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NON-OSCILLATION OF PERIODIC HALF-LINEAR EQUATIONS IN THE CRITICAL CASE

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ABSTRACT. Recently, it was shown that the Euler type half-linear differential equations

$$[r(t)t^{p-1}\Phi(x')]' + \frac{s(t)}{t\log^p t}\Phi(x) = 0$$

with periodic coefficients r, s are conditionally oscillatory and the critical oscillation constant was found. Nevertheless, the critical case remains unsolved. The objective of this article is to study the critical case. Thus, we consider the critical value of the coefficients and we prove that any considered equation is non-oscillatory. Moreover, we analyze the situation when the periods of coefficients r, s do not need to coincide.

1. Introduction

In this article, we study the oscillation behaviour of the equation

$$[r^{-p/q}(t)t^{p-1}\Phi(x')]' + \frac{s(t)}{t\log^p t}\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1}\operatorname{sgn} x, \tag{1.1}$$

where $p > 1$, log is the natural logarithm, $r > 0$ and s are continuous functions, and q is the number conjugated with p, i.e., $q = p/(p-1)$. The main motivation of the presented research comes from [25], where the equation

$$[r(t)t^{p-1}\Phi(x')]' + \frac{s(t)}{t\log^p t}\Phi(x) = 0 \tag{1.2}$$

is proved to be conditionally oscillatory. It means that there exists the so-called critical oscillation constant, which is a positive value given by coefficients r and s with the following property:

(1) If the coefficients indicate a value greater than the critical one, then (1.2) *is oscillatory*;
(2) If the coefficients indicate a value less than the critical one, then (1.2) *is non-oscillatory*.

We point out that for the equations studied here, all solutions are oscillatory if and only if a non-trivial solutions is oscillatory.

Note that, in [25], Equation (1.2) is considered without the power $-p/q$ in the first term. Nevertheless, since function r is positive, it does not have any impact.

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We consider (1.2) in the presented form only because of technical reasons, i.e., the technical parts of our processes are more transparent. The described result from [25] rewritten for (1.1) is explicitly mentioned in Theorem 4.2 below.

Since the case when the coefficients indicate exactly the critical value is open, the aim of this article is to fill this gap. We will consider (1.1) with periodic continuous coefficients. We will not require any common period for the coefficients r and s.

Now, let us give a short overview of the literature. The fundamental theory concerning half-linear differential equations can be found in books [1, 5]. As basic papers about half-linear equations, we refer to [7, 8]. For the analyzed conditional oscillation of half-linear differential equations, we mention, e.g., papers [4, 11, 13, 23, 27] and the paper [25] which we have already mentioned as the primary motivation. The corresponding results dealing with difference equations and with dynamic equations on time scales are also present in the literature, but they are still behind the continuous case. See [24, 26] for the discrete equations and [15] for the dynamic equations on time scales. In the linear case, there are many relevant results. We mention at least the most relevant papers [9, 12, 18, 28].

This article is organized as follows. In the next section, we give only necessary preliminaries including the half-linear trigonometric functions and the equation for the Prüfer angle, which will allow us to investigate the (non-)oscillation of (1.1). In Section 3, we prove auxiliary results and we mention the later used known results. Finally, in Section 4, we formulate, prove, and illustrate by examples the main result. To the best of our knowledge, the presented result is new in the linear case as well (see Corollary 4.4 below).

2. Preliminaries

In this section, we describe the equation for the modified half-linear Prüfer angle given by the studied type of equations. At first, we briefly recall the notion of half-linear trigonometric functions.

The half-linear sine function denoted by $\sin_p$ is introduced as the odd $2\pi_p$-periodic extension of the solution of the initial problem

$$[\Phi(x')]' + (p-1)\Phi(x) = 0, \quad x(0) = 0, \quad x'(0) = 1$$

on $[0, \pi_p]$, where

$$\pi_p := \frac{2\pi}{p\sin(\pi/p)}.$$

We denote the derivative of the half-linear sine function as $\cos_p$ and we call it the half-linear cosine function. It holds

$$|\cos_p a| \le 1, \quad |\sin_p a| \le 1, \quad a \in \mathbb{R}. \tag{2.1}$$

For more details about $\sin_p$ and $\cos_p$, we refer to [5, Section 1.1.2].

Now, let us turn our attention to the half-linear equation

$$[r^{-p/q}(t)\, t^{p-1}\Phi(x')]' + \frac{s(t)}{t\log^p t}\,\Phi(x) = 0 \tag{2.2}$$

and the corresponding equation for the Prüfer angle

$$\varphi'(t) = \frac{1}{t\log t}\Big[r(t)|\cos_p\varphi(t)|^p - \Phi\left(\cos_p\varphi(t)\right)\sin_p\varphi(t) + s(t)\,\frac{|\sin_p\varphi(t)|^p}{p-1}\Big], \tag{2.3}$$

where $r : \mathbb{R} \to \mathbb{R}$ is a continuous, positive, and α-periodic function and $s : \mathbb{R} \to \mathbb{R}$ is a continuous and β-periodic function.

We use the Riccati type transformation

$$w(t) = r^{-p/q}(t)t^{p-1}\Phi\Big(\frac{x'(t)}{x(t)}\Big)$$

to (2.2). This leads to the equation

$$w'(t) + \frac{s(t)}{t\log^p t} + (p-1)[r^{-p/q}(t)\,t^{p-1}]^{\frac{1}{1-p}}|w(t)|^{\frac{p}{p-1}} = 0. \tag{2.4}$$

Then, using the substitution

$$v(t) = (\log t)^{\frac{p}{q}} w(t), \quad t \in (\mathrm{e}, \infty),$$

in (2.4) and taking into account the modified Prüfer transformation

$$x(t) = \rho(t)\sin_p\varphi(t), \quad [r^{-p/q}(t)\,t^{p-1}]^{q-1}x'(t) = \frac{\rho(t)}{\log t}\cos_p\varphi(t),$$

we easily obtain (2.3). The more comprehensive description of the derivation of (2.3) is given in our previous paper [25].

Further, let us mention the definition of the mean value of an arbitrary periodic function which is essential for our results.

Definition 2.1. The mean value $M(f)$ of a periodic function $f : \mathbb{R} \to \mathbb{R}$ with period $P > 0$ is defined as

$$M(f) := \frac{1}{P}\int_0^P f(\tau)\,d\tau.$$

Finally, for the upcoming use, we put

$$\tilde{r} := \sup\{r(t) : t > \mathrm{e}\}, \quad \tilde{s} := \sup\{|s(t)| : t > \mathrm{e}\} \tag{2.5}$$

and we denote $2\varrho := \min\{p-1, 1\}$.

3. Auxiliary results

Let $\vartheta > 0$ be arbitrary. We define

$$\psi(t) := \frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}} \varphi(\tau)\ d\tau, \quad t \geq \mathrm{e} + \vartheta, \tag{3.1}$$

where φ is a solution of (2.3) on $[\mathrm{e}+\vartheta, \infty)$. Now, we formulate and prove auxiliary results concerning this function ψ.

Lemma 3.1. *If φ is a solution of (2.3) on $[\mathrm{e}+\vartheta, \infty)$, then the function $\psi : [\mathrm{e}+\vartheta, \infty) \to \mathbb{R}$ defined by (3.1) satisfies*

$$|\varphi(\tau) - \psi(t)| \leq \frac{C}{\sqrt{t}\log t}, \quad t \geq \mathrm{e}+\vartheta,\ \tau \in [t, t+\sqrt{t}], \tag{3.2}$$

for some constant $C > 0$.

Proof. The continuity of φ implies that, for any $t \geq \mathrm{e}+\vartheta$, there exists $\tilde{t} \in [t, t+\sqrt{t}]$ such that $\psi(t) = \varphi(\tilde{t})$. Hence, for all $t \geq \mathrm{e}+\vartheta$, $\tau \in [t, t+\sqrt{t}]$, we obtain

$$\begin{aligned}
|\varphi(\tau) - \psi(t)| &= |\varphi(\tau) - \varphi(\tilde{t})| \\
&\leq \int_t^{t+\sqrt{t}} |\varphi'(\tau)|\ d\tau
\end{aligned}$$

$$\leq \frac{1}{t\log t}\Big[\int_t^{t+\sqrt{t}} r(\tau)|\cos_p\varphi(\tau)|^p + |\Phi(\cos_p\varphi(\tau))\sin_p\varphi(\tau)|\ d\tau \\ + \int_t^{t+\sqrt{t}} \frac{|\sin_p\varphi(\tau)|^p}{p-1}|s(\tau)|\ d\tau\Big],$$

i.e., we obtain (see (2.1), (2.5))

$$|\varphi(\tau)-\psi(t)| \leq \frac{1}{t\log t}\int_t^{t+\sqrt{t}} \big(\tilde{r}+1+\frac{\tilde{s}}{p-1}\big)\,d\tau \leq \frac{C}{\sqrt{t}\log t},$$

where

$$C := \tilde{r}+1+\frac{\tilde{s}}{p-1}. \tag{3.3}$$

□

Lemma 3.2. *The inequality*

$$\Big|\psi'(t) - \frac{1}{t\log t}\Big[\frac{|\cos_p\psi(t)|^p}{\sqrt{t}}\int_t^{t+\sqrt{t}} r(\tau)\,d\tau \\ - \Phi(\cos_p\psi(t))\sin_p\psi(t) + \frac{|\sin_p\psi(t)|^p}{(p-1)\sqrt{t}}\int_t^{t+\sqrt{t}} s(\tau)\,d\tau\Big]\Big| \\ < \frac{D}{t^{1+\varrho}\log t}$$

holds for some $D>0$ and for all $t>\mathrm{e}+\vartheta$.

Proof. For all $t>\mathrm{e}+\vartheta$, we have

$$\begin{aligned} \psi'(t) &= \big(1+\frac{1}{2\sqrt{t}}\big)\frac{\varphi(t+\sqrt{t})}{\sqrt{t}} - \frac{\varphi(t)}{\sqrt{t}} - \frac{1}{2\sqrt{t^3}}\int_t^{t+\sqrt{t}}\varphi(\tau)\,d\tau \\ &= \frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}}\varphi'(\tau)\,d\tau + \frac{1}{2t}\varphi(t+\sqrt{t}) - \frac{1}{2\sqrt{t^3}}\int_t^{t+\sqrt{t}}\varphi(\tau)\,d\tau \\ &= \frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}}\frac{1}{\tau\log\tau}\Big[r(\tau)|\cos_p\varphi(\tau)|^p - \Phi\big(\cos_p\varphi(\tau)\big)\sin_p\varphi(\tau) \\ &\quad + s(\tau)\frac{|\sin_p\varphi(\tau)|^p}{p-1}\Big]\,d\tau + \frac{1}{2\sqrt{t^3}}\int_t^{t+\sqrt{t}}\big[\varphi(t+\sqrt{t})-\varphi(\tau)\big]\,d\tau. \end{aligned}$$

Since (see also (2.1), (2.5), and (3.3))

$$\begin{aligned} &\Big|\frac{1}{2\sqrt{t^3}}\int_t^{t+\sqrt{t}}[\varphi(t+\sqrt{t})-\varphi(\tau)]\,d\tau\Big| \\ &\leq \frac{1}{2\sqrt{t^3}}\int_t^{t+\sqrt{t}}\int_\tau^{t+\sqrt{t}}|\varphi'(\sigma)|\,d\sigma\,d\tau \\ &\leq \frac{1}{2\sqrt{t^3}}\int_t^{t+\sqrt{t}}\int_\tau^{t+\sqrt{t}}\frac{1}{\sigma\log\sigma}\Big|r(\sigma)|\cos_p\varphi(\sigma)|^p - \Phi\left(\cos_p\varphi(\sigma)\right)\sin_p\varphi(\sigma) \\ &\quad + s(\sigma)\,\frac{|\sin_p\varphi(\sigma)|^p}{p-1}\Big|\,d\sigma\,d\tau \\ &\leq \frac{1}{2\sqrt{t^5}\log t}\int_t^{t+\sqrt{t}}\int_t^{t+\sqrt{t}}[\tilde{r}+1+\frac{\tilde{s}}{p-1}]\,d\sigma\,d\tau \end{aligned}$$

$$\leq \frac{C}{2\sqrt{t^3}\log t},$$

it suffices to consider

$$\frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}} \frac{1}{\tau\log\tau}\Big[r(\tau)|\cos_p\varphi(\tau)|^p - \Phi\big(\cos_p\varphi(\tau)\big)\sin_p\varphi(\tau) + s(\tau)\,\frac{|\sin_p\varphi(\tau)|^p}{p-1}\Big]\,d\tau.$$

In fact, we will consider

$$\begin{aligned} &\frac{1}{\sqrt{t^3}\log t}\int_t^{t+\sqrt{t}}\Big[r(\tau)|\cos_p\varphi(\tau)|^p - \Phi\left(\cos_p\varphi(\tau)\right)\sin_p\varphi(\tau) \\ &\quad + s(\tau)\frac{|\sin_p\varphi(\tau)|^p}{p-1}\Big]\,d\tau, \end{aligned} \tag{3.4}$$

because

$$\begin{aligned} &\Big|\int_t^{t+\sqrt{t}} \frac{1}{\tau\log\tau}[r(\tau)|\cos_p\varphi(\tau)|^p - \Phi(\cos_p\varphi(\tau))\sin_p\varphi(\tau)]\,d\tau \\ &+ \int_t^{t+\sqrt{t}} \frac{1}{\tau\log\tau}\,\frac{|\sin_p\varphi(\tau)|^p}{p-1}\,s(\tau)\,d\tau \\ &- \int_t^{t+\sqrt{t}} \frac{1}{t\log t}[r(\tau)|\cos_p\varphi(\tau)|^p - \Phi(\cos_p\varphi(\tau))\sin_p\varphi(\tau)]\,d\tau \\ &- \int_t^{t+\sqrt{t}} \frac{1}{t\log t}\,\frac{|\sin_p\varphi(\tau)|^p}{p-1}s(\tau)\,d\tau\Big| \\ &\leq \int_t^{t+\sqrt{t}} [\tilde{r}+1+\frac{\tilde{s}}{p-1}][\frac{1}{t\log t}-\frac{1}{\tau\log\tau}]\,d\tau \\ &\leq C\sqrt{t}\frac{(t+\sqrt{t})\log(t+\sqrt{t}) - t\log t}{t(t+\sqrt{t})\log(t+\sqrt{t})\log t} \\ &\leq \frac{KC}{t\log t} \end{aligned}$$

for all $t \geq \mathrm{e}+\vartheta$, where $K > 0$ is such a constant that

$$\frac{(t+\sqrt{t})\log(t+\sqrt{t}) - t\log t}{\log(t+\sqrt{t})} \leq K\sqrt{t}, \quad t \geq \mathrm{e}+\vartheta.$$

Considering the form of (3.4), to finish the proof, it suffices to prove the following inequalities

$$\Big|\frac{|\cos_p\psi(t)|^p}{\sqrt{t}}\int_t^{t+\sqrt{t}} r(\tau)\,d\tau - \frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}} r(\tau)|\cos_p\varphi(\tau)|^p\,d\tau\Big| \leq \frac{E_1}{\sqrt{t}\log t}, \tag{3.5}$$

$$\begin{aligned} &\Big|\frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}} \Phi(\cos_p\psi(t))\sin_p\psi(t)\,d\tau - \frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}} \Phi(\cos_p\varphi(\tau))\sin_p\varphi(\tau)\,d\tau\Big| \\ &\leq \frac{E_2}{t^{\varrho}\log^{2\varrho} t}, \end{aligned} \tag{3.6}$$

$$\Big|\frac{|\sin_p\psi(t)|^p}{\sqrt{t}}\int_t^{t+\sqrt{t}} s(\tau)\,d\tau - \frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}} s(\tau)|\sin_p\varphi(\tau)|^p\,d\tau\Big| \leq \frac{E_3}{\sqrt{t}\log t} \tag{3.7}$$

for some constants $E_1, E_2, E_3 > 0$ and for all $t \geq \mathrm{e}+\vartheta$.

From [5, pp. 4-5], we know that there exists $A > 0$ for which

$$\big| |\cos_p a|^p - |\cos_p b|^p \big| \le A|a-b|, \quad a, b \in \mathbb{R}, \tag{3.8}$$

$$\big| |\sin_p a|^p - |\sin_p b|^p \big| \le A|a-b|, \quad a, b \in \mathbb{R}, \tag{3.9}$$

$$\big| \sin_p a - \sin_p b \big| \le A|a-b|, \quad a, b \in \mathbb{R}. \tag{3.10}$$

In addition, directly from the definition of Φ and $\cos_p$, it follows the existence of $B > 0$ such that

$$|\Phi(\cos_p a) - \Phi(\cos_p b)| \le [B|a-b|]^{\min\{1,p-1\}}, \quad a, b \in \mathbb{R}. \tag{3.11}$$

At first, we consider inequality (3.5) which follows from (see also (2.5), (3.2), and (3.8))

$$\begin{aligned}
&\Big| \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} r(\tau) \left(|\cos_p \psi(t)|^p - |\cos_p \varphi(\tau)|^p \right) d\tau \Big| \\
&\le \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} r(\tau) A |\psi(t) - \varphi(\tau)| \, d\tau \\
&\le \frac{\tilde{r} A C}{\sqrt{t} \log t}, \quad t \ge \mathrm{e} + \vartheta.
\end{aligned}$$

Similarly, we can obtain (3.7) from (see (2.5), (3.2), and (3.9))

$$\begin{aligned}
&\Big| \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} s(\tau) \left(|\sin_p \psi(t)|^p - |\sin_p \varphi(\tau)|^p \right) d\tau \Big| \\
&\le \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} |s(\tau)| A |\psi(t) - \varphi(\tau)| \, d\tau \\
&\le \frac{\tilde{s} A C}{\sqrt{t} \log t}, \quad t \ge \mathrm{e} + \vartheta.
\end{aligned}$$

It remains to show (3.6). We have (see (2.1))

$$\begin{aligned}
&\Big| \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} [\Phi(\cos_p \psi(t)) \sin_p \psi(t) - \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau)] \, d\tau \Big| \\
&\le \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} \big| \Phi(\cos_p \psi(t)) \sin_p \psi(t) - \Phi(\cos_p \psi(t)) \sin_p \varphi(\tau) \big| \, d\tau \\
&\quad + \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} \big| \Phi(\cos_p \psi(t)) \sin_p \varphi(\tau) - \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau) \big| \, d\tau \\
&\le \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} |\sin_p \psi(t) - \sin_p \varphi(\tau)| \, d\tau \\
&\quad + \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} |\Phi(\cos_p \psi(t)) - \Phi(\cos_p \varphi(\tau))| \, d\tau
\end{aligned}$$

for all $t \ge \mathrm{e} + \vartheta$ and, using (3.2), (3.10), and (3.11), we obtain

$$\begin{aligned}
&\Big| \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} [\Phi(\cos_p \psi(t)) \sin_p \psi(t) - \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau)] \, d\tau \Big| \\
&\le \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} A |\psi(t) - \varphi(\tau)| \, d\tau
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} [B|\psi(t) - \varphi(\tau)|]^{\min\{1,p-1\}}\, d\tau \\
&\le \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} \frac{AC}{\sqrt{t}\log t} + \Big[\frac{BC}{\sqrt{t}\log t}\Big]^{\min\{1,p-1\}}\, d\tau \\
&\le \frac{AC}{\sqrt{t}\log t} + \big(\frac{BC}{\sqrt{t}\log t}\big)^{\min\{1,p-1\}}
\end{aligned}
$$

for all $t \ge \mathrm{e} + \vartheta$, i.e., (3.6) is valid for

$$E_2 := AC + [BC]^{\min\{1,p-1\}}.$$

The proof is complete. □

Now we recall a known result and we provide its direct consequence which we will use in the proof of Theorem 4.1 in the next section.

Theorem 3.3. *If $M, N > 0$ are such that $M^{p-1}N = q^{-p}$, then the equation*

$$\Big[\big(M + \frac{1}{t}\big)^{-p/q}\Phi(x')\Big]' + \frac{1}{t^p}\big(N + \frac{1}{t}\big)\Phi(x) = 0 \tag{3.12}$$

is non-oscillatory.

For a proof of the above theorem, see [3].

Corollary 3.4. *If $M, N > 0$ are such that $M^{p-1}N = q^{-p}$, then the equation*

$$\Big[\big(M + \frac{1}{\log t}\big)^{-p/q} t^{p-1}\Phi(x')\Big]' + \frac{N + \frac{1}{\log t}}{t\log^p t}\Phi(x) = 0 \tag{3.13}$$

is non-oscillatory.

Proof. Let us consider (3.13), where $x = x(t)$ and $(\cdot)' = \frac{d}{dt}$. Using the transformation of the independent variable $s = \log t$ when $x(t) = y(s)$, we have

$$\frac{1}{t}\frac{d}{ds}\Big[\big(M + \frac{1}{s}\big)^{-p/q} t^{p-1}\Phi\Big(\frac{1}{t}\frac{dy}{ds}\Big)\Big]' + \frac{1}{ts^p}\big(N + \frac{1}{s}\big)\Phi(y) = 0.$$

This equation can be easily simplified into the form

$$\Big[\big(M + \frac{1}{s}\big)^{-p/q}\Phi(y')\Big]' + \frac{1}{s^p}\big(N + \frac{1}{s}\big)\Phi(y) = 0. \tag{3.14}$$

Hence (cf. (3.12) and (3.14)), it suffices to apply Theorem 3.3. □

4. Results

Applying Lemma 3.2 and Corollary 3.4, we prove the following theorem.

Theorem 4.1. *Let $\alpha, \beta > 0$. If $r : \mathbb{R} \to \mathbb{R}$ is α-periodic and $s : \mathbb{R} \to \mathbb{R}$ is β-periodic such that*

$$\Big[\frac{1}{\alpha}\int_0^\alpha r(\tau)\, d\tau\Big]^{p-1} \frac{1}{\beta}\int_0^\beta s(\tau)\, d\tau = [M(r)]^{p-1}M(s) = q^{-p}, \tag{4.1}$$

then (2.2) is non-oscillatory.

Proof. In this proof, we consider the equation for the Prüfer angle φ and the corresponding equation for ψ. The used method is based on the fact that the non-oscillation of solutions of (2.2) is equivalent to the boundedness from above of a solution φ of (2.3). We can refer to [25] or also to the papers [3, 4, 16, 17, 22]. In addition, Lemma 3.1 implies that a solution $\varphi : [\mathrm{e}+\vartheta, \infty) \to \mathbb{R}$ of (2.3) is bounded from above if and only if ψ given by (3.1) is bounded from above.

From Lemma 3.2, we have

$$\begin{aligned}\psi'(t) < \frac{1}{t\log t}\Big[&\frac{|\cos_p \psi(t)|^p}{\sqrt{t}}\int_t^{t+\sqrt{t}} r(\tau)\,d\tau - \Phi\left(\cos_p \psi(t)\right)\sin_p \psi(t)\\ &+\frac{|\sin_p \psi(t)|^p}{(p-1)\sqrt{t}}\int_t^{t+\sqrt{t}} s(\tau)\,d\tau + \frac{D}{t^{\varrho}}\Big]\end{aligned}$$

for all $t > \mathrm{e}+\vartheta$ and for some D. Especially,

$$\begin{aligned}\psi'(t) < \frac{1}{t\log t}\Big[&\frac{|\cos_p \psi(t)|^p}{\sqrt{t}}\int_t^{t+\sqrt{t}} r(\tau)\,d\tau - \Phi\left(\cos_p \psi(t)\right)\sin_p \psi(t)\\ &+\frac{|\sin_p \psi(t)|^p}{(p-1)\sqrt{t}}\int_t^{t+\sqrt{t}} s(\tau)\,d\tau + \frac{D}{\log^2 t}\Big]\end{aligned} \tag{4.2}$$

for all $t > \mathrm{e}+\vartheta$. Then, using the periodicity of coefficients r, s, we obtain (see (2.5) and (4.2))

$$\begin{aligned}\psi'(t) < \frac{1}{t\log t}\Big[&|\cos_p \psi(t)|^p\Big(M(r) + \frac{\tilde{r}\alpha}{\sqrt{t}}\Big) - \Phi\big(\cos_p \psi(t)\big)\sin_p \psi(t)\\ &+\frac{|\sin_p \psi(t)|^p}{p-1}\Big(M(s) + \frac{\tilde{s}\beta}{\sqrt{t}}\Big) + \frac{D}{\log^2 t}\Big]\end{aligned} \tag{4.3}$$

for all $t > \mathrm{e}+\vartheta$. Indeed, for any periodic continuous function f with period $P > 0$ and positive mean value $M(f)$, we have

$$\begin{aligned}\frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}} f(\tau)\,d\tau &= \frac{1}{\sqrt{t}}\Big(\int_t^{t+Pn} f(\tau)\,d\tau + \int_{t+Pn}^{t+\sqrt{t}} f(\tau)\,d\tau\Big)\\ &\le \frac{1}{Pn}\int_t^{t+Pn} f(\tau)\,d\tau + \frac{1}{\sqrt{t}}\int_{t+Pn}^{t+P(n+1)} |f(\tau)|\,d\tau \le M(f) + \frac{\tilde{f}P}{\sqrt{t}},\end{aligned}$$

where $\tilde{f} := \max\{|f(t)| : t \in [0, P]\}$ and $n \in \mathbb{N}\cup\{0\}$ is such that $Pn \le \sqrt{t}$ and that $P(n+1) > \sqrt{t}$.

For $R := \max\{1, p-1\}$, the well-known Pythagorean identity (see, e.g., [5, Section 1.1.2]) gives

$$R\Big(|\cos_p a|^p + \frac{|\sin_p a|^p}{p-1}\Big) \ge 1, \quad a \in \mathbb{R}. \tag{4.4}$$

Considering (4.3) and (4.4), we have

$$\begin{aligned}\psi'(t) < \frac{1}{t\log t}\Big[&|\cos_p \psi(t)|^p\Big(M(r) + \frac{\tilde{r}\alpha}{\sqrt{t}} + \frac{RD}{\log^2 t}\Big)\\ &- \Phi\left(\cos_p \psi(t)\right)\sin_p \psi(t) + \frac{|\sin_p \psi(t)|^p}{p-1}\Big(M(s) + \frac{\tilde{s}\beta}{\sqrt{t}} + \frac{RD}{\log^2 t}\Big)\Big]\end{aligned}$$

for all $t > \mathrm{e} + \vartheta$ and, consequently, we have

$$\begin{aligned}\psi'(t) < \frac{1}{t\log t}\Big[&|\cos_p \psi(t)|^p \Big(M(r) + \frac{1}{\log t}\Big)\\ &- \Phi\left(\cos_p \psi(t)\right)\sin_p \psi(t) + \frac{|\sin_p \psi(t)|^p}{p-1}\Big(M(s) + \frac{1}{\log t}\Big)\Big]\end{aligned} \tag{4.5}$$

for all large t.

The equation

$$\begin{aligned}\varphi'(t) = \frac{1}{t\log t}\Big[&|\cos_p \varphi(t)|^p \Big(M(r) + \frac{1}{\log t}\Big) - \Phi\left(\cos_p \varphi(t)\right)\sin_p \varphi(t)\\ &+ \frac{|\sin_p \varphi(t)|^p}{p-1}\Big(M(s) + \frac{1}{\log t}\Big)\Big]\end{aligned} \tag{4.6}$$

has the form of the equation for the Prüfer angle φ which corresponds to (3.13), where $M = M(r)$ and $N = M(s)$. Therefore (see (4.1)), Corollary 3.4 guarantees that any solution $\varphi : [\mathrm{e} + \vartheta, \infty) \to \mathbb{R}$ of (4.6) is bounded from above. Comparing (4.5) with (4.6) and considering the $2\pi_p$-periodicity of the half-linear trigonometric functions, we know that the considered function ψ is bounded from above. This means that any non-zero solution of (2.2) is non-oscillatory. □

Now we explicitly mention a result which is the basic motivation for our current research.

Theorem 4.2. *Let $r, s : \mathbb{R} \to \mathbb{R}$ be periodic.*

(i) *If $[M(r)]^{p-1}M(s) > q^{-p}$, then* (2.2) *is oscillatory.*
(ii) *If $[M(r)]^{p-1}M(s) < q^{-p}$, then* (2.2) *is non-oscillatory.*

The statements of the above theorem can be obtained immediately from the main results of [25]. Using Theorem 4.2, we can generalize Theorem 4.1 as follows.

Theorem 4.3. *Let $r, s : \mathbb{R} \to \mathbb{R}$ be periodic. Equation* (2.2) *is oscillatory if and only if $[M(r)]^{p-1}M(s) > q^{-p}$.*

We get a new result even for linear equations. Thus, we formulate the corollary below.

Corollary 4.4. *Let $r : \mathbb{R} \to \mathbb{R}$ be continuous, positive, and periodic function and let $s : \mathbb{R} \to \mathbb{R}$ be continuous and periodic function. The equation*

$$\Big[\frac{t}{r(t)}x'\Big]' + \frac{s(t)}{t\log^2 t}x = 0 \tag{4.7}$$

is oscillatory if and only if $4M(r)M(s) > 1$.

To illustrate the presented results, we give some examples of equations whose oscillation properties do not follow from previously known oscillation criteria. First, we mention an example to illustrate Theorem 4.1.

Example 4.5. For any $p > 1$, the equation

$$\Big[\Big(\frac{2 + \sin(\sqrt{q}t)}{2q}\Big)^{-p/q} t^{p-1}\Phi(x')\Big]' + \frac{p - 1 + \cos(pt)}{pt\log^p t}\Phi(x) = 0 \tag{4.8}$$

is in the critical case because

$$M(r) = M\Big(\frac{2 + \sin(\sqrt{q}t)}{2q}\Big) = \frac{1}{q} = M\Big(\frac{p - 1 + \cos(pt)}{p}\Big) = M(s).$$

Hence, $[M(r)]^{p-1}M(s) = q^{-p}$ and (4.8) is non-oscillatory due to Theorem 4.1.

Of course, the oscillation behaviour of (4.8) is solvable in many slightly modified situations as well. For example, its coefficients may involve parameters. Thus, we can apply Theorem 4.3 as follows.

Example 4.6. Let $a > 1$ and $b, c, d \neq 0$ be real parameters. We consider the equation

$$\Big[\Big(\frac{a + \sin(ct)}{q}\Big)^{-p/q} t^{p-1}\Phi(x')\Big]' + \frac{p-1+\cos(dt)}{bt\log^p t}\Phi(x) = 0 \tag{4.9}$$

with

$$\begin{gathered} M(r) = M\Big(\frac{a+\sin(ct)}{q}\Big) = \frac{a}{q}, \\ M(s) = M\Big(\frac{p-1+\cos(dt)}{b}\Big) = \frac{p-1}{b}. \end{gathered}$$

Therefore, by Theorem 4.3, Equation (4.9) is oscillatory for $a^{p-1}p/b > 1$ and non-oscillatory for $a^{p-1}p/b \leq 1$.

Finally, we mention the following simple example of linear equations whose oscillation properties are solvable by Corollary 4.4.

Example 4.7. Consider the equation

$$\begin{aligned} &\Big[\frac{t}{a_1 + b_1\sin(c_1 t) + d_1\cos(c_1 t)}\, x'\Big]' \\ &+ \frac{a_2 + b_2\sin(c_2 t)\cos(c_2 t) + d_2\arcsin[\cos(c_2 t)]}{t\log^2 t} x = 0, \end{aligned} \tag{4.10}$$

where $a_i, b_i, c_i, d_i \in \mathbb{R}$, $c_i \neq 0$, $i \in \{1, 2\}$, $a_1 > |b_1| + |d_1|$. It is seen that $M(r) = a_1$ and $M(s) = a_2$ (cf. (4.7) and (4.10)). Hence, (4.10) is oscillatory for $a_1a_2 > 1/4$ and non-oscillatory for $a_1a_2 \leq 1/4$. We emphasize that this conclusion remains valid even for, e.g., $c_1 = 1$ and $c_2 = \pi$ or $c_2 = \sqrt{2}$, when r and s do not possess any common period.

As a final remark, we consider again the critical case. In this paper, we deal with the critical case of equations with periodic coefficients. It is not possible to categorize as oscillatory and non-oscillatory equations in the critical case for "too general" coefficients. We can illustrate this fact by the Euler type half-linear equations

$$[r(t)\Phi(x')]' + \frac{s(t)}{t^p}\Phi(x) = 0.$$

We refer to [3, 6, 14, 22]. Concerning equations of the form given by (2.2), we conjecture that the critical case is not generally solvable even for almost periodic functions r, s (for the definition of almost periodicity, see, e.g., [2, 10]). This conjecture is based on constructions in [20] (see also [19, 21]).

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References

[1] R. P. Agarwal, A. R. Grace, D. O'Regan; *Oscillation theory for second order linear, half-linear, superlinear and sublinear dynamic equations.* Kluwer Academic Publishers, Dordrecht, 2002.

[2] C. Corduneanu; *Almost periodic functions.* John Wiley & Sons, New York, 1968.

[3] O. Došlý, H. Funková; *Euler type half-linear differential equation with periodic coefficients.* Abstract Appl. Anal. **2013** (2013), art. ID 714263, pp. 1–6.

[4] O. Došlý, P. Hasil; *Critical oscillation constant for half-linear differential equations with periodic coefficients.* Ann. Mat. Pura Appl. **190** (2011), no. 3, pp. 395–408.

[5] O. Došlý, P. Řehák; *Half-linear differential equations.* Elsevier, Amsterdam, 2005.

[6] O. Došlý, M. Veselý; *Oscillation and non-oscillation of Euler type half-linear differential equations.* J. Math. Anal. Appl. **429** (2015), no. 1, pp. 602–621.

[7] Á. Elbert; *Asymptotic behaviour of autonomous half-linear differential systems on the plane.* Studia Sci. Math. Hungar. **19** (1984), no. 2-4, pp. 447–464.

[8] Á. Elbert; *Oscillation and nonoscillation theorems for some nonlinear ordinary differential equations.* Ordinary and partial differential equations (Dundee, 1982), pp. 187–212, Lecture Notes in Math., vol. 964, Springer, Berlin, 1982.

[9] F. Gesztesy, M. Ünal; *Perturbative oscillation criteria and Hardy-type inequalities.* Math. Nachr. **189** (1998), pp. 121–144.

[10] A. M. Fink; *Almost periodic differential equations.* Springer-Verlag, Berlin, 1974.

[11] P. Hasil; *Conditional oscillation of half-linear differential equations with periodic coefficients.* Arch. Math. **44** (2008), no. 2, pp. 119–131.

[12] P. Hasil, M. Veselý; *Critical oscillation constant for difference equations with almost periodic coefficients.* Abstract Appl. Anal. **2012** (2012), art. ID 471435, pp. 1–19.

[13] P. Hasil, M. Veselý; *Non-oscillation of perturbed half-linear differential equations with sums of periodic coefficients.* Adv. Differ. Equ. **2015** (2015), no. 190, pp. 1–17.

[14] P. Hasil, M. Veselý; *Oscillation of half-linear differential equations with asymptotically almost periodic coefficients.* Adv. Differ. Equ. **2013** (2013), no. 122, pp. 1–15.

[15] P. Hasil, J. Vítovec; *Conditional oscillation of half-linear Euler-type dynamic equations on time scales.* Electron. J. Qual. Theory Differ. Equ. **2015** (2015), no. 6, pp. 1–24.

[16] H. Krüger, G. Teschl; *Effective Prüfer angles and relative oscillation criteria.* J. Differ. Equ. **245** (2008), no. 12, pp. 3823–3848.

[17] K. M. Schmidt; *Critical coupling constant and eigenvalue asymptotics of perturbed periodic Sturm-Liouville operators.* Commun Math. Phys. **211** (2000), pp. 465–485.

[18] K. M. Schmidt; *Oscillation of perturbed Hill equation and lower spectrum of radially periodic Schrödinger operators in the plane.* Proc. Amer. Math. Soc. **127** (1999), pp. 2367–2374.

[19] M. Veselý; *Almost periodic homogeneous linear difference systems without almost periodic solutions.* J. Differ. Equ. Appl. **18** (2012), no. 10, pp. 1623–1647.

[20] M. Veselý; *Construction of almost periodic functions with given properties.* Electron. J. Differ. Equ. **2011** (2011), no. 29, pp. 1–25.

[21] M. Veselý; *Construction of almost periodic sequences with given properties.* Electron. J. Differ. Equ. **2008** (2008), no. 126, pp. 1–22.

[22] M. Veselý, P. Hasil; *Conditional oscillation of Riemann-Weber half-linear differential equations with asymptotically almost periodic coefficients.* Studia Sci. Math. Hungar. **51** (2014), no. 3, pp. 303–321.

[23] M. Veselý, P. Hasil; *Non-oscillation of half-linear differential equations with periodic coefficients.* Electron. J. Qual. Theory Differ. Equ. **2015** (2015), no. 1, pp. 1–21.

[24] M. Veselý, P. Hasil; *Oscillation and non-oscillation of asymptotically almost periodic half-linear difference equations.* Abstract Appl. Anal. **2013** (2013), art. ID 432936, pp. 1–12.

[25] M. Veselý, P. Hasil; *Oscillation constant for modified Euler type half-linear equations.* Electron. J. Differ. Equ. **2015** (2015), no. 220, pp. 1–14.

[26] M. Veselý, P. Hasil; *Oscillation constants for half-linear difference equations with coefficients having mean values.* Adv. Differ. Equ. **2015** (2015), no. 210, pp. 1–18.

[27] M. Veselý, P. Hasil, R. Mařík; *Conditional oscillation of half-linear differential equations with coefficients having mean values.* Abstract Appl. Anal. **2014** (2014), art. ID 258159, pp. 1–14.

[28] J. Vítovec; *Critical oscillation constant for Euler-type dynamic equations on time scales.* Appl. Math. Comput. **243** (2014), pp. 838–848.

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