau_2)$, we obtain

$$\begin{aligned}c < 0 < &(\frac{K_0}{2} - \frac{1}{2^*})K_0^{(N-2)/2} S^{N/2} - \lambda(\frac{1}{q} - \frac{1}{2^*})|\Omega|^{\frac{(2^*-q)}{2}} \\ &\times \Big[\frac{q}{2^*}\lambda(\frac{1}{q} - \frac{1}{2^*})|\Omega|^{(2^*-q)/2^*}\Big((\frac{K_0}{2} - \frac{1}{2^*})\frac{1}{S^{2^*/2}}\Big)^{-1}\Big]^{\frac{q}{(2^*-q)}}\end{aligned}$$

and hence, up to a subsequence (u_n) is strongly convergent in $H_0^1(\Omega)$. □

Now, we construct an appropriate minimax sequence of negative critical values.

Lemma 4.4. *Given $k \in \mathbb{N}$, there exists $\epsilon = \epsilon(k) > 0$ such that*

$$\gamma(J_\lambda^{-\epsilon}) \ge k,$$

where $J_\lambda^{-\epsilon} = \{u \in H_0^1(\Omega) : J_\lambda(u) \le -\epsilon\}$.

Proof. Consider $k \in \mathbb{N}$ and let X_k be a k-dimensional subspace of $H_0^1(\Omega)$. Since in X_k all norms are equivalent, there exists $C(k) > 0$ such that

$$-C(k)\|u\|^q \ge -\int_\Omega |u|^q \, dx,$$

for all $u \in X_k$. We now use the inequality above to conclude that

$$J_\lambda(u) \le \frac{1}{2}\|u\|^2 - \frac{C(k)}{q}\|u\|^q = \|u\|^q\Big(\frac{1}{2}\|u\|^{2-q} - \frac{C(k)}{q}\Big).$$

Considering $R > 0$ sufficiently small, there exists $\epsilon = \epsilon(R) > 0$ such that

$$J_\lambda(u) < -\epsilon < 0,$$

for all $u \in \mathcal{S}_R = \{u \in X_k; \|u\| = R\}$. Since X_k and $\mathbb{R}^k$ are isomorphic and $\mathcal{S}_R$ and S^{k-1} are homeomorphic, we conclude from Corollary 3.3 that $\gamma(\mathcal{S}_R) = \gamma(S^{k-1}) = k$. Moreover, once that $\mathcal{S}_R \subset J_\lambda^{-\epsilon}$ and $J_\lambda^{-\epsilon}$ is symmetric and closed, we have

$$k = \gamma(\mathcal{S}_R) \le \gamma(J_\lambda^{-\epsilon}).$$

□

We define now, for each $k \in \mathbb{N}$, the sets

$$\Gamma_k = \{C \subset H : C \text{ is closed },C = -C \text{ and } \gamma(C) \geq k\},$$
$$K_c = \{u \in H : J'_\lambda(u) = 0 \text{ and } J_\lambda(u) = c\}$$

and the number

$$c_k = \inf_{C \in \Gamma_k} \sup_{u \in C} J_\lambda(u).$$

Lemma 4.5. *Given $k \in \mathbb{N}$, the number c_k is negative.*

Proof. From Lemma 4.4, for each $k \in \mathbb{N}$ there exists $\epsilon > 0$ such that $\gamma(J_\lambda^{-\epsilon}) \geq k$. Moreover, $0 \notin J_\lambda^{-\epsilon}$ and $J_\lambda^{-\epsilon} \in \Gamma_k$. On the other hand

$$\sup_{u \in J_\lambda^{-\epsilon}} J_\lambda(u) \leq -\epsilon.$$

Hence,

$$-\infty < c_k = \inf_{C \in \Gamma_k} \sup_{u \in C} J_\lambda(u) \leq \sup_{u \in J_\lambda^{-\epsilon}} J_\lambda(u) \leq -\epsilon < 0.$$

□

The next Lemma allows us to prove the existence of critical points of J_λ.

Lemma 4.6. *If $c = c_k = c_{k+1} = \dots = c_{k+r}$ for some $r \in \mathbb{N}$, then there exists $\lambda^* > 0$ such that*

$$\gamma(K_c) \geq r + 1,$$

for $\lambda \in (0, \lambda^*)$.

Proof. Since $c = c_k = c_{k+1} = \dots = c_{k+r} < 0$, for $\lambda^* = \min\{\tau_1, \tau_2\}$ and for all $\lambda \in (0, \lambda^*)$, from Lemma 4.2 and Lemma 4.5, we obtain that K_c is compact. Moreover, $K_c = -K_c$. If $\gamma(K_c) \leq r$, there exists a closed and symmetric set U with $K_c \subset U$ such that $\gamma(U) = \gamma(K_c) \leq r$. Note that we can choose $U \subset J_\lambda^0$ because $c < 0$. By the deformation lemma [2] we have an odd homeomorphism $\eta : H \to H$ such that $\eta(J_\lambda^{c+\delta} - U) \subset J_\lambda^{c-\delta}$ for some $\delta > 0$ with $0 < \delta < -c$. Thus, $J_\lambda^{c+\delta} \subset J_\lambda^0$ and by definition of $c = c_{k+r}$, there exists $A \in \Gamma_{k+r}$ such that $\sup_{u \in A} < c + \delta$; that is, $A \subset J_\lambda^{c+\delta}$ and

$$\eta(A - U) \subset \eta(J_\lambda^{c+\delta} - U) \subset J_\lambda^{c-\delta}. \tag{4.10}$$

But $\gamma(\overline{A-U}) \geq \gamma(A) - \gamma(U) \geq k$ and $\gamma(\eta(\overline{A-U})) \geq \gamma(\overline{A-U}) \geq k$. Then $\eta(\overline{A-U}) \in \Gamma_k$ which contradicts (4.10). □

5. Proof of Theorem 1.1

If $-\infty < c_1 < c_2 < \dots < c_k < \dots < 0$ with $c_i \neq c_j$, once each c_k is a critical value of J_λ, we obtain infinitely many critical points of J_λ and then, (2.2) has infinitely many solutions.

On the other hand, if $c_k = c_{k+r}$ for some k and r, then $c = c_k = c_{k+1} = \dots = c_{k+r}$ and from Lemma 4.6, there exists $\lambda^* > 0$ such that

$$\gamma(K_c) \geq r + 1 \geq 2$$

for all $\lambda \in (0, \lambda^*)$. From Proposition 3.4 K_c has infinitely many points; that is, (2.2) has infinitely many solutions.

Let λ^* be as in Lemma 4.6 and, for $\lambda \in (0, \lambda^*)$, let u_λ be a solution of (2.2). Thus $J_\lambda(u_\lambda) = I_\lambda(u_\lambda) < 0$. Hence,

$$\|u_\lambda\|^2 \leq R_0,$$

which together with (4.2) implies

$$\lim_{\lambda \to 0} \|u_\lambda\| = 0. \tag{5.1}$$

Now we use the Moser iteration technique in order to prove that there exists a positive constant C, independent on λ such that

$$\|u_\lambda\|_\infty \leq C\|u_\lambda\|. \tag{5.2}$$

Using (5.2) we can conclude that

$$\lim_{\lambda \to 0} \|u_\lambda\|_\infty = 0. \tag{5.3}$$

To save notation, from now on we denote u_λ by u. In what follows, we fix $R > R_1 > 0$, $R > 1$ and take a cut-off function $\eta_R \in C_0^\infty(\Omega)$ such that $0 \leq \eta_R \leq 1$, $\eta_R \equiv 0$ in B_R^c, $\eta_R \equiv 1$ in B_{R_1} and $|\nabla \eta_R| \leq C/R$, where $B_R \subset \Omega$ and $C > 0$ is a constant.

Let $h(t) = \lambda t^{q-1} + t^{2^*-1}$. Thus

$$\begin{gathered}
|h(t)| \to 0 \quad \text{as } t \to 0, \\
\frac{|h(t)|}{t^{2^*-1}} \to 1 \quad \text{as } t \to \infty.
\end{gathered}$$

Thus, for all $\delta > 0$ there is $C_\delta(\lambda) > 0$ such that

$$h(t) \leq \delta + C_\delta(\lambda) t^{2^*-1}. \tag{5.4}$$

Moreover, for $\lambda \in [0, \lambda_0]$, $C_\delta(\lambda)$ can be chosen uniformly in λ in such a way that (5.4) holds independently of λ. For each $L > 0$, define

$$u_L(x) = \begin{cases} u(x), & \text{if } u(x) \leq L \\ L, & \text{if } u(x) \geq L, \end{cases}$$

$$z_L = \eta_R^2 u_L^{2(\sigma-1)} u \quad \text{and} \quad w_L = \eta_R u u_L^{\sigma-1}$$

with $\sigma > 1$ to be determined later. In the course of this proof, $C_1, C_2 \dots$, denote constants independent of λ.

Taking z_L as a test function we obtain $I'_\lambda(u) z_L = 0$. More specifically,

$$\int_\Omega a(|\nabla u|^2) \nabla u \nabla z_L = \lambda \int_\Omega u^{q-1} z_L + \int_\Omega u^{2^*-1} z_L.$$

Hence

$$K_0 \int_\Omega \nabla u \nabla z_L \leq \int_\Omega h(u) z_L.$$

By (5.4) we obtain

$$\int_\Omega \nabla u \nabla z_L \leq \delta K_0^{-1} \int_\Omega z_L + K_0^{-1} C_\delta \int_\Omega u^{2^*-1} z_L.$$

Let us fix $\delta > 0$ small enough in such a way that

$$\int_\Omega \nabla u \nabla z_L \leq C \int_\Omega u^{2^*-1} z_L.$$

Using z_L we obtain

$$\begin{aligned}\int_\Omega \eta_R^2 u_L^{2(\sigma-1)}|\nabla u|^2\,dx \le &-\int_\Omega \eta_R u u_L^{2(\sigma-1)}\nabla\eta_R\nabla u\,dx\\ &-2(\sigma-1)\int_\Omega u_L^{(2\sigma-3)}u\nabla u\nabla u_L+\int_\Omega \eta_R^2 u^{2^*}u_L^{2(\sigma-1)}\,dx,\end{aligned}$$

and the definition of u_L implies

$$-2(\sigma-1)\int_\Omega u_L^{(2\sigma-3)}u\nabla u\nabla u_L\le 0.$$

Thus

$$\int_\Omega \eta_R^2 u_L^{2(\sigma-1)}|\nabla u|^2\,dx \le +\int_\Omega \eta_R u u_L^{2(\sigma-1)}|\nabla\eta_R||\nabla u|\,dx+\int_\Omega \eta_R^2 u^{2^*}u_L^{2(\sigma-1)}\,dx.$$

Taking z_L as a test function and using (5.4), we obtain

$$\int_\Omega \eta_R^2 u_L^{2(\sigma-1)}|\nabla u|^2\,dx \le C_1\int_\Omega \eta_R u u_L^{2(\sigma-1)}|\nabla\eta_R||\nabla u|\,dx+C_1\int_\Omega \eta_R^2 u^{2^*}u_L^{2(\sigma-1)}\,dx.$$

Fixing $\widetilde{\tau}>0$ and using Young's inequality, we obtain

$$\begin{aligned}\int_\Omega \eta_R^2 u_L^{2(\sigma-1)}|\nabla u|^2\,dx \le &\, C_1\int_\Omega \Big(\widetilde{\tau}\eta_R^2|\nabla u|^2+C_{\widetilde{\tau}}u^2|\nabla\eta_R|^2\Big)u_L^{2(\sigma-1)}\,dx\\ &+C_1\int_\Omega \eta_R^2 u^{2^*}u_L^{2(\sigma-1)}\,dx.\end{aligned}$$

Choosing $\widetilde{\tau}\le 1/4$, it follows that

$$\int_\Omega \eta_R^2 u_L^{2(\sigma-1)}|\nabla u|^2\,dx \le C_2\Big(\int_\Omega u^2 u_L^{2(\sigma-1)}|\nabla\eta_R|^2\,dx+\int_\Omega \eta_R^2 u^{2^*}u_L^{2(\sigma-1)}\,dx\Big). \tag{5.5}$$

On the other hand, we obtain

$$\begin{aligned}S\|w_L\|^2_{L^{2^*}(\Omega)} &\le \int_\Omega |\nabla(\eta_R u u_L^{\sigma-1})|^2\\ &\le \int_\Omega |u|^2 u_L^{2(\sigma-1)}|\nabla\eta_R|^2+\int_\Omega \eta_R^2\left|\nabla\left(u u_L^{\sigma-1}\right)\right|^2.\end{aligned}$$

But

$$\begin{aligned}\int_\Omega \eta_R^2\left|\nabla\left(u u_L^{\sigma-1}\right)\right|^2 &= \int_{\{|u|\le L\}}\eta_R^2\left|\nabla\left(u u_L^{\sigma-1}\right)\right|^2+\int_{\{|u|>L\}}\eta_R^2\left|\nabla\left(u u_L^{\sigma-1}\right)\right|^2\\ &= \int_{\{|u|\le L\}}\eta_R^2\left|\nabla u^\sigma\right|^2+\int_{\{|u|>L\}}\eta_R^2 L^{2(\sigma-1)}\left|\nabla u\right|^2\\ &\le \sigma^2\int_\Omega \eta_R^2 u_L^{2(\sigma-1)}|\nabla u|^2,\end{aligned}$$

and therefore

$$\|w_L\|^2_{L^{2^*}(\Omega)}\le C_3\sigma^2\Big(\int_\Omega |u|^2u_L^{2(\sigma-1)}|\nabla\eta_R|^2+\int_\Omega \eta_R^2u_L^{2(\sigma-1)}|\nabla u|^2\Big).$$

From this and (5.5),

$$\|w_L\|^2_{L^{2^*}(\Omega)}\le C_4\sigma^2\Big(\int_\Omega |u|^2u_L^{2(\sigma-1)}|\nabla\eta_R|^2+\int_\Omega \eta_R^2|u|^{2^*}u_L^{2(\sigma-1)}\Big), \tag{5.6}$$

for all $\sigma > 1$. The above expression, the properties of η_R and $u_L \le u$, imply

$$\|w_L\|^2_{L^{2^*}(\Omega)} \le C_4\sigma^2 \int_{B_R} \Big(|u|^{2\sigma}|\nabla \eta_R|^2 + |u|^{2^*-2}|u|^{2\sigma}\Big). \tag{5.7}$$

If we set

$$t := \frac{2^*2^*}{2(2^*-2)} > 1, \quad \alpha := \frac{2t}{t-1} < 2^*, \tag{5.8}$$

we can apply Hölder's inequality with exponents $t/(t-1)$ and t in (5.7) to get

$$\begin{aligned}\|w_L\|^2_{L^{2^*}(\Omega)} &\le C_4\sigma^2\|u\|^{2\sigma}_{L^{\sigma\alpha}(B_R)}\Big(\int_{B_R}|\nabla\eta_R|^{2t}\Big)^{1/t}\\ &\quad + C_4\sigma^2\|u\|^{2\sigma}_{L^{\sigma\alpha}(B_R)}\Big(\int_{B_R}|u|^{2^*(2^*/2)}\Big)^{1/t}.\end{aligned} \tag{5.9}$$

Since η_R is constant on $B_{R_1} \cup B_R^c$ and $|\nabla\eta_R| \le C/R$, we conclude that

$$\int_{B_R}|\nabla\eta_R|^{2t} = \int_{B_R\setminus B_{R_1}}|\nabla\eta_R|^{2t} \le \frac{C_5}{R^{2t-N}} \le C_5. \tag{5.10}$$

We have used $R > 1$ and $2t = \frac{2^*}{2}N > N$ in the last inequality.

Claim. There exist a constants $K > 0$ independent on λ such that,

$$\int_\Omega |u|^{2^*(2^*/2)} \le K.$$

Assuming the claim is true, we can use (5.9) and (5.10) to conclude that

$$\|w_L\|^2_{L^{2^*}(\Omega)} \le C_6\sigma^2\|u\|^{2\sigma}_{L^{\sigma\alpha}(B_R)}.$$

Since

$$\begin{aligned}\|u_L\|^{2\sigma}_{L^{\sigma 2^*}(B_R)} &= \Big(\int_{B_R} u_L^{\sigma 2^*}\Big)^{2/2^*}\\ &\le \Big(\int_\Omega \eta_R^{2^*}|u|^{2^*}u_L^{2^*(\sigma-1)}\Big)^{2/2^*}\\ &= \|w_L\|^2_{L^{2^*}(\Omega)} \le C_6\sigma^2\|u\|^{2\sigma}_{L^{\sigma\alpha}(\Omega)},\end{aligned}$$

we can apply Fatou's lemma in the variable L to obtain

$$\|u\|_{L^{\sigma 2^*}(B_R)} \le C_7^{1/\sigma}\sigma^{1/\sigma}\|u\|_{L^{\sigma\alpha}(\Omega)},$$

whenever $|u|^{\sigma\alpha} \in L^1(B_R)$. Here, C_7 is a positive constant independent on R. Iterating this process, for each $k \in \mathbb{N}$, it follows that

$$\|u\|_{L^{\sigma^k 2^*}(B_R)} \le C_7^{\sum_{i=1}^k \sigma^{-i}}\sigma^{\sum_{i=1}^m i\sigma^{-i}}\|u\|_{L^{2^*}(\Omega)}.$$

Since Ω can be covered by a finite number of balls B_R^j, we have that

$$\|u\|_{L^{\sigma^k 2^*}(\Omega)} \le \sum_j^{finite}\|u\|_{L^{\sigma^k 2^*}(B_R^j)} \le \sum_j^{finite} C_7^{\sum_{i=1}^k \sigma^{-i}}\sigma^{\sum_{i=1}^m i\sigma^{-i}}\|u\|_{L^{2^*}(\Omega)}.$$

Since $\sigma > 1$, we let $k \to \infty$ to obtain

$$\|u\|_{L^\infty(\Omega)} \le K_2\|u\|,$$

for some $K_2 > 0$ independent on λ.

It remains to prove the claim. From (5.6)

$$\|w_L\|^2_{L^{2^*}(\Omega)} \le C_9\sigma^2\Big(\int_\Omega |u|^2 u_L^{2(\sigma-1)}|\nabla\eta_R|^2 + \int_\Omega \eta_R^2|u|^{2^*}u_L^{2(\sigma-1)}\Big), \tag{5.11}$$

We set $\sigma := 2^*/2$ in (5.6) to obtain

$$\|w_L\|^2_{L^{2^*}(\Omega)} \le C_{10}\Big(\int_\Omega |u|^2 u_L^{(2^*-2)}|\nabla\eta_R|^2 + \int_{B_R} \eta_R^2|u|^2u_L^{(2^*-2)}|u|^{(2^*-2)}\Big).$$

By Hölder's inequality with exponents $2^*/2$ and $2^*/(2^*-2)$ we obtain

$$\begin{aligned}\|w_L\|^2_{L^{2^*}(\Omega)} &\le C_{10}\int_\Omega |u|^2u_L^{(2^*-2)}|\nabla\eta_R|^2\\ &\quad + C_{10}\Big(\int_{B_R}\big(\eta_R|u|u_L^{(2^*-2)/2}\big)^{2^*}\Big)^{2/2^*}\|u\|^{2^*-2}_{L^{2^*}(\Omega)}.\end{aligned}$$

From (5.1) and recalling that $\eta_R u u_L^{(2^*-2)/2} = w_L$, $u_L \le u$ and $\nabla\eta_R$ is bounded, we obtain

$$\|w_L\|^2_{L^{2^*}(\Omega)} \le C_{11}\int_\Omega |u|^2u_L^{(2^*-2)}|\nabla\eta_R|^2 \le C_{11}\int_\Omega |u|^{2^*} \le C_{12}.$$

The definition of η_R and w_L and the above inequality imply

$$\Big(\int_{B_R}|u|^{2^*}u_L^{2^*(2^*-2)/2}\Big)^{2/2^*} \le |w_L|^2_{L^{2^*}(\Omega)} \le C_{12}.$$

Using Fatou's lemma in the variable L, we have

$$\int_{B_R}|u|^{2^*(2^*/2)} \le K := C_{12}^{2^*/2}.$$

Since Ω can be covered by a finite number of balls B_R^j, we have

$$\int_\Omega |u|^{2^*(2^*/2)} \le \sum_j^n \int_{B_R}|u|^{2^*(2^*/2)} \le K_3,$$

for some $K_3 > 0$.

To estimate $\|\nabla u_\lambda\|_\infty$, we use the following result by Stampacchia [17].

Lemma 5.1. *Let $A(\eta)$ a given C^1 vector field in $\mathbb{R}^N$, and $f(x,s)$ a bounded Carathéodory function in $\Omega\times\mathbb{R}$. Let $u \in H_0^1(\Omega)$ be a solution of*

$$\int_\Omega (A(|\nabla u|)\nabla\varphi + f(x,u)\varphi) = 0,$$

for all $\varphi \in H_0^1(\Omega)$. Assume that there exist $0<\nu<M$ such that

$$\nu|\xi|^2 \le \frac{\partial A_i}{\partial\eta_j}(\nabla u)\xi_i\xi_j \quad \text{and} \quad \Big|\frac{\partial A_i}{\partial\eta_j}(\nabla u)\Big| \le M, \tag{5.12}$$

for $i,j = 1,\dots,N$ and $\xi\in\mathbb{R}^N$. Then $u \in W^{2,p}(\Omega)\cap C^{1,\alpha}(\overline{\Omega})$, for all $\alpha\in(0,1)$ and $p>1$. Moreover

$$\|u\|_{1,\alpha} \le C(\nu,M,\Omega)\|f(\cdot,u)\|_\infty. \tag{5.13}$$

Proof. By the definition of a, for r small enough (5.12) hold. This, together with the fact that $\|u_\lambda\|_\infty$ is bounded allow us to apply the last result. Then (5.3) implies

$$\|u\|_{1,\alpha} \leq \lambda \|u\|_\infty^{q-1} + \|u\|_\infty^{2^*-1} = o(\lambda), \tag{5.14}$$

as $\lambda \to 0$.

Then, there exists $\lambda^* > 0$ such that $\lambda \in (0, \lambda^*)$ implies $\|\nabla u\|_\infty \leq r$ and hence, u_λ is a solution of (1.1). □

Acknowledgments. G. M. Figueiredo is supported by PROCAD/CASADINHO: 552101/2011-7, CNPq/PQ 301242/2011-9 and CNPQ/CSF 200237/2012-8. M.T.O. Pimenta is supported by FAPESP: 2014/16136-1 and CNPq 442520/2014-0.

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