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EXISTENCE OF SOLUTIONS TO THREE-POINT BOUNDARY-VALUE PROBLEMS AT RESONANCE

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Abstract. Using Mawhin's continuation theorem, we prove the existence of solutions for a class of nonlinear second-order differential equations in $\mathbb{R}^n$ associated with a three-point boundary conditions at resonance.

1. Introduction

In solving linear partial differential equations by the method of separation of variables, one encounters differential equations containing several parameters with the auxiliary requirement that the solutions satisfy boundary conditions at several points. This has led to an extensive development of multiparameter spectral theory of linear operators (for example, Gregus et al [6]). Many examples of multipoint boundary value problems (briefly, BVPs) can be obtained when looking for solutions of one dimensional free-boundary problems (see Berger and Fraenkel [2]). Multi-point BVPs can arise in other ways. For instance, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities can be set up as a multi-point BVP (see Moshiinsky [14]). Also, many problems in the theory of elastic stability can be handled by the method of multi-point problems (see Timoshenko [16]). Also nonlinear multi-point BVPs have received much attention of many mathematicians in recent decades.

In this note, we are concerned with the existence of solutions to the three-point BVP in $\mathbb{R}^n$,

$$\begin{gathered} u''(t) = f(t, u(t), u'(t)), \quad t \in (0,1), \\ u'(0) = \theta, \quad u(1) = Au(\eta), \end{gathered} \tag{1.1}$$

where $\eta \in (0,1)$, θ is the zero vector in $\mathbb{R}^n$, A is a square matrix of order n and $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ satisfies the Carathéodory conditions:

(a) $f(\cdot, u, v)$ is Lebesgue measurable for every $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$,
(b) $f(t, \cdot, \cdot)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^n$ for almost every $t \in [0,1]$,
(c) for each compact set $K \subset \mathbb{R}^{2n}$, the function $h_K(t) = \sup\{|f(t, u, v)| : (u, v) \in K\}$ is Lebesgue integrable on $[0,1]$, where $|\cdot|$ is the max-norm in $\mathbb{R}^n$.

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As we will see in the next section, problem (1.1) can be rewritten in the operator form

$$Lu = Nu, \tag{1.2}$$

with L (resp. N) is a linear (resp. nonlinear) mapping between two Banach spaces X and Z that will be specified later.

Under certain boundary conditions, the differential (linear) operator $Lu = u''$ defined in a suitable Banach space is invertible. Such case is the so-called non-resonant; otherwise, they are called resonant. The problem (1.1), when $n = 1$ and $A \neq 1$, which was considered by Gupta et al [8], is non-resonant. When $A = 1$, the problem becomes resonant – the situation which is usually more complex.

For several decades, the multi-point BVPs at resonance for second-order differential equations have been extensively studied; see instance [3]-[11] and references therein. In 1992, Gupta first studied the three-point BVP at resonance [7]

$$\begin{gathered} u''(t) = f(t, u(t), u'(t)) - e(t), \quad t \in (0, 1), \\ u(0) = 0, \quad u(1) = u(\eta). \end{gathered} \tag{1.3}$$

And then, Feng [3], Webb [4] and Ma [11] investigated a similar problem with some improvement to the assumptions on the nonlinear term,

$$\begin{gathered} u''(t) = f(t, u(t), u'(t)) + e(t), \quad t \in (0, 1), \\ u'(0) = 0, \quad u(1) = u(\eta). \end{gathered} \tag{1.4}$$

The main tool in most of the above works is Mawhin continuation theorem, the method (in coincidence degree) which is basically relied on the dimension of $\ker L$. Because the construction of the projection Q (and sometime P) is often quite complicated when $\dim\ker L$ is large, authors have mainly discussed the case $\dim\ker L = 1$ only. When $\dim\ker L$ is large, there are few results about it, even $\dim\ker L = 2$, in which Kosmatov is a pioneer [10].

Recently, we have studied the three-point BVPs (1.1) at resonance [15], and generalizing some of the above results. Clearly, if the boundary condition at zero is $u(0) = \theta$, instead of $u'(0) = \theta$, we also obtain the similar results with some minor adjustment. In that paper, we showed a new technique to establish the existence result for the boundary problem at resonance when $\dim\ker L$ enabled to take value arbitrarily – depending on the assumption

$$\begin{gathered} A^2 = A, \quad \text{or} \\ A^2 = I \quad (I \text{ stands for the indentity matrix}). \end{gathered} \tag{1.5}$$

Our goal is omit the condition (1.5). We discover that the assumption (1.5) on A leads to the decomposition

$$\mathbb{R}^n = \operatorname{Im}(I - A) \oplus \ker(I - A). \tag{1.6}$$

This property is also expressed, more usefully as

$$\begin{gathered} P_A = \kappa(I - A) \text{ is a projection on } \operatorname{Im}(I - A), \text{ and} \\ I - P_A \text{ is a projection on } \ker(I - A), \end{gathered} \tag{1.7}$$

where

$$\kappa = \begin{cases} 1 & \text{if } A^2 = A, \\ 1/2 & \text{if } A^2 = I. \end{cases}$$

Inspired by this, in this note, having no longer (1.5), we substitute the nice decomposition (1.6) by the two following decompositions

$$\mathbb{R}^n = \operatorname{Im}(I-A) \oplus \ker U,$$
$$\mathbb{R}^n = \operatorname{Im} V \oplus \ker(I-A),$$

where U, V are two some matrices relating to A. More precisely, we can generalize (1.6) by the fact that

$$(I-A)(I-A)^+ \text{ is a projection on } \operatorname{Im}(I-A), \text{ and}$$
$$I-(I-A)^+(I-A) \text{ is a projection on } \ker(I-A),$$

where $(I-A)^+$ is denoted the Moore-Penrose pseudoinverse of $(I-A)$. Part of this idea derives from [9].

The rest of this article is organized as follows. In section 2, we provide some results regarding Mawhin's coincidence degree theory and several important lemmas which are motivation for obtaining our main result. In section 3 we state and prove the main theorem. Finally, we present an example to illustrate this result.

2. Preliminaries

We begin this section by recalling some definitions and preliminary results of coincidence degree theory due to Mawhin [5, 12, 13]. Suppose that X and Z are two Banach spaces.

Definition 2.1. A linear operator $L : \operatorname{dom} L \subset X \to Z$ is called to be a *Fredholm operator* provided that

(i) $\ker L$ is finite dimensional,
(ii) $\operatorname{Im} L$ is closed and has finite codimension.

In addition, the (Fredholm) *index* of L is defined by the integer number

$$\operatorname{ind} L = \dim \ker L - \operatorname{codim} \operatorname{Im} L.$$

From Definition 2.1, it follows that if L is a Fredholm operator, then there exist continuous projections $P : X \to X$ and $Q : Z \to Z$ such that

$$\operatorname{Im} P = \ker L, \quad \ker Q = \operatorname{Im} L, \quad X = \ker L \oplus \ker P, \quad Z = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

Furthermore, the restriction of L on $\operatorname{dom} L \cap \ker P, L_P : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$, is invertible. We denote by K_P the inverse of L_P and by $K_{P,Q} = K_P(I-Q)$ the generalized inverse of L. Moreover, if $\operatorname{ind} L = 0$, that is $\operatorname{Im} Q$ and $\ker L$ are isomorphic, the operator $JQ + K_{P,Q} : Z \to \operatorname{dom} L$ is isomorphic and

$$\Big(JQ + K_{P,Q}\Big)^{-1} = \Big(L + J^{-1}P\Big)\Big|_{\operatorname{dom} L},$$

for every isomorphism $J : \operatorname{Im} Q \to \ker L$. Hence, following Mawhin's equivalent theorem, $u \in \overline{\Omega}$ is a solution to equation $Lu = Nu$ if and only if it is a fixed point of Mawhin's operator

$$\Phi := P + (JQ + K_{P,Q})N,$$

where Ω is an given open bounded subset of X such that $\operatorname{dom} L \cap \Omega \neq \emptyset$.

Next, to get the compactness of Φ, Mawhin introduced a concept, weaker than compactness, to impose on N as follows.

Definition 2.2. Let L be a Fredholm operator of index zero. The operator $N : X \to Z$ is said to be *L-compact* in $\overline{\Omega}$ provided that

- the map $QN : \overline{\Omega} \to Z$ is continuous and $QN(\overline{\Omega})$ is bounded in Z,
- the map $K_{P,Q}N : \overline{\Omega} \to X$ is completely continuous.

In addition, we say that N is L-*completely continuous* if it is L-compact on every bounded set in X.

Finally, we state the fundamental theorem in coincidence degree theory, so-called Mawhin's continuation theorem [12]. It is the main tool to prove our result in the next section.

Theorem 2.3. *Let $\Omega \subset X$ be open and bounded, L be a Fredholm mapping of index zero and N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied*

(i) *$Lu \neq \lambda Nu$ for every $(u, \lambda) \in \big((\operatorname{dom} L \setminus \ker L) \cap \partial\Omega\big) \times (0, 1)$;*
(ii) *$QNu \neq 0$ for every $u \in \ker L \cap \partial\Omega$;*
(iii) *for some isomorphism $J : \operatorname{Im} Q \to \ker L$ we have*
$$\deg_B \big(JQN|_{\ker L}; \Omega \cap \ker L, \theta\big) \neq 0,$$
where $Q : Z \to Z$ is a projection given as above.

Then the equation $Lu = Nu$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

Next, to attain the solvability of problem (1.1) by using Theorem 2.3, we renew the spaces $X = C^1\big([0, 1]; \mathbb{R}^n\big)$ endowed with the norm
$$\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\},$$
where $\|\cdot\|_\infty$ stands for the sup-norm and $Z = L^1\big([0, 1]; \mathbb{R}^n\big)$ endowed with the Lebesgue norm denoted by $\|\cdot\|_1$. Further, we shall use the Sobolev space defined by
$$X_0 = \{u \in X : u'' \in Z\}.$$
Then we define the operator $L : \operatorname{dom} L \subset X \to Z$ by $Lu := u''$, where
$$\operatorname{dom} L = \{u \in X_0 : u'(0) = \theta, u(1) = Au(\eta)\}.$$
Because
$$u \in X_0 \Longleftrightarrow u(t) = u(0) + u'(0)t + I^2_{0^+}u''(t),$$
where
$$I^k_{0^+}z(t) = \int_0^t (t - s)^{k-1} z(s) ds, \quad t > 0, \text{ for } k \in \{1, 2\},$$
the domain of L is rewritten as
$$\operatorname{dom} L = \{u \in X_0 : u(t) = u(0) + I^2_{0^+}z(t) \text{ with } z \in Z \text{ satisfies } \mathcal{M}u(0) = \phi(z)\}, \tag{2.1}$$
where

- $\mathcal{M} = I - A$
- $\phi : Z \to \mathbb{R}^n$ is a continuous operator defined by
$$\phi(z) = AI^2_{0^+}z(\eta) - I^2_{0^+}z(1), \quad z \in Z. \tag{2.2}$$

Also, we easily have
$$\ker L = \{u \in X : u(t) = c,\, t \in [0, 1],\, c \in \ker \mathcal{M}\} \cong \ker \mathcal{M}. \tag{2.3}$$
Moreover, we claim that
$$\operatorname{Im} L = \phi^{-1}(\operatorname{Im} \mathcal{M}) = \{z \in Z : \phi(z) \in \operatorname{Im} \mathcal{M}\}. \tag{2.4}$$

Indeed, let $z \in \operatorname{Im} L$, so $z = Lu$ for some $u \in \operatorname{dom} L$. From (2.1) we have $\mathcal{M}u(0) = \phi(Lu) = \phi(z)$, which implies $\phi(z) \in \operatorname{Im} \mathcal{M}$. Thus $\operatorname{Im} L \subset \{z \in Z : \phi(z) \in \operatorname{Im} \mathcal{M}\}$.

Conversely, let $z \in Z$ such that $\phi(z) = \mathcal{M}\xi \in \operatorname{Im} \mathcal{M}$. Then it is clear to verify that $z = Lu$, where $u \in \operatorname{dom} L$ defined by $u(t) = \xi + I_{0+}^2 z(t)$. This implies $z \in \operatorname{Im} L$.

Remark 2.4. From (2.3) we show that when $\det(I - A) \neq 0$, i.e., L^{-1} exists, the problem is non-resonant. This is the classical case which has studied much by many authors. In this paper, we are only interested in resonant cases: $\det(I - A) = 0$.

Now we construct two continuous projections as in the framework of Mawhin's method, mentioned before, and then attain Fredholmness and index zero of operator L. As the introduction section, we denote by $\mathcal{M}^+$ the *Moore-Penrose pseudoinverse matrix* of $\mathcal{M}$, meaning the matrix satisfying

(i) $\mathcal{M}^+\mathcal{M}\mathcal{M}^+ = \mathcal{M}^+$,
(ii) $\mathcal{M}\mathcal{M}^+\mathcal{M} = \mathcal{M}$,
(iii) $\mathcal{M}\mathcal{M}^+$ is a (orthogonal) projection on $\operatorname{Im} \mathcal{M}$,
(iv) $I - \mathcal{M}^+\mathcal{M}$ is a (orthogonal) projection on $\ker \mathcal{M}$.

We can find more details of this concept in [1]. Next, we state and prove two important lemmas.

Lemma 2.5. *The operator $L : \operatorname{dom} L \subset X \to Z$ is Fredholm and has index zero.*

Proof. Since ϕ is continuous and $\operatorname{Im} \mathcal{M}$ is closed in $\mathbb{R}^n$, it is clear that $\operatorname{Im} L = \phi^{-1}(\operatorname{Im} \mathcal{M})$ is closed in Z. Further, we have $\dim\ker L = \dim\ker \mathcal{M} \leq n < \infty$. Hence, it remains to prove that

$$\dim\ker L - \operatorname{codim} \operatorname{Im} L = 0.$$

To do this, we formulate the continuous operator $Q : Z \to Z$ defined as, for $z \in Z$,

$$Qz(t) = \frac{2}{\eta^2 - 1}(I - \mathcal{M}\mathcal{M}^+)\phi(z), \quad t \in [0, 1], \tag{2.5}$$

It is necessary to note that if $z(t) = h \in \mathbb{R}^n$ for all $t \in [0, 1]$, then

$$\phi(z) = A\int_0^\eta (\eta - s)h\,ds - \int_0^1 (1 - s)h\,ds = \frac{1}{2}(\eta^2 A - I)h. \tag{2.6}$$

Also, we achieve

$$(I - \mathcal{M}\mathcal{M}^+)(\eta^2 A - I) = (\eta^2 - 1)(I - \mathcal{M}\mathcal{M}^+). \tag{2.7}$$

This is deduced from

$$(I - \mathcal{M}\mathcal{M}^+)(I - A) = (I - \mathcal{M}\mathcal{M}^+)\mathcal{M} = 0,$$

which is equivalent to

$$\begin{aligned}
(I - \mathcal{M}\mathcal{M}^+)A = (I - \mathcal{M}\mathcal{M}^+) &\Leftrightarrow (I - \mathcal{M}\mathcal{M}^+)\eta^2 A = \eta^2(I - \mathcal{M}\mathcal{M}^+)\\
&\Leftrightarrow (I - \mathcal{M}\mathcal{M}^+)(\eta^2 A - I) = (\eta^2 - 1)(I - \mathcal{M}\mathcal{M}^+).
\end{aligned}$$

The last equality is obtained trivially. Then, from (2.6)–(2.7), we have

$$\begin{aligned}
Q(Qz) &= \frac{2}{\eta^2 - 1}(I - \mathcal{M}\mathcal{M}^+)\phi(Qz) = \frac{1}{\eta^2 - 1}(I - \mathcal{M}\mathcal{M}^+)(\eta^2 A - I)Qz\\
&= (I - \mathcal{M}\mathcal{M}^+)Qz = \frac{2}{\eta^2 - 1}(I - \mathcal{M}\mathcal{M}^+)^2\phi(z)
\end{aligned}$$

$$= \frac{2}{\eta^2 - 1}(I - \mathcal{M}\mathcal{M}^+)\phi(z) = Qz,$$

and

$$\begin{aligned} z \in \ker Q &\Leftrightarrow \phi(z) \in \ker(I - \mathcal{M}\mathcal{M}^+) \\ &\Leftrightarrow \phi(z) \in \operatorname{Im}(\mathcal{M}\mathcal{M}^+) \\ &\Leftrightarrow \phi(z) \in \operatorname{Im} \mathcal{M}. \\ &\Leftrightarrow z \in \operatorname{Im}\, L. \end{aligned}$$

This means Q is a projection with $\ker Q = \operatorname{Im}\, L$. In addition, the continuity of Q is obvious. Thus, we have the decomposition $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. Then L is Fredholm due to Q has finite rank. Next, we check that $\operatorname{ind} L = 0$.

First, we claim that

$$\operatorname{Im} Q = \ker(\mathcal{M}\mathcal{M}^+). \tag{2.8}$$

Truly, the inclusion $\operatorname{Im}\, Q \subset \ker(\mathcal{M}\mathcal{M}^+)$ is clear since $\mathcal{M}\mathcal{M}^+$ is a projection. Inversely, let $\alpha \in \ker(\mathcal{M}\mathcal{M}^+)$, we have

$$\begin{aligned} Q\alpha &= \frac{2}{\eta^2 - 1}(I - \mathcal{M}\mathcal{M}^+)\phi(\alpha) \\ &= \frac{1}{\eta^2 - 1}(I - \mathcal{M}\mathcal{M}^+)(\eta^2 A - I)\alpha \\ &= (I - \mathcal{M}\mathcal{M}^+)\alpha = \alpha. \end{aligned}$$

Thus $\alpha \in \operatorname{Im} Q$. It follows from (2.8) that

$$\begin{aligned} \dim \operatorname{Im} Q &= \dim \ker(\mathcal{M}\mathcal{M}^+) = n - \dim \operatorname{Im}(\mathcal{M}\mathcal{M}^+) \\ &= n - \dim \operatorname{Im} \mathcal{M} = \dim \ker \mathcal{M} = \dim \ker L. \end{aligned}$$

The proof is complete. □

Remark 2.6. (i) The identity (2.7) is crucial to make the construction of Q quite simple.

(ii) The Fredholm index zero of L is deduced from the claim (2.8). This equality holds because the fact that the mapping ϕ is surjective. In more general contexts, lacking the surjective property of ϕ, the inclusion $\operatorname{Im}\, Q \subset \ker(\mathcal{M}\mathcal{M}^+)$ will not happen. Then L has positive index, and Mawhin theory will be no longer powerful.

Now, to establish the generalized inverse of L, we define the operator $P : X \to X$ by

$$Pu(t) = (I - \mathcal{M}^+\mathcal{M})u(0), \quad \forall t \in [0, 1]. \tag{2.9}$$

Lemma 2.7. *The following assertions hold:*

(i) *The mapping P defined by* (2.9) *is a continuous projection satisfying*

$$\operatorname{Im} P = \ker L, \quad X = \ker L \oplus \ker P.$$

(ii) *The linear operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ can be defined by*

$$K_P z(t) = \mathcal{M}^+\phi(z) + I^2_{0^+} z(t), \quad t \in [0, 1], \tag{2.10}$$

Moreover, K_P satisfies

$$K_P = L_P^{-1} \quad \text{and} \quad \|K_P z\| \le C\|z\|_1,$$

where $C = 1 + \|\mathcal{M}^+\mathcal{M}\|_*(1 + \eta\|A\|_*)$ ($\|\cdot\|_*$ is the maximum absolute column sum norm of matrices).

Proof. (i) It is useful to keep in mind that $I-\mathcal{M}^+\mathcal{M}$ is a projection on $\ker\mathcal{M}$. It follows that P is a continuous projection. Furthermore, Im $P=\ker L$. Indeed, we have

$$\begin{aligned} u(t)\in\ker L &\Leftrightarrow u(t)\equiv\alpha\in\ker\mathcal{M}=\operatorname{Im}(I-\mathcal{M}^+\mathcal{M})\\ &\Leftrightarrow \alpha=(I-\mathcal{M}^+\mathcal{M})\beta,\quad \beta\in\mathbb{R}^n\\ &\Leftrightarrow \alpha=P\beta,\ \beta\in\mathbb{R}^n\\ &\Leftrightarrow u(t)\equiv\alpha\in\operatorname{Im}P. \end{aligned}$$

This also implies the decomposition $X=\ker L\oplus\ker P$.

(ii) Let $z\in\operatorname{Im}L$, this deduces that $\phi(z)=\mathcal{M}\alpha$, $\alpha\in\mathbb{R}^n$. It follows from (2.9) and (2.10) that

$$\begin{gathered} PK_Pz(t)=(I-\mathcal{M}^+\mathcal{M})\mathcal{M}^+\phi(z)=(I-\mathcal{M}^+\mathcal{M})\mathcal{M}^+\mathcal{M}\alpha=\theta,\quad \forall t\in[0,1],\\ \mathcal{M}(K_Pz(0))=\mathcal{M}(\mathcal{M}^+\phi(z))=\mathcal{M}\mathcal{M}^+\mathcal{M}\alpha=\mathcal{M}\alpha=\phi(z). \end{gathered}$$

Thus $K_Pz\in\ker P\cap\operatorname{dom}L$, i.e., it is well defined. On the other hand, if $u\in\operatorname{dom}L\cap\ker P$ then $u(t)=u(0)+I_{0^+}^2Lu(t)$, in which

$$\begin{gathered} \mathcal{M}u(0)=\phi(Lu),\\ u(0)=(\mathcal{M}^+\mathcal{M})u(0). \end{gathered}$$

Hence

$$\begin{aligned} K_PL_Pu(t)&=\mathcal{M}^+\phi(Lu)+I_{0^+}^2Lu(t)\\ &=\mathcal{M}^+\mathcal{M}u(0)+I_{0^+}^2Lu(t)\\ &=u(0)+I_{0^+}^2Lu(t)\\ &=u(t). \end{aligned}$$

So, $K_P=L_P^{-1}$ since $LK_Pz(t)=z(t)$, $t\in[0,1]$, for all $z\in\operatorname{Im}L$. Finally, from the expression of K_P we have

$$(K_Pz)'(t)=I_{0^+}^1z(t),\quad t\in[0,1]. \tag{2.11}$$

Combining (2.2), (2.10) and (2.11) we have

- $\|K_Pz\|_\infty\le\|\mathcal{M}^+\|_*|\phi(z)|+\|z\|_1$,
- $|\phi(z)|\le(1+\eta\|A\|_*)\|z\|_1$,
- $\|(K_Pz)'\|_\infty\le\|z\|_1$.

These show that $\|K_Pz\|\le C\|z\|_1$. The lemma is proved. □

Lemma 2.8. *The operator $N:X\to Z$ is defined by*

$$Nu(t)=f(t,u(t),u'(t)),\quad \text{a.e. } t\in[0,1]$$

is L-completely continuous.

Proof. Let Ω be a bounded set in X. Put $R=\sup\{\|u\|:u\in\Omega\}$. From the hypotheses of the function f there exists a function $m_R\in Z$ such that, for all $u\in\Omega$, we have

$$|Nu(t)|=|f(t,u(t),u'(t))|\le m_R(t),\quad \text{a.e. } t\in[0,1], \tag{2.12}$$

It follows from (2.2), (2.12) and the identity

$$QNu(t)=\frac{2}{\eta^2-1}(I-\mathcal{M}\mathcal{M}^+)\phi(Nu) \tag{2.13}$$

that $QN(\overline{\Omega})$ is bounded and QN is continuous by using the Lebesgue's dominated convergence theorem. Now we shall prove that $K_{P,Q}N$ is completely continuous. First we note that, for every $u \in \Omega$, we have

$$\begin{aligned}
K_{P,Q}Nu(t) &= K_P(Id_Z - Q)Nu(t) \\
&= K_P(Nu - QNu)(t) \\
&= K_P\Big[Nu - \frac{2}{\eta^2-1}(I-\mathcal{M}\mathcal{M}^+)\phi(Nu)\Big](t) \\
&= I_{0^+}^2 Nu(t) - \frac{t^2}{2(\eta^2-1)}(\eta^2 A - I)(I-\mathcal{M}\mathcal{M}^+)\phi(Nu) \\
&\quad + \mathcal{M}^+\phi(Nu) - \frac{1}{(\eta^2-1)}\mathcal{M}^+(\eta^2 A - I)(I-\mathcal{M}\mathcal{M}^+)\phi(Nu),
\end{aligned} \tag{2.14}$$

and

$$(K_{P,Q}Nu)'(t) = I_{0^+}^1 Nu(t) - \frac{t}{(\eta^2-1)}(\eta^2 A - I)(I-\mathcal{M}\mathcal{M}^+)\phi(Nu). \tag{2.15}$$

Further, it follows from (2.12) and the definition of ϕ that

$$|\phi(Nu)| \le (1+\eta\|A\|_*)\|Nu\|_1 \le (1+\eta\|A\|_*)\|m_R\|_1. \tag{2.16}$$

Combining (2.12) and (2.14)–(2.16) we can find two positive constants C_1, C_2 such that

$$|K_{P,Q}Nu(t)| \le C_1\|m_R\|_1, \quad |(K_{P,Q}Nu)'(t)| \le C_2\|m_R\|_1, \tag{2.17}$$

for all $t \in [0,1]$ and for all $u \in \Omega$. This shows that

$$\|K_{P,Q}Nu\| \le \max\{C_1, C_2\}\|m_R\|_1,$$

that is, $K_{P,Q}N(\Omega)$ is uniformly bounded in X. On the other hand, for $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we have

$$\begin{aligned}
&|K_{P,Q}Nu(t_2) - K_{P,Q}Nu(t_1)| \\
&\le \int_{t_1}^{t_2} ds \int_0^s |Nu(\tau)|d\tau + |(t_2-t_1)\frac{t^2}{2(\eta^2-1)}(\eta^2 A - I)(I-\mathcal{M}\mathcal{M}^+)\phi(Nu)| \\
&\le C_3\|m_R\|_1|t_2 - t_1|,
\end{aligned}$$

and

$$|(K_{P,Q}Nu)'(t_2) - (K_{P,Q}Nu)'(t_1)| \le \int_{t_1}^{t_2} m_R(s)ds + C_4\|m_R\|_1|t_2-t_1|,$$

which prove that the family $K_{P,Q}N(\Omega)$ is equicontinuous in X. By Arzelà- Ascoli theorem $K_{P,Q}N(\Omega)$ is a relatively compact subset in X. Finally, it is easy to see that $K_{P,Q}N$ is continuous. Therefore the operator N is L-completely continuous. The proof of the theorem is complete. □

3. Main results

In this section we begin the search for appropriate open, bounded subset Ω for the application of the Mawhin continuation theorem 2.3 in proving the existence of the solutions of problem (1.1). For this purpose, it is essential to impose the standard hypotheses upon f to obtain a *prior estimate* for possible solutions of perturbation problems. Of course, this assumptions are somewhat technical but improving them is one of most difficult issues in applying topological degree methods to nonlinear functional analysis. By this reason, we always assume that the following three conditions hold.

(A1) There exist the positive functions $a, b, c \in Z$ with $(\|I-\mathcal{M}^{+}\mathcal{M}\|_{*}+C)(\|a\|_{1}+\|b\|_{1})<1$ such that
$$|f(t,u,v)| \le a(t)|u| + b(t)|v| + c(t), \tag{3.1}$$
for all $t \in [0,1]$ and $u, v \in \mathbb{R}^n$; where C is the constant given in Lemma 2.7;

(A2) There exists a positive constant Λ_1 such that for each $u \in \operatorname{dom} L$, if $|u(t)| > \Lambda_1$, for all $t \in [0,1]$, then
$$\int_\eta^1 ds \int_0^s f(\tau, u(\tau), u'(\tau))d\tau \notin \operatorname{Im}\mathcal{M}; \tag{3.2}$$

(A3) There exists a positive constant Λ_2 and an isomorphism $J : \operatorname{Im} Q \to \ker L$ such that for every $\alpha \in \ker \mathcal{M}$ with $|\alpha| > \Lambda_2$, then either
$$\langle \alpha, JQN(\alpha)\rangle \le 0 \quad \text{or} \quad \langle \alpha, JQN(\alpha)\rangle \ge 0, \tag{3.3}$$
where $\langle \cdot, \cdot \rangle$ stand for the scalar product in $\mathbb{R}^n$.

Lemma 3.1. *Let $\Omega_1 = \{u \in \operatorname{dom} L \setminus \ker L : Lu = \lambda Nu,\ \lambda \in (0,1]\}$. Then Ω_1 is bounded in X.*

Proof. Let $u \in \Omega_1$. Assume that $Lu = \lambda Nu$ for $0 < \lambda \le 1$. Then it is clear that $Nu \in \operatorname{Im} L = \ker Q$, which also implies $\phi(Nu) \in \operatorname{Im}\mathcal{M}$ by the characterization of $\operatorname{Im} L$. On the other hand, we have
$$\int_\eta^1 ds \int_0^s f(\tau, u(\tau), u'(\tau))d\tau = -\phi(Nu) - \mathcal{M}\int_0^\eta ds \int_0^s f(\tau, u(\tau), u'(\tau))d\tau.$$
Hence we deduce that
$$\int_\eta^1 ds \int_0^s f(\tau, u(\tau), u'(\tau))d\tau \in \operatorname{Im}\mathcal{M}.$$
By using the contraposition of assumption (A2), there exists $t_0 \in [0,1]$ such that $|u(t_0)| \le \Lambda_1$. Then
$$\begin{gathered} |u(t)| = \big|u(t_0) + \int_{t_0}^t u'(s)ds\big| \le \Lambda_1 + \|u'\|_\infty, \\ |u'(t)| \le \int_0^t |u''(s)|ds \le \|u''\|_1 \le \|Nu\|_1, \end{gathered} \tag{3.4}$$
uniformly on $[0,1]$. These imply that
$$\|Pu\| = |(I-\mathcal{M}^{+}\mathcal{M})u(0)| \le \|I-\mathcal{M}^{+}\mathcal{M}\|_{*}(\Lambda_1 + \|Nu\|_1). \tag{3.5}$$
On the other hand, we note that $(Id_X - P)u \in \operatorname{dom} L \cap \ker P$ because P is the projection on X. Then
$$\|(Id_X - P)u\| = \|K_P L(Id_X - P)u\| \le \|K_P Lu\| \le C\|Nu\|_1, \tag{3.6}$$
where the constant C is defined as in Lemma 2.7 and Id_X is the identity operator on X. Using (3.5), (3.6) we obtain
$$\begin{aligned} \|u\| &= \|Pu + (Id_X - P)u\| \le \|Pu\| + \|(Id_X - P)u\| \\ &\le \Lambda_1\|I-\mathcal{M}^{+}\mathcal{M}\|_{*} + (\|I-\mathcal{M}^{+}\mathcal{M}\|_{*} + C)\|Nu\|_1. \end{aligned} \tag{3.7}$$

By (A1) and the definition of N we have

$$\begin{aligned} \|Nu\|_1 &\le \int_0^1 |f(s,u(s),u'(s))|ds \\ &\le \|a\|_1\|u\|_\infty + \|b\|_1\|u'\|_\infty + \|c\|_1 \\ &\le (\|a\|_1 + \|b\|_1)\|u\| + \|c\|_1. \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8) we obtain

$$\|Nu\|_1 \le \frac{\Lambda_1\|I-\mathcal{M}^+\mathcal{M}\|_*(\|a\|_1+\|b\|_1)+\|c\|_1}{1-(\|I-\mathcal{M}^+\mathcal{M}\|_*+C)(\|a\|_1+\|b\|_1)}.$$

The last inequality combined with (3.4) allows us to deduce that

$$\sup_{u\in\Omega_1}\|u\| = \sup_{u\in\Omega_1}\max\{\|u\|_\infty, \|u'\|_\infty\} < +\infty.$$

Therefore Ω_1 is bounded in X. The lemma is proved. □

Lemma 3.2. *The set $\Omega_2 = \{u \in \ker L : Nu \in \operatorname{Im} L\}$ is a bounded subset in X.*

Proof. Let $u \in \Omega_2$. Assume that $u(t) = c$ for all $t \in [0,1]$, where $c \in \ker\mathcal{M}$. Since $Nu \in \operatorname{Im} L$ we have $\phi(Nu) \in \operatorname{Im}\mathcal{M}$. By the same arguments as in the proof of Lemma 3.1 we can point out that there exists $t_0 \in [0,1]$ such that $|u(t_0)| \le \Lambda_1$. Therefore

$$\|u\| = \|u\|_\infty = |u(t_0)| = |c| \le \Lambda_1.$$

So Ω_2 is bounded in X. The lemma is proved. □

Lemma 3.3. *The sets*

$$\begin{gathered} \Omega_3^- = \{u \in \ker L : -\lambda u + (1-\lambda)JQNu = \theta,\ \lambda \in [0,1]\}, \\ \Omega_3^+ = \{u \in \ker L : \lambda u + (1-\lambda)JQNu = \theta,\ \lambda \in [0,1]\} \end{gathered}$$

are bounded in X provided that the first and the second part of (3.3) *is satisfied, respectively.*

Proof. We consider two cases:
Case 1: $\langle\alpha, JQN\alpha\rangle \le 0$. Let $u \in \Omega_3^-$. Then there exists $\alpha \in \ker\mathcal{M}$ such that $u(t) = \alpha, \forall t \in [0,1]$, and

$$(1-\lambda)JQN\alpha = \lambda\alpha. \tag{3.9}$$

If $\lambda = 0$ then it follows from (3.9) that $N\alpha \in \ker JQ = \ker Q = \operatorname{Im} L$; that is, $u \in \Omega_2$. Lemma 3.2 deduce that $\|u\| \le \Lambda_1$. On the other hand, suppose in contrast that $|\alpha| > \Lambda_2$, then using (A3) we get a contradiction

$$0 < \lambda|\alpha|^2 = (1-\lambda)\langle\alpha, JQN\alpha\rangle \le 0, \quad \forall\lambda \in (0,1].$$

Thus $\|u\| = |\alpha| \le \Lambda_2$. Therefore we can conclude that Ω_3^- is bounded in X.
Case 2: $\langle\alpha, JQN\alpha\rangle \ge 0$. In this case, by using the same arguments as in above we are able to prove that Ω_3^+ is also bounded in X. □

Theorem 3.4. *Let* (A1)–(A3) *hold. Then problem* (1.1) *has at least one solution in X.*

Proof. We shall prove that all the conditions of Theorem 2.3 are satisfied, where Ω be open and bounded such that $\bigcup_{i=1}^{3}\overline{\Omega}_i \subset \Omega$, with

$$\Omega_3 = \begin{cases} \Omega_3^-, & \text{if the first inequality of (3.3) is satisfied,}\\ \Omega_3^+, & \text{if the second inequality of (3.3) is satisfied.} \end{cases}$$

It is clear that the conditions (i) and (ii) of Theorem 2.3 are fulfilled by using Lemma 3.1 and Lemma 3.2. So it remains to verify that the third condition holds. For this purpose we are going to employ invariant homotopy property of Brouwer degree. So we define a continuous homotopy as follows

$$H(u,\lambda) = \pm\lambda u + (1-\lambda)JQNu,$$

where $J : \operatorname{Im} Q \to \ker L$ is the isomorphism in hypothesis (A3). By Lemma 3.3 we have $H(u,\lambda) \neq \theta$ for all $(u,\lambda) \in (\ker L \cap \partial\Omega) \times [0,1]$. Hence

$$\begin{aligned} \deg(JQN|_{\ker L}; \Omega \cap \ker L, \theta) &= \deg(H(\cdot,0), \Omega \cap \ker L, \theta)\\ &= \deg(H(\cdot,1), \Omega \cap \ker L, \theta)\\ &= \deg(\pm Id, \Omega \cap \ker L, \theta)\\ &= \pm 1 \neq 0. \end{aligned}$$

Thus, Theorem 3.4 is proved. $\square$

To complete this paper, we introduce an example dealing with the solvability of a second-order system of differential equations associated with three-point boundary conditions by applying above our result.

Example 3.5. Consider the solvability of the boundary-value problem

$$\begin{gathered} x''(t) = f_1(t, x(t), y(t), x'(t), y'(t)), \quad t \in (0,1),\\ y''(t) = f_2(t, x(t), y(t), x'(t), y'(t)), \quad t \in (0,1),\\ x'(0) = y'(0) = 0,\\ x(1) = 4x(1/2) - 7y(1/2),\\ y(1) = 3x(1/2) - 6y(1/2), \end{gathered} \tag{3.10}$$

where the functions $f_i : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ $(i = 1,2)$ are

$$f_1(t, x_1, x_2, y_1, y_2) = \frac{t^3+3}{360}(x_1 + x_2) + \frac{t^5}{30}\ln(1 + \sqrt{y_1^2 + y_2^2}) + \frac{t^3+3}{36}, \tag{3.11}$$

$$f_2(t, x_1, x_2, y_1, y_2) = \frac{t^3+3}{360}(|x_1| + |x_2|) + \frac{t^5}{20}\sqrt{y_1^2 + y_2^2} + \frac{t^3+3}{18}, \tag{3.12}$$

for all $t \in [0,1]$ and $(x_1,x_2), (y_1,y_2) \in \mathbb{R}^2$.

Next we prove that (3.10) has at least one solution by using Theorem 3.4. For this we put

$$\eta = 1/2, \quad A = \begin{bmatrix} 4 & -7\\ 3 & -6 \end{bmatrix},$$

and the function $f : [0,1] \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$f(t,u,v) = (f_1(t,u,v), f_2(t,u,v)), \tag{3.13}$$

for all $t \in [0,1]$ and $u = (x_1,x_2)$, $v = (y_1,y_2) \in \mathbb{R}^2$. Then problem (3.10) has one solution if and only if (1.1) (with η, A and f defined as above) has one solution. Therefore, we need show that the conditions of Theorem 3.4 hold.

First, from (3.11)–(3.13), the function f satisfies the Carathéodory condition. Next we check conditions (A1)–(A3). It follows from (3.11), (3.12) and (3.13) that

$$|f(t,u,v)| \le a(t)|u| + b(t)|v| + c(t),$$

for all $t \in [0,1]$ and $u, v \in \mathbb{R}^2$, where

$$a(t) = \frac{t^3+3}{180}, \quad b(t) = \frac{t^5}{10}, \quad c(t) = \frac{t^3+3}{18}$$

are the positive integrable functions on $[0,1]$. By some simple computation, we get

$$\mathcal{M} = \begin{bmatrix} -3 & 7 \\ -3 & 7 \end{bmatrix}, \quad \mathcal{M}^+ = \begin{bmatrix} -3/116 & -3/116 \\ 7/116 & 7/116 \end{bmatrix}$$

and

$$(\|I - \mathcal{M}^+\mathcal{M}\|_* + C)(\|a\|_1 + \|b\|_1) = 0.39092 < 1.$$

Hence (A1) is satisfied. In order to check (A2) we note that

$$f_1(t,u(t),u'(t)) < f_2(t,u(t),u'(t)),$$

for all $u \in C^1([0,1];\mathbb{R}^2)$ and all $t \in [0,1]$. This implies that

$$\int_\eta^1 ds \int_0^s f(t,u(t),u'(t))dt \notin \mathrm{Im}\ \mathcal{M}$$

because $\mathrm{Im}\,\mathcal{M} = \langle (1,1) \rangle = \{(q,q) : q \in \mathbb{R}\}$. It means that (A2) holds. Finally, we choose a suitable isomorphism $J : \mathrm{Im}\, Q = \langle (0,1) \rangle \to \ker L \cong \langle (7,3) \rangle$ defined by the following non-degenerate matrix

$$J \equiv \begin{bmatrix} 1 & 7 \\ 0 & 3 \end{bmatrix}.$$

Thus, from (2.5)-(2.6), we have

$$JQ(z) = \begin{bmatrix} 28/3 & -28/3 \\ 4 & -4 \end{bmatrix} \phi(z),$$

for all $z \in L^1([0,1];\mathbb{R}^2)$, where

$$\phi(z) = A \int_0^{1/2} ds \int_0^s z(\tau)d\tau - \int_0^1 ds \int_0^s z(\tau)d\tau.$$

Let $\alpha = (7a, 3a) \in \ker \mathcal{M} = \langle (7,3) \rangle$, we have

$$(N\alpha)(t) = (f_1(t,\alpha,0),\ f_2(t,\alpha,0)) = \frac{t^3+3}{36}(a+1, |a|+2),$$

and

$$\phi(N\alpha) = \Big(\frac{-7}{5760}a - \frac{1687}{23040}|a| - \frac{189}{1280}, \frac{241}{7680}a - \frac{1219}{11520}|a| - \frac{3161}{23040} \Big).$$

So we obtain

$$JQ(N\alpha) = \Big(\frac{5257}{17280}|a| - \frac{5257}{17280}a - \frac{1687}{17280}, \frac{751}{5760}|a| - \frac{751}{5760}a - \frac{241}{5760} \Big).$$

Therefore,

$$\langle \alpha, JQN\alpha \rangle = -\frac{29}{8640}(751a^2 - 751a|a| + 241a) < 0,$$

for all real number a such that $|a| > \frac{241}{1502}$. That means (A3) is verified. Thanks to Theorem 3.4, the problem (3.10) has at least one solution.

References

[1] A. Ben-Israel, Thomas N.E. Greville; *Generalized Inverses: Theory and Applications*, Springer-Verlag, New York, 2003.

[2] M. S. Berger, L. E. Fraenkel; Nonlinear desingularization in certain free-boundary problems, Comm. Math. Phys., **77** (2) (1980), 149–172.

[3] W. Feng; On an m-point boundary value problems, Nonlinear Anal., **30** (1997), 5369–5374.

[4] W. Feng, J. R. L. Webb; Solvability of three-point boundary value problems at resonance, *Nonlinear Anal.*, **30** (6) (1997), 3227–3238.

[5] R. E. Gaines, J. Mawhin; *Coincidence Degree and Nonlinear Differential Equations*, vol. 568, Lecture Notes in Math., Springer-Verlag, Berlin, 1977.

[6] M. Gregus, F. Neuman, F. M. Arscott; Three-point boundary value problems in differential equations, *J. London Math. Soc.*, **2** (1971) 429–436.

[7] C. P. Gupta; Solvability of a three-point boundary value problem for a second order ordinary differential equation, *J. Math. Anal. Appl.*, **168** (1992), 540–551.

[8] C. P. Gupta, S. K. Ntouyas, P. Ch. Tsamatos; Solvability of an m-point boundary value problem for second order ordinary differential equations, *J. Math. Anal. Appl.*, **189** (1995), 575–584.

[9] D. H. Hoang, L. X. Truong; Fredholm alternative for the second order differential operator associated to a class of boundary conditions, *preprint*.

[10] N. Kosmatov; A multi-point boundary value problem with two critical conditions, *Nonlinear Anal.*, **65** (2006), 622–633.

[11] R. Ma; Existence results of a m-point boundary value problem at resonance, *J. Math. Anal. Appl.*, **294** (2004), 147–157.

[12] J. Mawhin; Theorems for nonlinear operator equations and coincidence degree theory for some mappings in locally convex topological vector spaces, *J. Differential Equations*, **12** (1972), 610–636.

[13] J. Mawhin; *Topological Degree Methods in Nonlinear Boundary Value Problems*, CBMS Series No. 40, American Math. Soc., Providence, 1979.

[14] M. Moshiinsky; Sobre los problemas de condiciones a la frontiera en una dimension de caracteristicas discontinuas, *Boletin De La Sociedad Matematica Mexicana*, **7** (1950), 1–25.

[15] P. D. Phung, L. X. Truong; On the existence of a three point boundary value problem at resonance in R^n, *J. Math. Anal. Appl.*, **416** (2014), 522–533.

[16] S. Timoshenko; *Theory of Elastic Stability*, McGraw-Hill, New York, 1961.

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