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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF GLOBAL REGULAR SOLUTIONS FOR A 3-D KAZHIKHOV-SMAGULOV MODEL WITH KORTEWEG STRESS

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Abstract. In this article, we consider a 3-D multiphasic incompressible fluid model, called the Kazhikhov-Smagulov model, with a specific Korteweg stress tensor. We prove the existence of a global unique regular solution to the Kazhikhov-Smagulov-Korteweg model provided that initial data and external force are sufficiently small. Furthermore, in the absence of external forcing, the solution decays exponentially in time to the equilibrium solution.

1. Introduction

In this article, we study a 3-D Kazhikhov-Smagulov-Korteweg (KSK) model describing the motion of a viscous incompressible mixture of two fluids having different densities. This type model can be derived from the compressible Navier-Stokes system. Let Ω be a bounded open set in $\mathbb{R}^3$ with boundary Γ that is regular enough. We denote by $[0,T]$ the time interval, for $T>0$. The mixture of two fluids is described by the density $\rho(t,\mathbf{x}) \geq 0$, the mass velocity field $\mathbf{v}(t,\mathbf{x})$ and the pressure $p(t,\mathbf{x})$, depending on the time and space variables $(t,\mathbf{x}) \in [0,T] \times \Omega$. According to Dunn and Serrin [8] (see also Bresch et al [6]), we consider the compressible Navier-Stokes system

$$
\begin{gathered}
\frac{\partial}{\partial t}(\rho\mathbf{v}) + \operatorname{div}\big(\rho\mathbf{v}\otimes\mathbf{v}\big) = \rho\mathbf{g} + \operatorname{div}\big(\mathbf{S}+\mathbf{K}\big), \\
\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{v}) = 0,
\end{gathered}
\tag{1.1}
$$

where $\mathbf{g}$ stands for the gravity acceleration (but it can include further external forces). The viscous stress tensor $\mathbf{S}$ and the Korteweg stress tensor $\mathbf{K}$ given by

$$
\begin{gathered}
\mathbf{S} = (\nu \operatorname{div}\mathbf{v} - p)\mathbf{I} + 2\mu\mathbf{D}(\mathbf{v}), \\
\mathbf{K} = (\alpha\Delta\rho + \beta|\nabla\rho|^2)\mathbf{I} + \delta(\nabla\rho\otimes\nabla\rho) + \gamma D_x^2\rho,
\end{gathered}
\tag{1.2}
$$

where $\mathbf{D}(\mathbf{v}) = (\nabla\mathbf{v} + \nabla\mathbf{v}^T)/2$ is the strain tensor and $D_x^2\rho$ is the hessian matrix of the density ρ. The pressure p and the coefficients α, β, γ, δ, ν and μ are functions

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of ρ. As in [9], choosing the viscosity coefficients ν and μ constants in the viscous stress tensor $\mathbf{S}$, we have

$$\operatorname{div}\mathbf{S} = \nu\nabla(\operatorname{div}\mathbf{v}) - \nabla p + 2\mu\operatorname{div}\big(\mathbf{D}(\mathbf{v})\big). \tag{1.3}$$

In the Korteweg stress tensor $\mathbf{K}$, we consider the special case:

$$\alpha = \kappa\rho, \quad \beta = \frac{\kappa}{2}, \quad \delta = -\kappa, \quad \gamma = 0,$$

for some constant $\kappa > 0$, called Korteweg's constant. This choice corresponds essentially to the Korteweg's original assumptions connected with the variational theory of Van Der Waals (see [10]). Therefore, the Korteweg stress tensor yields

$$\mathbf{K} = \frac{\kappa}{2}(\Delta\rho^2 - |\nabla\rho|^2)\mathbf{I} - \kappa(\nabla\rho\otimes\nabla\rho), \tag{1.4}$$

and we obtain

$$\operatorname{div}\mathbf{K} = \kappa\rho\nabla(\Delta\rho) = \kappa\nabla(\rho\Delta\rho) - \kappa\Delta\rho\nabla\rho. \tag{1.5}$$

On another side, Fick's law which relates the velocity to the derivatives of the density (see [11, 1]), gives

$$\mathbf{v} = \mathbf{u} - \lambda\nabla\ln(\rho), \tag{1.6}$$

with a volume velocity field $\mathbf{u}$ that is solenoidal ($\operatorname{div}\mathbf{u} = 0$) and $\lambda > 0$ is interpreted as a diffusion coefficient. Consequently, we use (1.6) in the compressible Navier-Stokes system (1.1), and after some calculations, we obtain the following system, that we call the Kazhikhov-Smagulov-Korteweg (KSK) model,

$$\begin{gathered}
\rho\Big(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big) - \mu\boldsymbol{\Delta}\mathbf{u} - \lambda\big(\nabla\rho\cdot\nabla\big)\mathbf{u} - \lambda\big(\mathbf{u}\cdot\nabla\big)\nabla\rho \\
+ \nabla P + \frac{\lambda^2}{\rho}\Big(\Delta\rho\nabla\rho + \big(\nabla\rho\cdot\nabla\big)\nabla\rho - \frac{|\nabla\rho|^2}{\rho}\nabla\rho\Big) = \rho\mathbf{g} - \kappa\Delta\rho\nabla\rho, \\
\frac{\partial\rho}{\partial t} + \mathbf{u}\cdot\nabla\rho = \lambda\Delta\rho, \\
\operatorname{div}\mathbf{u} = 0.
\end{gathered} \tag{1.7}$$

With $\mathcal{Q}_T = (0,T)\times\Omega$ and $\Sigma = (0,T)\times\Gamma$, the unknowns for the model (1.7) are $\rho : \mathcal{Q}_T \to \mathbb{R}$ the density of the fluid, $\mathbf{u} : \mathcal{Q}_T \to \mathbb{R}^3$ the incompressible velocity field and $P : \mathcal{Q}_T \to \mathbb{R}$ the modified pressure. We attach to (1.7) the following boundary and initial conditions:

$$\mathbf{u}(t,\mathbf{x}) = 0, \quad \frac{\partial\rho}{\partial\mathbf{n}}(t,\mathbf{x}) = 0, \quad (t,\mathbf{x})\in\Sigma, \tag{1.8}$$

$$\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \quad \rho(0,\mathbf{x}) = \rho_0(\mathbf{x}), \quad \mathbf{x}\in\Omega, \tag{1.9}$$

with the compatibility condition $\operatorname{div}\mathbf{u}_0 = 0$, where $\rho_0 : \Omega\to\mathbb{R}$ and $\mathbf{u}_0 : \Omega\to\mathbb{R}^3$ are given functions. We denote by $\mathbf{n}$ the unit outward normal on the boundary Γ. Throughout this work, we assume the hypothesis

$$0 < m \le \rho_0(\mathbf{x}) \le M < +\infty, \quad \mathbf{x}\in\Omega. \tag{1.10}$$

Let us mention some known results about the Kazhikhov-Smagulov model without the Korteweg stress tensor. Taking $\kappa = 0$, many authors study the global existence of solution for the so-called Kazhikhov-Smagulov model. We can refer for instance to [1, 11, 7, 14]. In [2], Beirão da Veiga considered the same model (1.7) without Korteweg term and proved the existence of a unique local solution for arbitrary initial data and external force and the existence of a unique global regular solution for small initial data and external force. Moreover, he proved that if $\mathbf{g} = 0$, the solution decay exponentially in time to the equilibrium solution with zero velocity field. In [5], Beirão da Veiga et al. have previously found the same results obtained in [2], in the non-viscous case for an Euler system.

The aim of this work is to establish the same kind of results given in [2] for (1.7). That is existence of a unique global in time regular solution of the Kazhikhov-Smagulov-Korteweg model (1.7) for small initial data and external force. Also, we study the longtime behavior of the solution and show that it converges to a constant solution with zero velocity field.

We think that the results presented here can be extended if we replace the Laplace operator by the p-Laplace operator $\operatorname{div}\big(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\big)$, $1 < p < \infty$, in the momentum equation $(1.7)_1$ [3]. Moreover, one aims to study the full regularity of the steady KSK model in the framework of functional spaces $C^{0,\lambda}_{\alpha}(\overline{\Omega})$ introduced recently by Beirão da Veiga in [4]. These will be investigated in future works.

The outline of the paper is as follows. In section 2 we present the functional setting and the main result of this paper, that will be proved in section 3.

2. Functional setup and main results

Let us introduce the following functional spaces (see [12, 15] for their properties):

$$\begin{gathered}
\mathcal{V} = \{\mathbf{u} \in \mathcal{D}(\Omega)^3 : \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega\},\\
\mathbf{V} = \{\mathbf{u} \in \mathbf{H}^1_0(\Omega) : \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega\},\\
\mathbf{H} = \{\mathbf{u} \in \mathbf{L}^2(\Omega) : \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega,\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma\}.
\end{gathered}$$

The spaces $\mathbf{V}$ and $\mathbf{H}$ are the closures of $\mathcal{V}$ in $\mathbf{H}^1_0(\Omega)$ and $\mathbf{L}^2(\Omega)$ respectively. Denoting by $\mathbb{P}$ the orthogonal projection operator of $\mathbf{L}^2(\Omega)$ onto $\mathbf{H}$, we define the Stokes operator $\mathbb{A} = -\mathbb{P}\Delta$ on $\mathbf{H}^2(\Omega) \cap \mathbf{V}$. The norms $\|\mathbf{u}\|_{H^1(\Omega)}$ and $\|\nabla\mathbf{u}\|_{L^2(\Omega)}$ are equivalent in $\mathbf{V}$, and the norms $\|\mathbf{u}\|_{H^2(\Omega)}$ and $\|\mathbb{A}\mathbf{u}\|_{L^2(\Omega)}$ are equivalent in $\mathbf{H}^2(\Omega)\cap\mathbf{V}$. Next, we consider the affine spaces

$$H^s_N = \{\rho \in H^s(\Omega) : \frac{\partial\rho}{\partial\mathbf{n}} = 0 \text{ on } \Gamma,\ \int_\Omega \rho(\mathbf{x})d\mathbf{x} = \int_\Omega \rho_0(\mathbf{x})\,d\mathbf{x}\}.$$

Evidently, $H^s_N = \widehat{\rho} + H^s_{N,0}$, where $\widehat{\rho} = \frac{1}{|\Omega|}\int_\Omega \rho_0(\mathbf{x})d\mathbf{x}$ and

$$H^s_{N,0} = \{\rho \in H^s(\Omega) : \frac{\partial\rho}{\partial\mathbf{n}} = 0 \text{ on } \Gamma,\ \int_\Omega \rho(\mathbf{x})\,d\mathbf{x} = 0\}.$$

Thus, $H^s_{N,0}$, for $s = 2, 3$, is a closed subspace of H^s_N. The norms $\|\rho\|_{H^2(\Omega)}$ and $\|\Delta\rho\|_{L^2(\Omega)}$ are equivalent in H^2_N, and the norms $\|\rho\|_{H^3(\Omega)}$ and $\|\nabla\Delta\rho\|_{L^2(\Omega)}$ are equivalent in H^3_N.

Next we state and prove the main result of this article.

Theorem 2.1. *Let $\mathbf{u}_0 \in \mathbf{V}$, $\rho_0 \in H^2(\Omega)$ satisfy (1.10), $T > 0$, $\mathbf{g} \in L^2\big(0, T; \mathbf{L}^2(\Omega)\big)$ and*

$$\widehat{\rho} = \frac{1}{|\Omega|}\int_\Omega \rho_0(\mathbf{x})\,d\mathbf{x}.$$

There exist positive constants γ_1, γ_2, γ_3 depending on Ω, λ, μ, κ, M, m, such that if

$$\begin{gathered}
\|\nabla\mathbf{u}_0\|^2_{L^2(\Omega)} + \|\rho_0 - \widehat{\rho}\|^2_{H^2(\Omega)} \le \gamma_1,\\
\|\mathbf{g}\|^2_{L^\infty(0,+\infty;L^2(\Omega))} \le \gamma_2,
\end{gathered} \tag{2.1}$$

then there exists a unique regular solution $(\mathbf{u},\rho)$ *of problem* (1.7), (1.8), (1.9)*, global in time such that*

$$\mathbf{u}\in L^2\big(0,T;\mathbf{H}^2(\Omega)\big)\cap\mathcal{C}\big([0,T];\mathbf{V}\big),$$
$$\rho\in L^2\big(0,T;H^3_N\big)\cap\mathcal{C}\big([0,T];H^2_N\big).$$

Moreover if $\mathbf{g}=\mathbf{0}$*, the solution* $(\mathbf{u},\rho)$ *decays exponentially in time to the equilibrium solution* $(\mathbf{0},\widehat{\rho})$*, such that* $\forall t\geq 0$*,*

$$\|\nabla\mathbf{u}(t)\|^2_{L^2(\Omega)}+\|\rho(t)-\widehat{\rho}\|^2_{H^2(\Omega)}\leq\big(\|\nabla\mathbf{u}_0\|^2_{L^2(\Omega)}+\|\rho_0-\widehat{\rho}\|^2_{H^2(\Omega)}\big)\mathrm{e}^{-\gamma_3 t}. \tag{2.2}$$

3. Proof of Theorem 2.1

Intermediate results. In this section we present some results to be used in proving Theorem 2.1. First of all, integrating the convection-diffusion equation $(1.7)_2$ over Ω, we see that

$$\frac{d}{dt}\int_\Omega\rho(t,\mathbf{x})\,d\mathbf{x}=0,$$

and we note that the mean value of ρ is conserved:

$$\int_\Omega\rho(t,\mathbf{x})\,d\mathbf{x}=\int_\Omega\rho_0(\mathbf{x})\,d\mathbf{x}.$$

Therefore, we set

$$\sigma=\rho-\widehat{\rho}, \tag{3.1}$$

such that $\widehat{\rho}=\frac{1}{|\Omega|}\int_\Omega\rho_0(\mathbf{x})\,d\mathbf{x}$ and $\int_\Omega\sigma(t,\mathbf{x})\,d\mathbf{x}=0$.

Next, the KSK model (1.7) is equivalent to find $(\mathbf{u},\sigma)$ satisfying

$$\begin{gathered}\mathbb{P}\big(\rho\frac{\partial\mathbf{u}}{\partial t}\big)-\mu\mathbb{P}\boldsymbol{\Delta}\mathbf{u}=\mathbf{F}(\mathbf{u},\sigma),\\ \frac{\partial\sigma}{\partial t}-\lambda\Delta\sigma=G(\mathbf{u},\sigma),\\ \operatorname{div}\mathbf{u}=0,\end{gathered} \tag{3.2}$$

where

$$\begin{gathered}\mathbf{F}(\mathbf{u},\sigma)=\mathbb{P}\Big(\rho\mathbf{g}-\kappa\Delta\rho\nabla\rho-\rho\big(\mathbf{u}\cdot\nabla\big)\mathbf{u}+\lambda\big(\nabla\rho\cdot\nabla\big)\mathbf{u}+\lambda\big(\mathbf{u}\cdot\nabla\big)\nabla\rho\\ -\frac{\lambda^2}{\rho}\Delta\rho\nabla\rho-\frac{\lambda^2}{\rho}\big(\nabla\rho\cdot\nabla\big)\nabla\rho+\lambda^2\frac{|\nabla\rho|^2}{\rho^2}\nabla\rho\Big),\\ G(\mathbf{u},\sigma)=-\mathbf{u}\cdot\nabla\sigma,\end{gathered} \tag{3.3}$$

Problem (3.2) is coupled with the boundary and initial conditions

$$\begin{gathered}\mathbf{u}(t,\mathbf{x})=0,\quad\frac{\partial\sigma}{\partial\mathbf{n}}(t,\mathbf{x})=0,\quad(t,\mathbf{x})\in\Sigma,\\ \mathbf{u}(0,\mathbf{x})=\mathbf{u}_0(\mathbf{x}),\quad\sigma(0,\mathbf{x})=\sigma_0(\mathbf{x}),\quad\mathbf{x}\in\Omega,\end{gathered}$$

where $\sigma_0(\mathbf{x})=\rho_0(\mathbf{x})-\widehat{\rho}$. We introduce the spaces:

$$\begin{aligned}\mathcal{X}_1=\Big\{\bar{\mathbf{u}}:\bar{\mathbf{u}}\in L^2\big(0,T;\mathbf{H}^2(\Omega)\big)\cap\mathcal{C}\big([0,T];\mathbf{V}\big);\frac{\partial\bar{\mathbf{u}}}{\partial t}\in L^2\big(0,T;\mathbf{H}\big);\bar{\mathbf{u}}(0)=\mathbf{u}_0;\\ \|\bar{\mathbf{u}}\|^2_{\mathcal{C}([0,T];\mathbf{V})}+\|\bar{\mathbf{u}}\|^2_{L^2(0,T;H^2(\Omega))}+\|\frac{\partial\bar{\mathbf{u}}}{\partial t}\|^2_{L^2(0,T;\mathbf{H})}\leq 2C_4\|\nabla\mathbf{u}_0\|^2_{L^2(\Omega)}\Big\}\end{aligned}$$

and

$$\mathcal{X}_2=\Big\{\bar{\sigma}:\bar{\sigma}\in L^2\big(0,T;H^3_{N,0}\big)\cap\mathcal{C}\big([0,T];H^2_{N,0}\big);\frac{\partial\bar{\sigma}}{\partial t}\in L^2\big(0,T;H^1(\Omega)\big);$$

$$
\begin{aligned}
&\bar\sigma(0)=\sigma_0;\ \|\bar\sigma\|^2_{\mathcal{C}([0,T];H^2(\Omega))}+\|\bar\sigma\|^2_{L^2(0,T;H^3(\Omega))}\le 2\|\sigma_0\|^2_{H^2(\Omega)};\\
&\|\frac{\partial\bar\sigma}{\partial t}\|^2_{L^2(0,T;H^1(\Omega))}\le K_0;\ \|\bar\sigma-\sigma_0\|_{\mathcal{C}(\bar{Q}_T)}\le\frac{m}{2}\Big\}.
\end{aligned}
$$

Here C_4 is a positive constant depending on μ, $\bar{M}$, $\bar{m}$ and we denote by K_0 a positive constant depending on norms of initial data $\|\nabla\mathbf{u}_0\|_{L^2(\Omega)}$ and $\|\sigma_0\|_{H^2(\Omega)}$.

Now, we define the linearized problem as follows:

Given $(\bar{\mathbf{u}},\bar\sigma)\in\mathcal{X}_1\times\mathcal{X}_2$ such that $\bar\sigma=\bar\rho-\widehat\rho$, find $(\mathbf{u},\sigma)\in\mathcal{X}_1\times\mathcal{X}_2$ such that $\sigma=\rho-\widehat\rho$ satisfying

$$
\begin{gathered}
\mathbb{P}(\bar\rho\frac{\partial\mathbf{u}}{\partial t})+\mu\mathbb{A}\mathbf{u}=\mathbf{F}(\bar{\mathbf{u}},\bar\sigma),\\
\frac{\partial\sigma}{\partial t}-\lambda\ \Delta\sigma=G(\bar{\mathbf{u}},\bar\sigma),\\
\operatorname{div}\mathbf{u}=0,\\
\int_\Omega\sigma(t,\mathbf{x})\,d\mathbf{x}=0,
\end{gathered}
\tag{3.4}
$$

For $(\bar{\mathbf{u}},\bar\sigma)\in\mathcal{X}_1\times\mathcal{X}_2$, we define the map

$$
\Phi:\mathcal{X}_1\times\mathcal{X}_2\to\mathcal{X}_1\times\mathcal{X}_2,
$$

such that $\Phi(\bar{\mathbf{u}},\bar\sigma)=(\mathbf{u},\sigma)$ defined by (3.4). Since (3.4) is a linear problem with respect to $\mathbf{u}$ and σ, it is clear that Φ is well defined (see [2, §2], [13, Vol.I, Chap.1, Theorem 3.1] and [13, Vol.II, Chap.4, Theorem 5.2]).

Analogously as in [2], we can prove *the existence of a local regular solution in time* to (1.7) for arbitrary initial data and external force in the three-dimensional case. For this, we consider the linearized problem (3.4) and we prove via an application of Schauder fixed point theorem, the existence of a fixed point $(\bar{\mathbf{u}},\bar\sigma)\in\mathcal{X}_1\times\mathcal{X}_2$ for the map Φ, such that

$$
(\bar{\mathbf{u}},\bar\sigma)=(\mathbf{u},\sigma).
$$

(See [2] for a detailed study.) To prove the main result of this article, Theorem 2.1, we need some useful results. On one hand, from the estimate (1.10) for the initial density ρ_0 follows a similar estimate for $\bar\rho$.

Proposition 3.1. *Let $\bar\sigma\in\mathcal{X}_2$. Then the function $\bar\rho=\bar\sigma+\widehat\rho$ satisfies*

$$
\bar m\equiv\frac{m}{2}\le\bar\rho(t,\mathbf{x})\le M+\frac{m}{2}\equiv\bar M,\quad (t,\mathbf{x})\in\mathcal{Q}_T.
\tag{3.5}
$$

On the other hand, the right-hand side $\mathbf{F}(\bar{\mathbf{u}},\bar\sigma)$ of (3.4), defined by (3.3), satisfies the following property.

Proposition 3.2. *Let $\mathbf{g}\in L^2\big(0,T,\mathbf{L}^2(\Omega)\big)$ and $(\bar{\mathbf{u}},\bar\sigma)\in\mathcal{X}_1\times\mathcal{X}_2$. Then $\mathbf{F}(\bar{\mathbf{u}},\bar\sigma)$ defined by (3.3), satisfies*

$$
\begin{aligned}
\|\mathbf{F}(\bar{\mathbf{u}},\bar\sigma)\|^2_{L^2(\Omega)}\le C\Big(&\|\nabla\bar{\mathbf{u}}\|^{2(1+\beta)}_{L^2(\Omega)}\|\nabla\bar{\mathbf{u}}\|^{2(1-\beta)}_{H^1(\Omega)}+\|\nabla\bar\sigma\|^{2(1+\beta)}_{H^1(\Omega)}\|\Delta\bar\sigma\|^{2(1-\beta)}_{H^1(\Omega)}\\
&+\|\nabla\nabla\bar\sigma\|^{2\beta}_{L^2(\Omega)}\|\nabla\nabla\bar\sigma\|^{2(1-\beta)}_{H^1(\Omega)}\|\nabla\bar{\mathbf{u}}\|^2_{L^2(\Omega)}+\|\nabla\bar\sigma\|^6_{H^1(\Omega)}\\
&+\|\nabla\bar{\mathbf{u}}\|^{2\beta}_{L^2(\Omega)}\|\nabla\bar{\mathbf{u}}\|^{2(1-\beta)}_{H^1(\Omega)}\|\nabla\bar\sigma\|^2_{H^1(\Omega)}+\|\mathbf{g}\|^2_{L^2(\Omega)}\Big),
\end{aligned}
\tag{3.6}
$$

where $C=C\big(\lambda,\kappa,\bar M,\bar m\big)$, and

$$
\beta=\begin{cases}1/2 & \text{if } d=2,\\ 1/4 & \text{if } d=3.\end{cases}
$$

Lemma 3.3. *Let $(\bar{\mathbf{u}},\bar{\sigma}) \in \mathcal{X}_1 \times \mathcal{X}_2$ and $\mathbf{F}(\bar{\mathbf{u}},\bar{\sigma}) \in \mathbf{L}^2(\Omega)$ satisfy (3.3). Then a solution $(\mathbf{u},\sigma)$ of the linearized problem (3.4) satisfies the following estimates:*

$$\begin{aligned} &\frac{\mu}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} + \frac{\mu\varepsilon_0}{2}\|\mathbb{A}\mathbf{u}\|^2_{L^2(\Omega)} + \big(\frac{3m}{4} - \frac{\varepsilon_0 M^2}{\mu}\big)\|\frac{\partial\mathbf{u}}{\partial t}\|^2_{L^2(\Omega)} \\ &\le \big(\frac{1}{m} + \frac{\varepsilon_0}{\mu}\big)\|\mathbf{F}(\bar{\mathbf{u}},\bar{\sigma})\|^2_{L^2(\Omega)}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\frac{d}{dt}\|\Delta\sigma\|^2_{L^2(\Omega)} + \lambda\|\nabla\Delta\sigma\|^2_{L^2(\Omega)} \\ &\le C_1\varepsilon_1\Big(\|\nabla\bar{\mathbf{u}}\|^2_{H^1(\Omega)} + \|\nabla\nabla\bar{\sigma}\|^2_{H^1(\Omega)}\Big) + 2C_2\varepsilon_1^{-k_d}\Big(\|\bar{\mathbf{u}}\|^{k_d+3}_{H^1(\Omega)} + \|\nabla\bar{\sigma}\|^{k_d+3}_{H^1(\Omega)}\Big), \end{aligned} \tag{3.8}$$

where ε_0, ε_1 being arbitrary, C_1, C_2 are positive constants depending only on Ω, and

$$k_d = \begin{cases} 3 & \text{if } d = 2, \\ 7 & \text{if } d = 3. \end{cases}$$

Global solutions. Let $(\mathbf{u},\rho)$ be a local solution of (1.7), such that $\rho = \sigma + \hat{\rho}$. We will prove that this local solution is, in fact, a global solution. On the one hand, we choose $\varepsilon_0 = \frac{m\mu}{4M^2}$ in (3.7) to obtain

$$\frac{\mu}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} + \frac{m}{2}\|\frac{\partial\mathbf{u}}{\partial t}\|^2_{L^2(\Omega)} + \frac{m\mu^2}{8M^2}\|\mathbb{A}\mathbf{u}\|^2_{L^2(\Omega)} \le \big(\frac{1}{m} + \frac{m}{4M^2}\big)\|\mathbf{F}\|^2_{L^2(\Omega)}.$$

Next, we use (3.6) for $\beta = \frac{1}{4}$ as follows:

$$\begin{aligned} \|\mathbf{F}\|^2_{L^2(\Omega)} \le C\Big(&\|\nabla\mathbf{u}\|^{5/2}_{L^2(\Omega)}\|\nabla\mathbf{u}\|^{3/2}_{H^1(\Omega)} + \|\nabla\sigma\|^{5/2}_{H^1(\Omega)}\|\Delta\sigma\|^{3/2}_{H^1(\Omega)} \\ &+ \|\nabla\mathbf{u}\|^{1/2}_{L^2(\Omega)}\|\nabla\mathbf{u}\|^{3/2}_{H^1(\Omega)}\|\nabla\sigma\|^2_{H^1(\Omega)} + \|\nabla\sigma\|^6_{H^1(\Omega)} \\ &+ \|\nabla\nabla\sigma\|^{1/2}_{L^2(\Omega)}\|\nabla\nabla\sigma\|^{3/2}_{H^1(\Omega)}\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} + \|\mathbf{g}\|^2_{L^2(\Omega)}\Big). \end{aligned}$$

Applying the Young inequality $\big(ab \le \frac{a^5}{5} + \frac{4}{5}b^{5/4}\big)$, we obtain

$$\begin{aligned} \|\mathbf{F}\|^2_{L^2(\Omega)} \le C\Big(&\big(\|\nabla\mathbf{u}\|^{5/2}_{L^2(\Omega)} + \|\nabla\sigma\|^{5/2}_{H^1(\Omega)}\big)\big(\|\nabla\mathbf{u}\|^{3/2}_{H^1(\Omega)} + \|\Delta\sigma\|^{3/2}_{H^1(\Omega)}\big) \\ &+ \|\mathbf{g}\|^2_{L^2(\Omega)} + \|\nabla\sigma\|^6_{H^1(\Omega)}\Big). \end{aligned}$$

Consequently,

$$\begin{aligned} &\frac{\mu}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} + \frac{m}{2}\|\frac{\partial\mathbf{u}}{\partial t}\|^2_{L^2(\Omega)} + \frac{m\mu^2}{8M^2}\|\mathbb{A}\mathbf{u}\|^2_{L^2(\Omega)} \\ &\le C\big(\|\nabla\mathbf{u}\|^{5/2}_{L^2(\Omega)} + \|\nabla\sigma\|^{5/2}_{H^1(\Omega)}\big)\big(\|\nabla\mathbf{u}\|^{3/2}_{H^1(\Omega)} + \|\Delta\sigma\|^{3/2}_{H^1(\Omega)}\big) \\ &\quad + C\|\nabla\sigma\|^6_{H^1(\Omega)} + C\|\mathbf{g}\|^2_{L^2(\Omega)}, \end{aligned} \tag{3.9}$$

where $C = C(\lambda,\kappa,M,m)$. On the other hand, using (3.8) for $k_d = 7$ and taking $\varepsilon_1 = \min\big(\frac{\lambda}{2C_1}, \frac{m\mu^2}{32M^2C_1}\big)$, we obtain

$$\begin{aligned} &\frac{d}{dt}\|\Delta\sigma\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|\nabla\Delta\sigma\|^2_{L^2(\Omega)} \\ &\le \frac{m\mu^2}{32M^2}\|\nabla\mathbf{u}\|^2_{H^1(\Omega)} + C\big(\|\mathbf{u}\|^{10}_{H^1(\Omega)} + \|\nabla\sigma\|^{10}_{H^1(\Omega)}\big), \end{aligned} \tag{3.10}$$

where $C = C(\lambda, \mu, M, m, \Omega)$. From (3.9) and (3.10), and recalling the equivalent norms $\|\mathbf{u}\|_{H^2(\Omega)}$ and $\|\mathbb{A}\mathbf{u}\|_{L^2(\Omega)}$ in $\mathbf{H}^2(\Omega) \cap \mathbf{V}$, it follows easily that

$$\begin{aligned}
&\frac{d}{dt}\Big(\frac{\mu}{2}\|\nabla \mathbf{u}\|^2_{L^2(\Omega)} + \|\Delta\sigma\|^2_{L^2(\Omega)}\Big) + \frac{m}{2}\|\frac{\partial \mathbf{u}}{\partial t}\|^2_{L^2(\Omega)} \\
&+ \frac{3m\mu^2}{32M^2}\|\mathbb{A}\mathbf{u}\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|\nabla\Delta\sigma\|^2_{L^2(\Omega)} \\
&\le C\big(\|\nabla \mathbf{u}\|^{10}_{L^2(\Omega)} + \|\Delta\sigma\|^{10}_{L^2(\Omega)}\big) + C\big(\|\nabla \mathbf{u}\|^{5/2}_{L^2(\Omega)} + \|\Delta\sigma\|^{5/2}_{L^2(\Omega)}\big) \\
&\quad \times \big(\|\mathbb{A}\mathbf{u}\|^{3/2}_{L^2(\Omega)} + \|\nabla\Delta\sigma\|^{3/2}_{L^2(\Omega)}\big) + C\|\Delta\sigma\|^6_{L^2(\Omega)} + C\|\mathbf{g}\|^2_{L^2(\Omega)}.
\end{aligned} \tag{3.11}$$

Using the Young inequality $\big(ab \le \frac{a^4}{4} + \frac{3}{4}b^{4/3}\big)$, inequality (3.11) is rewritten as

$$\begin{aligned}
&\frac{d}{dt}\Big(\frac{\mu}{2}\|\nabla \mathbf{u}\|^2_{L^2(\Omega)} + \|\Delta\sigma\|^2_{L^2(\Omega)}\Big) + \frac{m}{2}\|\frac{\partial \mathbf{u}}{\partial t}\|^2_{L^2(\Omega)} \\
&+ \frac{m\mu^2}{16M^2}\|\mathbb{A}\mathbf{u}\|^2_{L^2(\Omega)} + \frac{\lambda}{4}\|\nabla\Delta\sigma\|^2_{L^2(\Omega)} \\
&\le C\Big(\|\nabla \mathbf{u}\|^{10}_{L^2(\Omega)} + \|\Delta\sigma\|^{10}_{L^2(\Omega)} + \|\mathbf{g}\|^2_{L^2(\Omega)} + \|\Delta\sigma\|^6_{L^2(\Omega)}\Big),
\end{aligned}$$

where $C = C(\lambda, \mu, \kappa, M, m, \Omega)$. Then, put $\alpha = \min(\frac{\mu}{2}, 1)$ and we write the above inequality as

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\nabla \mathbf{u}\|^2_{L^2(\Omega)} + \|\Delta\sigma\|^2_{L^2(\Omega)}\Big) + \frac{m}{2\alpha}\|\frac{\partial \mathbf{u}}{\partial t}\|^2_{L^2(\Omega)} \\
&+ \frac{m\mu^2}{16M^2\alpha}\|\mathbb{A}\mathbf{u}\|^2_{L^2(\Omega)} + \frac{\lambda}{4\alpha}\|\nabla\Delta\sigma\|^2_{L^2(\Omega)} \\
&\le \frac{C}{\alpha}\big(\|\nabla \mathbf{u}\|^2_{L^2(\Omega)} + \|\Delta\sigma\|^2_{L^2(\Omega)}\big)^4\big(\|\nabla \mathbf{u}\|^2_{L^2(\Omega)} + \|\Delta\sigma\|^2_{L^2(\Omega)}\big) \\
&\quad + \frac{C}{\alpha}\big(\|\nabla \mathbf{u}\|^2_{L^2(\Omega)} + \|\Delta\sigma\|^2_{L^2(\Omega)}\big)^2\big(\|\nabla \mathbf{u}\|^2_{L^2(\Omega)} + \|\Delta\sigma\|^2_{L^2(\Omega)}\big) + \frac{C}{\alpha}\|\mathbf{g}\|^2_{L^2(\Omega)}.
\end{aligned}$$

Since $\|\mathbb{A}\mathbf{u}\|_{L^2(\Omega)} \ge C_\Omega\|\nabla \mathbf{u}\|_{L^2(\Omega)}$ and $\|\nabla\Delta\sigma\|_{L^2(\Omega)} \ge C_\Omega\|\Delta\sigma\|_{L^2(\Omega)}$, it follows that for some positive constants c_1, c_2 depending on Ω, λ, μ, κ, M, m, we have

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\nabla \mathbf{u}\|^2_{L^2(\Omega)} + \|\Delta\sigma\|^2_{L^2(\Omega)}\Big) \\
&\le c_2\|\mathbf{g}\|^2_{L^2(\Omega)} - \Big[c_1 - c_2\big(\|\nabla \mathbf{u}\|^2_{L^2(\Omega)} + \|\Delta\sigma\|^2_{L^2(\Omega)}\big)^4 \\
&\quad - c_2\big(\|\nabla \mathbf{u}\|^2_{L^2(\Omega)} + \|\Delta\sigma\|^2_{L^2(\Omega)}\big)^2\Big]\big(\|\nabla \mathbf{u}\|^2_{L^2(\Omega)} + \|\Delta\sigma\|^2_{L^2(\Omega)}\big).
\end{aligned} \tag{3.12}$$

Integrating in time from 0 to $t < T_1$, and taking into account that $(\mathbf{u}, \sigma) \in \mathcal{X}_1 \times \mathcal{X}_2$, we find for every $t \in [0, T_1)$,

$$\begin{aligned}
&\|\nabla \mathbf{u}(t)\|^2_{L^2(\Omega)} + \|\Delta\sigma(t)\|^2_{L^2(\Omega)} \\
&\le \|\nabla \mathbf{u}_0\|^2_{L^2(\Omega)} + \|\Delta\sigma_0\|^2_{L^2(\Omega)} - 2\big(C_4\|\nabla \mathbf{u}_0\|^2_{L^2(\Omega)} + \|\Delta\sigma_0\|^2_{L^2(\Omega)}\big) \\
&\quad \times \Big[c_1 - 16c_2\big(C_4\|\nabla \mathbf{u}_0\|^2_{L^2(\Omega)} + \|\Delta\sigma_0\|^2_{L^2(\Omega)}\big)^4 - 4c_2\big(C_4\|\nabla \mathbf{u}_0\|^2_{L^2(\Omega)} \\
&\quad + \|\Delta\sigma_0\|^2_{L^2(\Omega)}\big)^2\Big]T_1 + c_2\|\mathbf{g}\|^2_{L^\infty(0,T_1,L^2(\Omega))}T_1.
\end{aligned}$$

Consequently, for every $t \in [0, T_1)$,

$$\|\nabla \mathbf{u}(t)\|^2_{L^2(\Omega)} + \|\Delta\sigma(t)\|^2_{L^2(\Omega)} \le \|\nabla \mathbf{u}_0\|^2_{L^2(\Omega)} + \|\Delta\sigma_0\|^2_{L^2(\Omega)},$$

provided that

$$
\begin{gathered}
C_4\|\nabla\mathbf{u}_0\|^2_{L^2(\Omega)}+\|\Delta\sigma_0\|^2_{L^2(\Omega)}<\frac{1}{2}\Big(\frac{\sqrt{\frac{c_1}{2c_2}+1}-1}{2}\Big)^{1/2},\\
c_2\|\mathbf{g}\|^2_{L^\infty(0,+\infty;L^2(\Omega))}<\frac{7}{8}\ c_1\Big(\frac{\sqrt{\frac{c_1}{2c_2}+1}-1}{2}\Big)^{1/2}.
\end{gathered}
\tag{3.13}
$$

Finally, we conclude that $(\mathbf{u},\sigma)$, such that $\sigma=\rho-\widehat{\rho}$, is a global solution of (3.2), and for all $T>0$, we have

$$
\begin{gathered}
\mathbf{u}\in L^2\big(0,T;\mathbf{H}^2(\Omega)\big)\cap\mathcal{C}\big([0,T];\mathbf{V}\big),\\
\rho-\widehat{\rho}\in L^2\big(0,T;H^3_{N,0}\big)\cap\mathcal{C}\big([0,T];H^2_{N,0}\big).
\end{gathered}
$$

Uniqueness. Let $(\mathbf{u}_1,\rho_1)$, $(\mathbf{u}_2,\rho_2)$ be two solutions of (1.7) such that $\mathbf{u}_1(0,\mathbf{x})=\mathbf{u}_2(0,\mathbf{x})=\mathbf{u}_0(\mathbf{x})$ and $\rho_1(0,\mathbf{x})=\rho_2(0,\mathbf{x})=\rho_0(\mathbf{x})$. We put $\mathbf{u}=\mathbf{u}_1-\mathbf{u}_2$ and $\rho=\rho_1-\rho_2$. The system verified by $(\mathbf{u},\rho)$ reads

$$
\begin{gathered}
\mathbb{P}\Big(\rho_1\frac{\partial\mathbf{u}}{\partial t}\Big)+\mathbb{P}\Big(\rho\frac{\partial\mathbf{u}_2}{\partial t}\Big)+\mu\mathbb{A}\mathbf{u}=\mathbf{F}_1-\mathbf{F}_2,\\
\frac{\partial\rho}{\partial t}+\mathbf{u}_1\cdot\nabla\rho+\mathbf{u}\cdot\nabla\rho_2=\lambda\Delta\rho,\\
\operatorname{div}\mathbf{u}=0,\\
\mathbf{u}(0,\mathbf{x})=0,\quad \rho(0,\mathbf{x})=0,
\end{gathered}
\tag{3.14}
$$

where

$$
\begin{aligned}
\mathbf{F}_1&\equiv\mathbf{F}(\mathbf{u}_1,\rho_1)\\
&=\mathbb{P}\Big(\rho_1\ \mathbf{g}-\kappa\Delta\rho_1\nabla\rho_1-\rho_1(\mathbf{u}_1\cdot\nabla)\mathbf{u}_1+\lambda(\nabla\rho_1\cdot\nabla)\mathbf{u}_1\\
&\quad+\lambda(\mathbf{u}_1\cdot\nabla)\nabla\rho_1-\frac{\lambda^2}{\rho_1}\Delta\rho_1\nabla\rho_1-\frac{\lambda^2}{\rho_1}\big(\nabla\rho_1\cdot\nabla\big)\nabla\rho_1+\lambda^2\frac{|\nabla\rho_1|^2}{\rho_1^2}\nabla\rho_1\Big),\\
\mathbf{F}_2&\equiv\mathbf{F}(\mathbf{u}_2,\rho_2)\\
&=\mathbb{P}\Big(\rho_2\mathbf{g}-\kappa\Delta\rho_2\nabla\rho_2-\rho_2(\mathbf{u}_2\cdot\nabla)\mathbf{u}_2+\lambda(\nabla\rho_2\cdot\nabla)\mathbf{u}_2\\
&\quad+\lambda(\mathbf{u}_2\cdot\nabla)\nabla\rho_2-\frac{\lambda^2}{\rho_2}\Delta\rho_2\nabla\rho_2-\frac{\lambda^2}{\rho_2}\big(\nabla\rho_2\cdot\nabla\big)\nabla\rho_2+\lambda^2\frac{|\nabla\rho_2|^2}{\rho_2^2}\nabla\rho_2\Big).
\end{aligned}
$$

First, taking the inner product of $(3.14)_1$ with $\mathbf{u}$ in $\mathbf{H}$, we have

$$
\Big(\mathbb{P}\big(\rho_1\frac{\partial\mathbf{u}}{\partial t}\big),\mathbf{u}\Big)+\Big(\mathbb{P}\big(\rho\frac{\partial\mathbf{u}_2}{\partial t}\big),\mathbf{u}\Big)+\mu\Big(\mathbb{A}\mathbf{u},\mathbf{u}\Big)=\Big(\mathbf{F}_1-\mathbf{F}_2,\mathbf{u}\Big).
$$

Then, by using the definition of operator $\mathbb{P}$, such that

$$
\Big(\mathbb{P}\mathbf{u},\mathbf{v}\Big)=\big(\mathbf{u},\mathbf{v}\big),\quad \forall\mathbf{u}\in\mathbf{L}^2(\Omega),\ \forall\mathbf{v}\in\mathbf{H},
$$

we have

$$
\Big(\rho_1\frac{\partial\mathbf{u}}{\partial t},\mathbf{u}\Big)=\frac{1}{2}\frac{d}{dt}\Big(\rho_1\mathbf{u},\mathbf{u}\Big)-\frac{1}{2}\Big(\frac{\partial\rho_1}{\partial t}\mathbf{u},\mathbf{u}\Big).
$$

Since ρ_1 is a solution of the convection-diffusion equation $(1.7)_2$, we obtain

$$
\frac{1}{2}\frac{d}{dt}\Big(\rho_1\mathbf{u},\mathbf{u}\Big)+\mu\|\nabla\mathbf{u}\|^2_{L^2(\Omega)}
$$

$$= \frac{\lambda}{2}\Big(\Delta\rho_1, \mathbf{u}^2\Big) - \frac{1}{2}\Big(\mathbf{u}_1\cdot\nabla\rho_1, \mathbf{u}^2\Big) - \Big(\rho\frac{\partial \mathbf{u}_2}{\partial t}, \mathbf{u}\Big) + \Big(\mathbf{F}_1 - \mathbf{F}_2, \mathbf{u}\Big).$$

By using Green's theorem and Cauchy-Schwarz and Young inequalities, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\rho_1\mathbf{u}, \mathbf{u}\big) + \frac{\mu}{2}\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} \\
&\le \frac{\lambda}{4}\|\Delta\rho\|^2_{L^2(\Omega)} + \Big(\frac{C}{\lambda}\|\frac{\partial\mathbf{u}_2}{\partial t}\|^2_{L^2(\Omega)} + \frac{C\lambda^2}{2\mu}\|\nabla\rho_1\|^2_{L^\infty(\Omega)} \\
&\quad + \frac{1}{2}\|\nabla\rho_1\|_{L^\infty(\Omega)}\|\mathbf{u}_1\|_{L^\infty(\Omega)}\Big)\|\mathbf{u}\|^2_{L^2(\Omega)} + \big(\mathbf{F}_1 - \mathbf{F}_2, \mathbf{u}\big).
\end{aligned} \tag{3.15}$$

Second, taking the inner product of $(3.14)_2$ with $-\Delta\rho$ in $L^2(\Omega)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\rho\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|\Delta\rho\|^2_{L^2(\Omega)} \\
&\le \frac{1}{\lambda}\|\mathbf{u}_1\|^2_{L^\infty(\Omega)}\|\nabla\rho\|^2_{L^2(\Omega)} + \frac{1}{\lambda}\|\nabla\rho_2\|^2_{L^\infty(\Omega)}\|\mathbf{u}\|^2_{L^2(\Omega)}.
\end{aligned} \tag{3.16}$$

By adding (3.15) and (3.16), it follows that

$$\begin{aligned}
&\frac{d}{dt}\Big(\big(\rho_1\mathbf{u}, \mathbf{u}\big) + \|\nabla\rho\|^2_{L^2(\Omega)}\Big) + \mu\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|\Delta\rho\|^2_{L^2(\Omega)} \\
&\le \Psi_1(t)\ \Big(m\|\mathbf{u}\|^2_{L^2(\Omega)} + \|\nabla\rho\|^2_{L^2(\Omega)}\Big) + 2\big(\mathbf{F}_1 - \mathbf{F}_2, \mathbf{u}\big),
\end{aligned} \tag{3.17}$$

where $\Psi_1 \in L^1\big([0,T]\big)$ dependent on $\mathbf{u}_1$, $\mathbf{u}_2$, ρ_1, ρ_2. In particular, applying Cauchy-Schwarz and Young inequalities $\big(ab \le \varepsilon a^2 + \frac{b^2}{\varepsilon}\big)$, the embedding $H^2(\Omega) \subset L^\infty(\Omega)$ and the equivalent norms, we obtain the inequality

$$2\Big|\big(\mathbf{F}_1 - \mathbf{F}_2, \mathbf{u}\big)\Big| \le \Psi_2(t)\Big(m\|\mathbf{u}\|^2_{L^2(\Omega)} + \|\nabla\rho\|^2_{L^2(\Omega)}\Big) + \varepsilon\Big(\|\mathbf{u}\|^2_{H^1(\Omega)} + \|\rho\|^2_{H^2(\Omega)}\Big),$$

where $\Psi_2 \in L^1\big([0,T]\big)$ dependent on ε, $\mathbf{u}_1$, $\mathbf{u}_2$, ρ_1, ρ_2, $\mathbf{g}$, with $\varepsilon > 0$ being arbitrary. Therefore, using this last estimate in (3.17) and choosing $\varepsilon > 0$ such that $\varepsilon < \min\big(\mu, \frac{\lambda}{2}\big)$, we arrive at

$$\frac{d}{dt}\Big(\big(\rho_1\mathbf{u}, \mathbf{u}\big) + \|\nabla\rho\|^2_{L^2(\Omega)}\Big) \le \Big(\Psi_1(t) + \Psi_2(t)\Big)\Big(m\|\mathbf{u}\|^2_{L^2(\Omega)} + \|\nabla\rho\|^2_{L^2(\Omega)}\Big).$$

Since ρ_1 is a solution of (1.7) satisfying the maximum principle, we have $\|\mathbf{u}\|^2_{L^2(\Omega)} \le m^{-1}\big(\rho_1\mathbf{u}, \mathbf{u}\big)$ and we obtain

$$\frac{d}{dt}\Big(\big(\rho_1\mathbf{u}, \mathbf{u}\big) + \|\nabla\rho\|^2_{L^2(\Omega)}\Big) \le \Big(\Psi_1(t) + \Psi_2(t)\Big)\Big(\big(\rho_1\mathbf{u}, \mathbf{u}\big) + \|\nabla\rho\|^2_{L^2(\Omega)}\Big).$$

Finally, from the Gronwall Lemma and from $\mathbf{u}(0) = 0$, $\rho(0) = 0$, we deduce the uniqueness of the solution of (1.7).

Asymptotic behavior. Let us prove the inequality (2.2) in Theorem 2.1. Assume that $\mathbf{g} = \mathbf{0}$. Then under hypothesis $(3.13)_1$, the inequality (3.12) is rewritten as

$$\frac{d}{dt}\Big(\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} + \|\Delta\sigma\|^2_{L^2(\Omega)}\Big) \le -\frac{7}{8}c_1\Big(\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} + \|\Delta\sigma\|^2_{L^2(\Omega)}\Big).$$

Consequently, since $\sigma = \rho - \widehat{\rho}$ and from Gronwall Lemma, we obtain (2.2). Finally, from this inequality (2.2), we conclude that the solution $(\mathbf{u}, \rho)$ of (1.7), converges to a constant solution as $t \to +\infty$:

$$\mathbf{u}(t, \mathbf{x}) \to \mathbf{0} \quad \text{in } \mathbf{V},$$

$$\rho(t, \mathbf{x}) \to \widehat{\rho} \quad \text{in } H_N^2.$$

The convergence is exponential in time. The proof of Theorem 2.1 is complete.

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