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ASYMPTOTIC BEHAVIOR OF INTERMEDIATE SOLUTIONS OF FOURTH-ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH REGULARLY VARYING COEFFICIENTS

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Abstract. We study the fourth-order nonlinear differential equation

$$\big(p(t)|x''(t)|^{\alpha-1}\, x''(t)\big)'' + q(t)|x(t)|^{\beta-1}\, x(t) = 0, \quad \alpha > \beta,$$

with regularly varying coefficient p, q satisfying

$$\int_a^\infty t\Big(\frac{t}{p(t)}\Big)^{1/\alpha}\, dt < \infty.$$

in the framework of regular variation. It is shown that complete information can be acquired about the existence of all possible intermediate regularly varying solutions and their accurate asymptotic behavior at infinity.

1. Introduction

We study the equation

$$\big(p(t)|x''(t)|^{\alpha-1}x''(t)\big)'' + q(t)|x(t)|^{\beta-1}x(t) = 0, \quad t \geq a > 0, \tag{1.1}$$

where

(i) α and β are positive constants such that $\alpha > \beta$,
(ii) $p, q : [a, \infty) \to (0, \infty)$ are continuous functions and p satisfies

$$\int_a^\infty \frac{t^{1+(1/\alpha)}}{p(t)^{1/\alpha}}\, dt < \infty. \tag{1.2}$$

Equation (1.1) is called *sub-half-linear* if $\beta < \alpha$ and *super-half-linear* if $\beta > \alpha$. By a solution of (1.1) we mean a function $x : [T, \infty) \to \mathbb{R}, T \geq a$, which is twice continuously differentiable together with $p|x''|^{\alpha-1}x''$ on $[T, \infty)$ and satisfies the equation (1.1) at every point in $[T, \infty)$. A solution x of (1.1) is said to be *nonoscillatory* if there exists $T \geq a$ such that $x(t) \neq 0$ for all $t \geq T$ and *oscillatory* otherwise. It is clear if x is a solution of (1.1), then so does $-x$, and so in studying nonoscillatory solutions of (1.1) it suffices to restrict our attention to its (eventually) positive solutions.

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Throughout this paper extensive use is made of the symbol $\sim$ to denote the asymptotic equivalence of two positive functions, i.e.,

$$f(t) \sim g(t),\ t \to \infty \ \Leftrightarrow\ \lim_{t\to\infty} \frac{g(t)}{f(t)} = 1.$$

We also use the symbol $\prec$ to denote the dominance relation between two positive functions in the sense that

$$f(t) \prec g(t),\ t \to \infty \ \Leftrightarrow\ \lim_{t\to\infty} \frac{g(t)}{f(t)} = \infty.$$

In our analysis of positive solutions of (1.1) a special role is played by the four functions

$$\varphi_1(t) = \int_t^\infty \frac{s-t}{p(s)^{1/\alpha}}\,ds, \quad \varphi_2(t) = \int_t^\infty (s-t)(\frac{s}{p(s)})^{1/\alpha} ds, \quad \psi_1(t) = 1, \quad \psi_2(t) = t,$$

which are the particular solutions of the unperturbed differential equation

$$(p(t)|x''(t)|^{\alpha-1}x''(t))'' = 0.$$

Note that the functions φ_i and ψ_i, $i = 1, 2$ defined above satisfy the dominance relation

$$\varphi_1(t) \prec \varphi_2(t) \prec \psi_1(t) \prec \psi_2(t), \quad t \to \infty.$$

Asymptotic and oscillatory behavior of solutions of (1.1) have been previously considered in [9, 19, 16, 21, 26, 30, 31]. Kusano and Tanigawa in [19] made a detailed classification of all positive solutions of the equation (1.1) under the condition (1.2) and established conditions for the existence of such solutions. It was proved that the following four types of combination of the signs of x', x'' and $\big(p|x''|^{\alpha-1}\,x''\big)'$ are possible for an eventually positive solution $x(t)$ of (1.1):

$$(p(t)|x''(t)|^{\alpha-1}x''(t))' > 0, \quad x''(t) > 0, \quad x'(t) > 0 \quad \text{for all large } t, \tag{1.3}$$
$$(p(t)|x''(t)|^{\alpha-1}x''(t))' > 0, \quad x''(t) > 0, \quad x'(t) < 0 \quad \text{for all large } t, \tag{1.4}$$
$$(p(t)|x''(t)|^{\alpha-1}x''(t))' > 0, \quad x''(t) < 0, \quad x'(t) > 0 \quad \text{for all large } t, \tag{1.5}$$
$$(p(t)|x''(t)|^{\alpha-1}x''(t))' < 0, \quad x''(t) < 0, \quad x'(t) > 0 \quad \text{for all large } t. \tag{1.6}$$

As a results of further analysis of the four types of solutions mentioned above, Kusano and Tanigawa in [19] have shown that the following six types are possible for the asymptotic behavior of positive solutions of (1.1):

- (P1) $x(t) \sim c_1\varphi_1(t)$,
- (P2) $x(t) \sim c_2\varphi_2(t)$ as $t \to \infty$,
- (P3) $x(t) \sim c_3$ as $t \to \infty$,
- (P4) $x(t) \sim c_4 t$ as $t \to \infty$,
- (I1) $\varphi_1(t) \prec x(t) \prec \varphi_2(t)$ as $t \to \infty$,
- (I2) $1 \prec x(t) \prec t$ as $t \to \infty$,

where $c_i > 0$, $i = 1, 2, 3, 4$ are constants. Positive solutions of (1.1) having the asymptotic behavior (P1)–(P4) are collectively called *primitive positive solutions* of the equation (1.1), while the solutions having the asymptotic behavior (I1) and (I2) are referred to as *intermediate solutions* of the equation (1.1).

The interrelation between the types (1.3)-(1.6) of the derivatives of solutions and the types (P1)–(P4), (I1) and (I2) of the asymptotic behavior of solutions is as follows:

(i) All solutions of type (1.3) have the asymptotic behavior of type (P1);

(ii) A solution of type (1.4) has the asymptotic behavior of one of the types (P1), (P2), (P3) and (I1);
(iii) A solution of type (1.5) has the asymptotic behavior of one of the types (P3) and (P4);
(iv) A solution of type (1.6) has the asymptotic behavior of one of the types (P3), (P4) and (I2).

The existence of four types of primitive solutions has been completely characterized for both sub-half-linear and super-half-linear case of (1.1) with continuous coefficients p and q as the following theorems proven in [19] show.

Theorem 1.1. *Let $p, q \in C[a,\infty)$. Equation* (1.1) *has a positive solution x satisfying* (P3) *if and only if*

$$\mathcal{J}_1 = \int_a^\infty t\Big(\frac{1}{p(t)}\int_a^t (t-s)q(s)\,ds\Big)^{1/\alpha}\,dt < \infty. \tag{1.7}$$

Theorem 1.2. *Let $p, q \in C[a,\infty)$. Equation* (1.1) *has a positive solution x satisfying* (P4) *if and only if*

$$\mathcal{J}_2 = \int_a^\infty \Big(\frac{1}{p(t)}\int_a^t (t-s)s^\beta\, q(s)\,ds\Big)^{1/\alpha}\,dt < \infty. \tag{1.8}$$

Theorem 1.3. *Let $p, q \in C[a,\infty)$. Equation* (1.1) *has a positive solution x satisfying* (P1) *if and only if*

$$\mathcal{J}_3 = \int_a^\infty tq(t)\varphi_1(t)^\beta\,dt < \infty. \tag{1.9}$$

Theorem 1.4. *Let $p, q \in C[a,\infty)$. Equation* (1.1) *has a positive solution x satisfying* (P2) *if and only if*

$$\mathcal{J}_4 = \int_a^\infty q(t)\varphi_2(t)^\beta\,dt < \infty. \tag{1.10}$$

Unlike primitive solutions, establishing necessary and sufficient conditions for the existence of the intermediate solutions seems to be much more difficult task. Thus, only sufficient conditions for the existence of these solutions was obtained in [19].

Theorem 1.5. *If* (1.10) *holds and if*

$$\mathcal{J}_3 = \int_a^\infty tq(t)\varphi_1(t)^\beta\,dt = \infty,$$

then equation (1.1) *has a positive solution x such that $\varphi_1(t) \prec x(t) \prec \varphi_2(t)$, $t \to \infty$.*

Theorem 1.6. *If* (1.8) *holds and*

$$\mathcal{J}_1 = \int_a^\infty t\Big(\frac{1}{p(t)}\int_a^t (t-s)q(s)\,ds\Big)^{1/\alpha}\,dt = \infty,$$

then (1.1) *has a positive solution x such that $1 \prec x(t) \prec t$ as $t \to \infty$.*

However, sharp conditions for the oscillation of all solutions of (1.1) in both cases (sub-half-linear and super-half-linear) have been obtained in [16].

Theorem 1.7. *Let $\beta < 1 \le \alpha$. All solutions of* (1.1) *are oscillatory if and only if*

$$\mathcal{J}_2 = \int_a^\infty \Big(\frac{1}{p(t)} \int_a^t (t-s)s^\beta\, q(s)\, ds\Big)^{1/\alpha} dt = \infty\,.$$

Thus, our task is to establish necessary and sufficient conditions for (1.1) to possess intermediate solutions of types $(I1)$ and $(I2)$ and to determine precisely their asymptotic behavior at infinity. Since this problem is very difficult for equation (1.1) with general continuous coefficients p and q, we will make an attempt to solve the problem in the framework of regular variation, that is, we limit ourselves to the case where p and q are regularly varying functions and focus our attention on regularly varying solutions of (1.1). The recent development of asymptotic analysis of differential equations by the means of regularly varying functions, which was initiated by the monograph of Marić [22], has shown that there exists a variety of nonlinear differential equations for which the problem mentioned above can be solved completely. The reader is referred to the papers [8, 10, 13, 14, 18, 20, 28] for the second order differential equations, to [11, 12, 15, 17, 25] for the fourth order differential equations and to [3]-[7], [23, 24, 27] for some systems of differential equations. The present work can be considered as a continuation of the previous papers [11, 12, 15], which are the special cases of (1.1) with $\alpha = 1$ or $p(t) \equiv 1$ but has features different from them in the sense that the *generalized regularly varying functions* (or generalized Karamata functions) introduced in [2] will be used in order to make clear the dependence of asymptotic behavior of intermediate solutions on the coefficient p.

For reader's convenience the definition of generalized regularly varying functions and some of their basic properties are summarized in Section 2. In Sections 3 we consider equation (1.1) with generalized regularly varying p and q, and after showing that each of two classes of its intermediate generalized regularly varying solutions of type (I1) and (I2) can be divided into three disjoint subclasses according to their asymptotic behavior at infinity, we establish necessary and sufficient conditions for the existence of solutions and determine the asymptotic behavior of solutions contained in each of the six subclasses explicitly and precisely. In the final Section 4 it is shown that our main results, when specialized to the case where p and q are regularly varying functions in the sense of Karamata, provide complete information about the existence and asymptotic behavior of regularly varying solutions in the sense of Karamata for that equation (1.1). This information combined with that of the primitive solutions of (1.1) (cf. Theorems 1.1-1.4) enables us to present full structure of the set of regularly varying solutions for equations of the form (1.1) with regularly varying coefficients.

2. Basic properties of regularly varying functions

We recall that the set of regularly varying functions of index $\rho \in \mathbb{R}$ is introduced by the following definition.

Definition 2.1. A measurable function $f : (a,\infty) \to (0,\infty)$ for some $a > 0$ is said to be regularly varying at infinity of index $\rho \in \mathbb{R}$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{for all } \lambda > 0.$$

The totality of all regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. In the special case when $\rho = 0$, we use the notation SV instead of $\mathrm{RV}(0)$ and refer to members of SV as *slowly varying functions*. Any function $f \in \mathrm{RV}(\rho)$ is written as $f(t) = t^\rho\, g(t)$ with $g \in \mathrm{SV}$, and so the class SV of slowly varying functions is of fundamental importance in the theory of regular variation. If

$$\lim_{t\to\infty} \frac{f(t)}{t^\rho} = \lim_{t\to\infty} g(t) = \text{const} > 0$$

then f is said to be a *trivial* regularly varying function of the index ρ and it is denoted by $f \in \mathrm{tr}-\mathrm{RV}(\rho)$. Otherwise, f is said to be a *nontrivial* regularly varying function of the index ρ and it is denoted by $f \in \mathrm{ntr}-\mathrm{RV}(\rho)$.

The reader is referred to Bingham et al. [1] and Seneta [29] for a complete exposition of theory of regular variation and its application to various branches of mathematical analysis.

To properly describe the possible asymptotic behavior of nonoscillatory solutions of the self-adjoint second-order linear differential equation $(p(t)x'(t))' + q(t)x(t) = 0$, which are essentially affected by the function $p(t)$, Jaroš and Kusano introduced in [2] the class of generalized Karamata functions with the following definition.

Definition 2.2. Let R be a positive function which is continuously differentiable on (a,∞) and satisfies $R'(t) > 0$, $t > a$ and $\lim_{t\to\infty} R(t) = \infty$. A measurable function $f : (a,\infty) \to (0,\infty)$ for some $a > 0$ is said to be *regularly varying of index $\rho \in \mathbb{R}$ with respect to R if $f \circ R^{-1}$ is defined for all large t and is regularly varying function of index ρ in the sense of Karamata, where R^{-1} denotes the inverse function of R.*

The symbol $\mathrm{RV}_R(\rho)$ is used to denote the totality of regularly varying functions of index $\rho \in \mathbb{R}$ with respect to R. The symbol SV_R is often used for $\mathrm{RV}_R(0)$. It is easy to see that if $f \in \mathrm{RV}_R(\rho)$, then $f(t) = R(t)^\rho\, \ell(t)$, $\ell \in \mathrm{SV}_R$. If

$$\lim_{t\to\infty} \frac{f(t)}{R(t)^\rho} = \lim_{t\to\infty} \ell(t) = \text{const} > 0$$

then f is said to be a *trivial* regularly varying function of index ρ with respect to R and it is denoted by $f \in \mathrm{tr}-\mathrm{RV}_R(\rho)$. Otherwise, f is said to be a *nontrivial* regularly varying function of index ρ with respect to R and it is denoted by $f \in \mathrm{ntr}-\mathrm{RV}_R(\rho)$. Also, from Definition 2.2 it follows that $f \in \mathrm{RV}_R(\rho)$ if and only if it is written in the form $f(t) = g(R(t))$, $g \in \mathrm{RV}(\rho)$. It is clear that $\mathrm{RV}(\rho) = \mathrm{RV}_t(\rho)$. We emphasize that there exists a function which is regularly varying in generalized sense, but is not regularly varying in the sense of Karamata, so that, roughly speaking, the class of generalized Karamata functions is larger than that of classical Karamata functions.

To help the reader we present here some elementary properties of generalized regularly varying functions.

Proposition 2.3. (i) *If $g_1 \in \mathrm{RV}_R(\sigma_1)$, then $g_1^\alpha \in \mathrm{RV}_R(\alpha\sigma_1)$ for any $\alpha \in \mathbb{R}$.*

(ii) *If $g_i \in \mathrm{RV}_R(\sigma_i)$, $i = 1,2$, then $g_1 + g_2 \in \mathrm{RV}_R(\sigma)$, $\sigma = \max(\sigma_1,\sigma_2)$.*

(iii) *If $g_i \in \mathrm{RV}_R(\sigma_i)$, $i = 1,2$, then $g_1 g_2 \in \mathrm{RV}_R(\sigma_1 + \sigma_2)$.*

(iv) *If $g_i \in \mathrm{RV}_R(\sigma_i)$, $i = 1,2$ and $g_2(t) \to \infty$ as $t \to \infty$, then $g_1 \circ g_2 \in \mathrm{RV}_R(\sigma_1\sigma_2)$.*

(v) *If $\ell \in \mathrm{SV}_R$, then for any $\varepsilon > 0$,*

$$\lim_{t\to\infty} R(t)^\varepsilon \ell(t) = \infty, \quad \lim_{t\to\infty} R(t)^{-\varepsilon} \ell(t) = 0.$$

Next, we present a fundamental result (see [2]), called *generalized Karamata integration theorem*, which will be used throughout the paper and play a central role in establishing our main results.

Proposition 2.4. *Let $\ell \in \mathrm{SV}_R$. Then:*

(i) *If $\alpha > -1$,*
$$\int_a^t R'(s)R(s)^\alpha \ell(s)\,ds \sim \frac{R(t)^{\alpha+1}\,\ell(t)}{\alpha+1}, \quad t \to \infty;$$

(ii) *If $\alpha < -1$,*
$$\int_t^\infty R'(s)\;R(s)^\alpha\;\ell(s)\,ds \sim -\frac{R(t)^{\alpha+1}\,\ell(t)}{\alpha+1}, \quad t \to \infty;$$

(iii) *If $\alpha = -1$, then functions*
$$\int_a^t R'(s)R(s)^{-1}\,\ell(s)\,ds \quad \text{and} \quad \int_t^\infty R'(s)R(s)^{-1}\,\ell(s)\,ds$$
are slowly varying with respect to R.

3. Asymptotic behavior of intermediate generalized regularly varying solutions

In what follows it is always assumed that functions p and q are generalized regularly varying of index η and σ with respect to R, with $R(t)$ is defined with
$$R(t) = \Big(\int_t^\infty \frac{s^{1+\frac{1}{\alpha}}}{p(s)^{1/\alpha}}\,ds\Big)^{-1}, \tag{3.1}$$
and expressed as
$$p(t) = R(t)^\eta l_p(t),\ l_p \in \mathrm{SV}_R \quad \text{and} \quad q(t) = R(t)^\sigma l_q(t),\ l_q \in \mathrm{SV}_R\,. \tag{3.2}$$
From (3.1) and (3.2) we have that
$$t^{1+\frac{1}{\alpha}} = R'(t)R(t)^{\frac{\eta}{\alpha}-2} l_p(t)^{1/\alpha}. \tag{3.3}$$
Integrating (3.3) from a to t we have
$$\frac{t^{2+\frac{1}{\alpha}}}{2+\frac{1}{\alpha}} = \int_a^t R'(s)R(s)^{\frac{\eta}{\alpha}-2} l_p(s)^{1/\alpha} ds, \quad t \to \infty, \tag{3.4}$$
implying that $\frac{\eta}{\alpha} \geq 1$. In what follows we limit ourselves to the case where $\eta > \alpha$ excluding the other possibilities because of computational difficulty. Applying the generalized Karamata integration theorem (Proposition 2.4) at the right hand side of (3.4) we obtain
$$t \sim \Big(\frac{\eta-\alpha}{2\alpha+1}\Big)^{-\frac{\alpha}{2\alpha+1}} R(t)^{\frac{\eta-\alpha}{2\alpha+1}} l_p(t)^{\frac{1}{2\alpha+1}}, \quad t \to \infty. \tag{3.5}$$
From (3.3) and (3.5) we can express $R'(t)$ as follows
$$R'(t) \sim \Big(\frac{\eta-\alpha}{2\alpha+1}\Big)^{-\frac{\alpha+1}{2\alpha+1}} R(t)^{\frac{3\alpha+1-\eta}{2\alpha+1}} l_p(t)^{-\frac{1}{2\alpha+1}}, \quad t \to \infty\,, \tag{3.6}$$
which can be rewritten in the form
$$1 \sim \Big(\frac{\eta-\alpha}{2\alpha+1}\Big)^{\frac{\alpha+1}{2\alpha+1}} R'(t)R(t)^{m_2(\alpha,\eta)-1} l_p(t)^{\frac{1}{2\alpha+1}}, \quad t \to \infty. \tag{3.7}$$

The next lemma, following directly from the generalized Karamata integration theorem using (3.7), will be frequently used in our later discussions. To that end and to further simplify formulation of our main results we introduce the notation:

$$m_1(\alpha,\eta)=\frac{-2\alpha^2-\eta}{\alpha(2\alpha+1)},\quad m_2(\alpha,\eta)=\frac{\eta-\alpha}{2\alpha+1}. \tag{3.8}$$

It is clear that $m_1(\alpha,\eta)<-1<0<m_2(\alpha,\eta)$ and

$$m_1(\alpha,\eta)=2m_2(\alpha,\eta)-\frac{\eta}{\alpha};\quad \frac{m_2(\alpha,\eta)-\eta}{\alpha}=-2m_2(\alpha,\eta)-1\,. \tag{3.9}$$

In proofs of our main results constants $m_i(\alpha,\eta)$, $i=1,2$, will be abbreviated as m_i, $i=1,2$, respectively.

Lemma 3.1. *Let $f(t)=R(t)^{\mu}L_f(t)$, $L_f\in \mathrm{SV}_R$. Then:*

(i) *If $\mu>-m_2(\alpha,\eta)$,*
$$\int_a^t f(s)\,ds\sim\frac{m_2(\alpha,\eta)^{\frac{\alpha+1}{2\alpha+1}}}{\mu+m_2(\alpha,\eta)}\,R(t)^{\mu+m_2(\alpha,\eta)}L_f(t)l_p(t)^{\frac{1}{2\alpha+1}},\quad t\to\infty;$$

(ii) *If $\mu<-m_2(\alpha,\eta)$,*
$$\int_t^\infty f(s)\,ds\sim\frac{m_2(\alpha,\eta)^{\frac{\alpha+1}{2\alpha+1}}}{-(\mu+m_2(\alpha,\eta))}R(t)^{\mu+m_2(\alpha,\eta)}L_f(t)l_p(t)^{\frac{1}{2\alpha+1}},\quad t\to\infty;$$

(iii) *If $\mu=-m_2(\alpha,\eta)$, then functions*
$$\int_a^t f(s)\,ds=\int_a^t R(s)^{-m_2(\alpha,\eta)}L_f(s)\,ds,$$
$$\int_t^\infty f(s)\,ds=\int_t^\infty R(s)^{-m_2(\alpha,\eta)}L_f(s)\,ds$$
are slowly varying with respect to R.

To make an in depth analysis of intermediate solutions of type (I1) and (I2) of (1.1) we need a fair knowledge of the structure of the functions ψ_1,ψ_2,φ_1 and φ_2 regarded as generalized regularly varying functions with respect to R. From (3.5), (3.6) and (3.7) it is clear that $\psi_1\in\mathrm{SV}_R$ and $\psi_2\in\mathrm{RV}_R(m_2(\alpha,\eta))$. Using (3.2) and applying Lemma 3.1 twice, we obtain

$$\begin{aligned}\varphi_1(t)&=\int_t^\infty\int_s^\infty R(r)^{-\eta/\alpha}l_p(r)^{-1/\alpha}\,dr\,ds\\&\sim\frac{m_2(\alpha,\eta)^{\frac{2(\alpha+1)}{2\alpha+1}}}{m_1(\alpha,\eta)(m_1(\alpha,\eta)-m_2(\alpha,\eta))}R(t)^{m_1(\alpha,\eta)}\,l_p(t)^{-\frac{1}{\alpha(2\alpha+1)}},\quad t\to\infty,\end{aligned} \tag{3.10}$$

which shows that $\varphi_1\in\mathrm{RV}_R(m_1(\alpha,\eta))$. Further, by (3.2) and (3.5), in view of (3.9)-(ii), another two applications of Lemma 3.1 yield

$$\begin{aligned}\varphi_2(t)&\sim m_2(\alpha,\eta)^{-\frac{1}{2\alpha+1}}\int_t^\infty\int_s^\infty R(r)^{-2m_2(\alpha,\eta)-1}\,l_p(r)^{-\frac{2}{2\alpha+1}}\,dr\,ds\\&\sim\frac{m_2(\alpha,\eta)}{m_2(\alpha,\eta)+1}R(t)^{-1},\quad t\to\infty,\end{aligned} \tag{3.11}$$

implying $\varphi_2\in\mathrm{RV}_R(-1)$.

3.1. **Regularly varying solutions of type (I1).** The first subsection is devoted to the study of the existence and asymptotic behavior of generalized regularly varying solutions with respect to R of type (I1) with p and q satisfying (3.2). Expressing such solution x of (1.1) in the form

$$x(t) = R(t)^{\rho}\, l_x(t), \quad l_x \in \mathrm{SV}_R, \tag{3.12}$$

since $\varphi_1(t) \prec x(t) \prec \varphi_2(t)$, $t \to \infty$, the regularity index ρ of x must satisfy

$$m_1(\alpha,\eta) \le \rho \le -1.$$

If $\rho = m_1(\alpha,\eta)$, then since $x(t)/R(t)^{m_1(\alpha,\eta)} = l_x(t) \to \infty$, $t \to \infty$, x is a member of $\mathrm{ntr}-\mathrm{RV}_R(m_1(\alpha,\eta))$, while if $\rho = -1$, then since $x(t)/R(t)^{-1} = l_x(t) \to 0$, $t \to \infty$, x is a member of $\mathrm{ntr}-\mathrm{RV}_R(-1)$. Thus the set of all generalized regularly varying solutions of type (I1) is naturally divided into the three disjoint classes

$$\mathrm{ntr}-\mathrm{RV}_R(m_1(\alpha,\eta)) \quad \text{or}$$
$$\mathrm{RV}_R(\rho) \ \text{ with } \rho \in (m_1(\alpha,\eta)\ ,\ -1) \quad \text{or} \quad \mathrm{ntr}-\mathrm{RV}_R\,(-1)\,.$$

Our aim is to establish necessary and sufficient conditions for each of the above classes to have a member and furthermore to show that the asymptotic behavior of all members of each class is governed by a unique explicit formula describing the decay order at infinity accurately.

Main results.

Theorem 3.2. *Let $p \in \mathrm{RV}_R(\eta), q \in \mathrm{RV}_R(\sigma)$. Equation* (1.1) *has intermediate solutions $x \in \mathrm{ntr}-\mathrm{RV}_R(m_1(\alpha,\eta))$ satisfying* (I1) *if and only if*

$$\sigma = -\beta m_1(\alpha,\eta) - 2m_2(\alpha,\eta) \quad \text{and} \quad \int_a^\infty t q(t)\varphi_1(t)^\beta\,dt = \infty. \tag{3.13}$$

The asymptotic behavior of any such solution x is governed by the unique formula $x(t) \sim X_1(t)$, $t \to \infty$, where

$$X_1(t) = \varphi_1(t)\Big(\frac{\alpha-\beta}{\alpha}\int_a^t s q(s)\varphi_1(s)^\beta\,ds\Big)^{\frac{1}{\alpha-\beta}}. \tag{3.14}$$

Theorem 3.3. *Let $p \in \mathrm{RV}_R(\eta), q \in \mathrm{RV}_R(\sigma)$. Equation* (1.1) *has intermediate solutions $x \in \mathrm{RV}_R(\rho)$ with $\rho \in (m_1(\alpha,\eta),-1)$ if and only if*

$$-\beta m_1(\alpha,\eta) - 2m_2(\alpha,\eta) < \sigma < \beta - m_2(\alpha,\eta), \tag{3.15}$$

in which case

$$\rho = \frac{\sigma + m_2(\alpha,\eta) - \alpha}{\alpha-\beta} \tag{3.16}$$

and the asymptotic behavior of any such solution x is given by the unique formula $x(t) \sim X_2(t)$, $t \to \infty$, where

$$X_2(t) = \bigg(\Big(\frac{m_2(\alpha,\eta)^{\frac{(\alpha+1)^2}{2\alpha+1}}}{\alpha}\Big)^2 \frac{p(t)^{\frac{1}{2\alpha+1}} q(t) R(t)^{-2\frac{\alpha(\alpha+1)}{2\alpha+1}}}{(m_1(\alpha,\eta)-\rho)(\rho+1)(\rho(\rho-m_2(\alpha,\eta)))^\alpha}\bigg)^{\frac{1}{\alpha-\beta}}. \tag{3.17}$$

Theorem 3.4. *Let $p \in \mathrm{RV}_R(\eta)$, $q \in \mathrm{RV}_R(\sigma)$. Equation* (1.1) *has intermediate solutions $x \in \mathrm{ntr}-\mathrm{RV}_R(-1)$ satisfying* (I1) *if and only if*

$$\sigma = \beta - m_2(\alpha,\eta) \quad \text{and} \quad \int_a^\infty q(t)\varphi_2(t)^\beta\,dt < \infty. \tag{3.18}$$

The asymptotic behavior of any such solution x is given by the unique formula $x(t) \sim X_3(t)$, $t \to \infty$, where

$$X_3(t) = \varphi_2(t)\Big(\frac{\alpha-\beta}{\alpha}\int_t^\infty q(s)\,\varphi_2(s)^\beta\,ds\Big)^{\frac{1}{\alpha-\beta}}. \tag{3.19}$$

Preparatory results. Let x be a solution of (1.1) on $[t_0,\infty)$ such that $\varphi_1(t) \prec x(t) \prec \varphi_2(t)$ as $t\to\infty$. Since

$$\lim_{t\to\infty}\big(p(t)(x''(t))^\alpha\big)' = \lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} p(t)(x''(t))^\alpha = \infty, \tag{3.20}$$

integrating (1.1) first on $[t,\infty)$, and then on $[t_0,t]$ and finally twice on $[t,\infty)$ we obtain

$$x(t) = \int_t^\infty \frac{s-t}{p(s)^{1/\alpha}}\Big(\xi_2 + \int_{t_0}^s\int_r^\infty q(u)x(u)^\beta\,du\,dr\Big)^{1/\alpha} ds, \quad t \ge t_0, \tag{3.21}$$

where $\xi_2 = p(t_0)x''(t_0)^\alpha$.

To prove the existence of intermediate solutions of type (I1) it is sufficient to prove the existence of a positive solution of the integral equation (3.21) for some constants $t_0 \ge a$ and $\xi_2 > 0$, which is most commonly achieved by application of Schauder-Tychonoff fixed point theorem. Denoting by $\mathcal{G}x(t)$ the right-hand side of (3.21), to find a fixed point of $\mathcal{G}$ it is crucial to choose a closed convex subset $\mathcal{X} \subset C[t_0,\infty)$ on which $\mathcal{G}$ is a self-map. Since our primary goal is not only proving the existence of generalized RV intermediate solutions, but establishing a precise asymptotic formula for such solutions, a choice of such a subset $\mathcal{X}$ must be made appropriately. It will be shown that such a choice of $\mathcal{X}$ is possible by solving the integral asymptotic relation

$$x(t) \sim \int_t^\infty \frac{s-t}{p(s)^{1/\alpha}}\Big(\int_b^s\int_r^\infty q(u)x(u)^\beta\,du\,dr\Big)^{1/\alpha} ds, \quad t\to\infty, \tag{3.22}$$

for some $b \ge t_0$, which can be considered as an approximation (at infinity) of (3.21) in the sense that it is satisfied by all possible solutions of type $(I1)$ of (1.1). Theory of regular variation will in fact ensure the solvability of (3.22) in the framework of generalized Karamata functions.

As preparatory steps toward the proofs of Theorems 3.2-3.4 we show that the generalized regularly varying functions $X_i, i = 1,2,3$ defined respectively by (3.14), (3.17) and (3.19) satisfy the asymptotic relation (3.22).

Lemma 3.5. *Suppose that* (3.13) *holds. Function X_1 given by* (3.14) *satisfies the asymptotic relation* (3.22) *for any $b \ge a$ and belongs to* $\mathrm{ntr}-\mathrm{RV}_R(m_1(\alpha,\eta))$.

Proof. From (3.2), (3.5) and (3.10), we have

$$tq(t)\varphi_1(t)^\beta \sim \frac{m_2^{\frac{2\beta(\alpha+1)-\alpha}{2\alpha+1}}}{(m_1(m_1-m_2))^\beta} R(t)^{\sigma+\beta m_1+m_2} l_p(t)^{\frac{\alpha-\beta}{\alpha(2\alpha+1)}} l_q(t), \quad t\to\infty\,,$$

and applying (iii) of Lemma 3.1, in view of (3.13), we obtain

$$\int_a^t sq(s)\varphi_1(s)^\beta\,ds \sim \frac{m_2^{\frac{2\beta(\alpha+1)-\alpha}{2\alpha+1}}}{(m_1(m_1-m_2))^\beta}\int_a^t R(s)^{-m_2} l_p(s)^{\frac{\alpha-\beta}{\alpha(2\alpha+1)}} l_q(s)\,ds \in \mathrm{SV}_R, \tag{3.23}$$

as $t\to\infty$, which together with (3.14) gives

$$X_1(t)\sim\varphi_1(t)\Big(\frac{m_2^{\frac{2\beta(\alpha+1)-\alpha}{2\alpha+1}}}{(m_1(m_1-m_2))^{\beta}}\frac{\alpha-\beta}{\alpha}J_1(t)\Big)^{\frac{1}{\alpha-\beta}},\quad t\to\infty,$$

where

$$J_1(t)=\int_a^t R(s)^{-m_2}l_p(s)^{\frac{\alpha-\beta}{\alpha(2\alpha+1)}}l_q(s)\,ds. \tag{3.24}$$

Thus, since $J_1\in \mathrm{SV}_R$, we conclude that $X_1\in \mathrm{ntr}-\mathrm{RV}_R(m_1(\alpha,\eta))$ and rewrite the previous relation, using (3.10), as

$$X_1(t)\sim R(t)^{m_1}l_p(t)^{-\frac{1}{\alpha(2\alpha+1)}}\Big(\Big(\frac{m_2}{m_1(m_1-m_2)}\Big)^{\alpha}\frac{\alpha-\beta}{\alpha}J_1(t)\Big)^{\frac{1}{\alpha-\beta}},\quad t\to\infty. \tag{3.25}$$

To prove that (3.22) is satisfied by X_1, we first integrate $q(t)X_1(t)^{\beta}$ on $[t,\infty)$, applying Lemma 3.1 and using (3.13) we have

$$\begin{aligned}
&\int_t^{\infty}q(s)\,X_1(s)^{\beta}\,ds\\
&\sim m_2^{-\frac{\alpha}{2\alpha+1}}\Big(\Big(\frac{m_2}{m_1(m_1-m_2)}\Big)^{\alpha}\frac{\alpha-\beta}{\alpha}\Big)^{\frac{\beta}{\alpha-\beta}}R(t)^{-m_2}l_p(t)^{\frac{\alpha-\beta}{\alpha(2\alpha+1)}}l_q(t)J_1(t)^{\frac{\beta}{\alpha-\beta}},
\end{aligned}$$

as $t\to\infty$. Integrating the above relation on $[b,t]$, for any $b\ge a$, we obtain

$$\begin{aligned}
\int_b^t\int_s^{\infty}q(r)\,X_1(r)^{\beta}\,dr\,ds&\sim m_2^{-\frac{\alpha}{2\alpha+1}}\Big(\Big(\frac{m_2}{m_1(m_1-m_2)}\Big)^{\alpha}\frac{\alpha-\beta}{\alpha}\Big)^{\frac{\beta}{\alpha-\beta}}\\
&\times\int_b^t R(s)^{-m_2}l_p(s)^{\frac{\alpha-\beta}{\alpha(2\alpha+1)}}l_q(s)J_1(s)^{\frac{\beta}{\alpha-\beta}}\,ds\\
&=m_2^{-\frac{\alpha}{2\alpha+1}}\Big(\Big(\frac{m_2}{m_1(m_1-m_2)}\Big)^{\alpha}\frac{\alpha-\beta}{\alpha}\Big)^{\frac{\beta}{\alpha-\beta}}\int_b^t J_1(s)^{\frac{\beta}{\alpha-\beta}}\,dJ_1(s)\\
&=m_2^{-\frac{\alpha}{2\alpha+1}}\Big(\Big(\frac{m_2}{m_1(m_1-m_2)}\Big)^{\beta}\frac{\alpha-\beta}{\alpha}\Big)^{\frac{\alpha}{\alpha-\beta}}J_1(t)^{\frac{\alpha}{\alpha-\beta}},\quad t\to\infty.
\end{aligned}$$

Integrating the above relation multiplied by $p(t)^{-1}$ and powered by $\frac{1}{\alpha}$ twice on $[t,\infty)$, applying Lemma 3.1 and using (3.9)-(i), we obtain

$$\begin{aligned}
&\int_t^{\infty}\int_s^{\infty}\Big(\frac{1}{p(r)}\int_a^r\int_u^{\infty}q(\omega)X_1(\omega)^{\beta}\,d\omega du\Big)^{1/\alpha}\,dr\,ds\\
&\sim\Big(\Big(\frac{m_2}{m_1(m_1-m_2)}\Big)^{\beta}\frac{\alpha-\beta}{\alpha}\Big)^{\frac{1}{\alpha-\beta}}\frac{m_2}{m_1(m_1-m_2)}R(t)^{m_1}l_p(t)^{-\frac{1}{\alpha(2\alpha+1)}}J_1(t)^{\frac{1}{\alpha-\beta}},
\end{aligned}$$

as $t\to\infty$, which due to (3.25) proves that X_1 satisfies the desired asymptotic relation (3.22) for any $b\ge a$. □

Lemma 3.6. *Suppose that* (3.15) *holds and let* ρ *be defined by* (3.16)*. Function* X_2 *given by* (3.17) *satisfies the asymptotic relation* (3.22) *for any* $b\ge a$ *and belongs to* $\mathrm{RV}_R(\rho)$.

Proof. Using (3.8) and (3.16) we obtain

$$\sigma+\rho\beta+m_2=\alpha(\rho+1),\quad \sigma+\rho\beta+2m_2=\alpha(\rho-m_1)\,. \tag{3.26}$$

The function X_2 given by (3.17) can be expressed in the form

$$X_2(t) \sim (\lambda\alpha^2)^{-\frac{1}{\alpha-\beta}}\, m_2^{\frac{2(\alpha+1)^2}{(2\alpha+1)(\alpha-\beta)}}\, R(t)^{\rho} \Big(l_p(t)^{\frac{1}{2\alpha+1}}\, l_q(t)\Big)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty, \tag{3.27}$$

where

$$\lambda = (\rho(\rho - m_2))^{\alpha}\,(m_1 - \rho)\,(\rho + 1)\,.$$

Thus, $X_2 \in \mathrm{RV}_R(\rho)$. Using (3.26) and (3.27), applying Lemma 3.1 twice, we find

$$\int_t^{\infty} q(s)\, X_2(s)^{\beta}\, ds \sim -\frac{m_2^{\frac{(\alpha+1)(2\alpha\beta+\alpha+\beta)}{(2\alpha+1)(\alpha-\beta)}}}{\big(\lambda\alpha^2\big)^{\frac{\beta}{\alpha-\beta}}\,(\sigma+\rho\beta+m_2)} R(t)^{\sigma+\rho\beta+m_2}\Big(l_p(t)^{\frac{1}{2\alpha+1}}\, l_q(t)\Big)^{\frac{\alpha}{\alpha-\beta}},$$

and for any $b \ge a$,

$$\begin{aligned}
&\int_b^t \int_s^{\infty} q(r)\, X_2(r)^{\beta}\, dr\, ds\\
&\sim \frac{m_2^{\frac{2\alpha(\alpha+1)(\beta+1)}{(2\alpha+1)(\alpha-\beta)}}}{\big(\lambda\alpha^2\big)^{\frac{\beta}{\alpha-\beta}}\,(-(\sigma+\rho\beta+m_2))(\sigma+\rho\beta+2m_2)} R(t)^{\sigma+\rho\beta+2m_2}\Big(l_p(t)^{\frac{2\alpha-\beta}{2\alpha+1}}\, l_q(t)^{\alpha}\Big)^{\frac{1}{\alpha-\beta}}\\
&= \frac{m_2^{\frac{2\alpha(\alpha+1)(\beta+1)}{(2\alpha+1)(\alpha-\beta)}}}{\big(\lambda\alpha^2\big)^{\frac{\beta}{\alpha-\beta}}\,\alpha^2(-(\rho+1))(\rho-m_1)} R(t)^{\alpha(\rho-m_1)}\Big(l_p(t)^{\frac{2\alpha-\beta}{2\alpha+1}}\, l_q(t)^{\alpha}\Big)^{\frac{1}{\alpha-\beta}}\\
&= \frac{m_2^{\frac{2\alpha(\alpha+1)(\beta+1)}{(2\alpha+1)(\alpha-\beta)}}}{\big(\lambda\alpha^2\big)^{\frac{\beta}{\alpha-\beta}}\,\alpha^2(\rho+1)(m_1-\rho)} R(t)^{\alpha(\rho-2m_2+\frac{\eta}{\alpha})}\Big(l_p(t)^{\frac{2\alpha-\beta}{2\alpha+1}}\, l_q(t)^{\alpha}\Big)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty\,,
\end{aligned}$$

where we have used (3.9)-(i) in the last step. We now multiply the last relation by $p(t)^{-1}$, raise to the exponent $1/\alpha$ and integrate the obtained relation twice on $[t,\infty)$. As a result of application of Lemma 3.1, we obtain for $t \to \infty$

$$\begin{aligned}
&\int_t^{\infty} \Big(\frac{1}{p(s)} \int_b^s \int_r^{\infty} q(u)\, X_2(u)^{\beta}\, du\, dr\Big)^{1/\alpha} ds\\
&\sim -\frac{m_2^{\frac{(\alpha+1)(\alpha+\beta+2)}{(\alpha-\beta)(2\alpha+1)}}}{\big(\lambda\alpha^2\big)^{\frac{\beta}{\alpha(\alpha-\beta)}}\,(\alpha^2(m_1-\rho)(\rho+1))^{1/\alpha}(\rho-m_2)} R(t)^{\rho-m_2}\Big(l_p(t)^{\frac{\beta-\alpha+1}{2\alpha+1}}\, l_q(t)\Big)^{\frac{1}{\alpha-\beta}},
\end{aligned}$$

and

$$\begin{aligned}
&\int_t^{\infty} \int_s^{\infty} \Big(\frac{1}{p(r)} \int_b^r \int_u^{\infty} q(\omega)\, X_2(\omega)^{\beta}\, d\omega du\Big)^{1/\alpha} dr\, ds\\
&\sim \frac{m_2^{\frac{2(\alpha+1)^2}{(\alpha-\beta)(2\alpha+1)}}}{\big(\lambda\alpha^2\big)^{\frac{\beta}{\alpha-\beta}}\,\rho(\rho-m_2)(\alpha^2(m_1-\rho)(\rho+1))^{1/\alpha}} R(t)^{\rho}\Big(l_p(t)^{\frac{1}{2\alpha+1}}\, l_q(t)\Big)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty.
\end{aligned}$$

This, due to (3.27), completes the proof of Lemma 3.6. □

Lemma 3.7. *Suppose that* (3.18) *holds. Then the function* X_3 *given by* (3.19) *satisfies the asymptotic relation* (3.22) *for any* $b \ge a$ *and belongs to* $\mathrm{ntr}-\mathrm{RV}_R(-1)$.

Proof. Using (3.2), (3.11), (3.18) and applying (iii) of Lemma 3.1, we obtain

$$\int_t^{\infty} q(s)\, \varphi_2(s)^{\beta}\, ds \sim \Big(\frac{m_2}{m_2+1}\Big)^{\beta} J_3(t), \quad t \to \infty, \tag{3.28}$$

where

$$J_3(t) = \int_t^\infty R(s)^{-m_2} l_q(s)\,ds, \quad J_3 \in \mathrm{SV}_R, \tag{3.29}$$

implying, from (3.19),

$$X_3(t) \sim \Big(\frac{m_2}{m_2+1}\Big)^{\frac{\alpha}{\alpha-\beta}} R(t)^{-1}\Big(\frac{\alpha-\beta}{\alpha}\,J_3(t)\Big)^{\frac{1}{\alpha-\beta}}, \quad t\to\infty. \tag{3.30}$$

This shows that $X_3 \in \mathrm{RV}_R(-1)$. Next, we integrate $q(t)\,X_3(t)^\beta$ on $[t,\infty)$, using (3.18) we obtain

$$\begin{aligned}
\int_t^\infty q(s)\,X_3(s)^\beta\,ds &\sim \Big(\frac{m_2}{m_2+1}\Big)^{\frac{\alpha\beta}{\alpha-\beta}}\Big(\frac{\alpha-\beta}{\alpha}\Big)^{\frac{\beta}{\alpha-\beta}}\int_t^\infty R(s)^{-m_2} l_q(s)\,J_3(s)^{\frac{\beta}{\alpha-\beta}}\,ds\\
&= \Big(\frac{m_2}{m_2+1}\Big)^{\frac{\alpha\beta}{\alpha-\beta}}\Big(\frac{\alpha-\beta}{\alpha}\Big)^{\frac{\beta}{\alpha-\beta}}\int_t^\infty J_3(s)^{\frac{\beta}{\alpha-\beta}}(-dJ_3(s))\\
&= \Big(\frac{m_2}{m_2+1}\Big)^{\frac{\alpha\beta}{\alpha-\beta}}\Big(\frac{\alpha-\beta}{\alpha}\Big)^{\frac{\alpha}{\alpha-\beta}} J_3(t)^{\frac{\alpha}{\alpha-\beta}} \in \mathrm{SV}_R, \quad t\to\infty.
\end{aligned}$$

Further, integrating previous relation on $[b,t]$ for any fixed $b \ge a$, by Lemma 3.1, we have

$$\begin{aligned}
&\int_b^t\int_s^\infty q(r)X_3(r)^\beta\,dr\,ds\\
&\sim \Big(\frac{m_2}{m_2+1}\Big)^{\frac{\alpha\beta}{\alpha-\beta}}\Big(\frac{\alpha-\beta}{\alpha}\Big)^{\frac{\alpha}{\alpha-\beta}} m_2^{-\frac{\alpha}{2\alpha+1}} R(t)^{m_2} l_p(t)^{\frac{1}{2\alpha+1}} J_3(t)^{\frac{\alpha}{\alpha-\beta}}, \quad t\to\infty.
\end{aligned}$$

Multiply the above by $p(t)^{-1}$ and raise to the exponent $1/\alpha$, integrating obtained relation twice on $[t,\infty)$, using (3.9)-(ii), as a result of application of Lemma 3.1, we obtain

$$\begin{aligned}
&\int_t^\infty\Big(\frac{1}{p(s)}\int_b^s\int_r^\infty q(u)X_3(u)^\beta\,du\,dr\Big)^{1/\alpha}ds\\
&\sim \Big(\frac{m_2}{m_2+1}\Big)^{\frac{\beta}{\alpha-\beta}}\Big(\frac{\alpha-\beta}{\alpha}\Big)^{\frac{1}{\alpha-\beta}}\frac{m_2^{\frac{\alpha}{2\alpha+1}}}{m_2+1} R(t)^{-m_2-1} l_p(t)^{-\frac{\alpha}{\alpha(2\alpha+1)}} J_3(t)^{\frac{1}{\alpha-\beta}}, \quad t\to\infty,
\end{aligned}$$

and

$$\begin{aligned}
&\int_t^\infty\int_s^\infty\Big(\frac{1}{p(r)}\int_b^r\int_u^\infty q(\omega)X_3(\omega)^\beta\,d\omega du\Big)^{1/\alpha}\,dr\,ds\\
&\sim \Big(\frac{m_2}{m_2+1}\Big)^{\frac{\beta}{\alpha-\beta}}\Big(\frac{\alpha-\beta}{\alpha}\Big)^{\frac{1}{\alpha-\beta}}\frac{m_2}{m_2+1}R(t)^{-1}J_3(t)^{\frac{1}{\alpha-\beta}} \sim X_3(t), \quad t\to\infty,
\end{aligned}$$

which in view of (3.30), completes the proof of Lemma 3.7. □

The above theorems are a basis for applying the Schauder-Tychonoff fixed point theorem to establish the existence of intermediate solutions of the equation (1.1). In fact, intermediate solutions will be constructed by means of fixed point techniques, and afterwards we confirm that they are really generalized regularly varying functions with the help of the generalized L'Hospital rule formulated below.

Lemma 3.8. *Let $f,g \in C^1[T,\infty)$. Let*

$$\lim_{t\to\infty} g(t) = \infty \quad \textit{and} \quad g'(t) > 0 \quad \textit{for all large } t. \tag{3.31}$$

Then
$$\liminf_{t\to\infty}\frac{f'(t)}{g'(t)}\le\liminf_{t\to\infty}\frac{f(t)}{g(t)}\le\limsup_{t\to\infty}\frac{f(t)}{g(t)}\le\limsup_{t\to\infty}\frac{f'(t)}{g'(t)}.$$
If we replace (3.31) *with the condition*
$$\lim_{t\to\infty}f(t)=\lim_{t\to\infty}g(t)=0\quad\textit{and}\quad g'(t)<0\quad\textit{for all large } t,$$
then the same conclusion holds.

Proofs of main results.

Proof of the "only if" part of Theorems 3.2, 3.3 and 3.4. Suppose that (1.1) has a type (I1) intermediate solution $x\in\mathrm{RV}_R(\rho)$ on $[t_0,\infty)$. Clearly, $\rho\in[m_1,-1]$. Using (3.2) and (3.12), we obtain integrating (1.1) on $[t,\infty)$
$$(p(t)(x''(t))^\alpha)'=\int_t^\infty q(s)x(s)^\beta\,ds=\int_t^\infty R(s)^{\sigma+\beta\rho}l_q(s)l_x(s)^\beta\,ds. \tag{3.32}$$
Noting that the last integral is convergent, we conclude that $\sigma+\beta\rho+m_2\le 0$ and distinguish the two cases:
(1) $\sigma+\beta\rho+m_2=0$ and (2) $\sigma+\beta\rho+m_2<0$.

Assume that (1) holds. Since by Lemma 3.1-(iii) function S_3 defined with
$$S_3(t)=\int_t^\infty R(s)^{-m_2}l_q(s)l_x(s)^\beta\,ds, \tag{3.33}$$
is slowly varying with respect to R, integration of (3.32) on $[t_0,t]$ shows that
$$p(t)(x''(t))^\alpha\sim m_2^{-\frac{\alpha}{2\alpha+1}}R(t)^{m_2}l_p(t)^{\frac{1}{2\alpha+1}}S_3(t),\qquad t\to\infty, \tag{3.34}$$
which is rewritten using (3.9)-(ii) as
$$x''(t)\sim m_2^{-\frac{1}{2\alpha+1}}R(t)^{-2m_2-1}l_p(t)^{-\frac{2}{2\alpha+1}}S_3(t)^{1/\alpha},\quad t\to\infty.$$
Integrability of $x''(t)$ on $[t,\infty)$, and $-m_2-1<0$, allows us to integrate the previous relation on $[t,\infty)$, implying
$$-x'(t)\sim\frac{m_2^{\frac{\alpha}{2\alpha+1}}}{m_2+1}R(t)^{-m_2-1}l_p(t)^{-\frac{1}{2\alpha+1}}S_3(t)^{1/\alpha},\quad t\to\infty,$$
which we may integrate once more on $[t,\infty]$ to obtain
$$x(t)\sim\frac{m_2}{m_2+1}R(t)^{-1}S_3(t)^{1/\alpha},\quad t\to\infty. \tag{3.35}$$
This shows that $x\in\mathrm{RV}_R(-1)$.

Assume next that (2) holds. From (3.32) we find that
$$(p(t)(x''(t))^\alpha)'\sim-\frac{m_2^{\frac{\alpha+1}{2\alpha+1}}}{\sigma+\beta\rho+m_2}R(t)^{\sigma+\beta\rho+m_2}l_p(t)^{\frac{1}{2\alpha+1}}l_q(t)l_x(t)^\beta,\quad t\to\infty,$$
which by integration on $[t_0,t]$ implies
$$p(t)(x''(t))^\alpha\sim-\frac{m_2^{\frac{\alpha+1}{2\alpha+1}}}{\sigma+\beta\rho+m_2}\int_{t_0}^t R(s)^{\sigma+\beta\rho+m_2}l_p(s)^{\frac{1}{2\alpha+1}}l_q(s)l_x(s)^\beta ds, \tag{3.36}$$
as $t\to\infty$. In view of (3.20), integral on right-hand side is divergent, so $\sigma+\beta\rho+2m_2\ge 0$. We distinguish the two cases:
(2.a) $\sigma+\beta\rho+2m_2=0$ and (2.b) $\sigma+\beta\rho+2m_2>0$.

Assume that (2.a) holds. Denote by

$$S_1(t)=\int_{t_0}^{t} R(s)^{-m_2} l_p(s)^{\frac{1}{2\alpha+1}} l_q(s) l_x(s)^{\beta} ds\,. \tag{3.37}$$

Then $S_1 \in \mathrm{SV}_R$ and using (3.2) we rewrite (3.36) as

$$x''(t)\sim m_2^{-\frac{1}{2\alpha+1}} R(t)^{-\eta/\alpha} l_p(t)^{-1/\alpha} S_1(t)^{1/\alpha}, \quad t\to\infty. \tag{3.38}$$

Because of integrability of $x''(t)$ on $[t,\infty]$ and the fact that $-\frac{\eta}{\alpha}+m_2=m_1-m_2<0$, via Lemma 3.1 we conclude by integration of (3.38) on $[t,\infty]$ that

$$-x'(t)\sim -\frac{m_2^{\frac{\alpha}{2\alpha+1}}}{m_1-m_2} R(t)^{m_1-m_2}\, l_p(t)^{-\frac{\alpha+1}{\alpha(2\alpha+1)}} S_1(t)^{1/\alpha}, \quad t\to\infty.$$

which because integrability of $x'(t)$ on $[t,\infty)$ and $m_1<0$, we may integrate once more on $[t,\infty)$ to get

$$x(t)\sim \frac{m_2}{m_1(m_1-m_2)} R(t)^{m_1} l_p(t)^{-\frac{1}{\alpha(2\alpha+1)}} S_1(t)^{1/\alpha}, \quad t\to\infty. \tag{3.39}$$

implying that $x\in \mathrm{RV}_R(m_1)$.

Assume that (2.b) holds. From (3.36), application of Lemma 3.1 gives

$$p(t)(x''(t))^{\alpha}\sim -\frac{m_2^{\frac{2(\alpha+1)}{2\alpha+1}}}{(\sigma+\beta\rho+m_2)(\sigma+\beta\rho+2m_2)} R(t)^{\sigma+\beta\rho+2m_2} l_p(t)^{\frac{2}{2\alpha+1}} l_q(t) l_x(t)^{\beta},$$

as $t\to\infty$, which yields

$$\begin{aligned} x''(t)\sim{}& \frac{m_2^{\frac{2(\alpha+1)}{\alpha(2\alpha+1)}}}{(-(\sigma+\beta\rho+m_2)(\sigma+\beta\rho+2m_2))^{1/\alpha}}\\ &\times R(t)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}} l_p(t)^{\frac{1-2\alpha}{\alpha(2\alpha+1)}} l_q(t)^{1/\alpha} l_x(t)^{\beta/\alpha}, \quad t\to\infty. \end{aligned}$$

Integrability of $x''(t)$ on $[t,\infty]$ allows us to integrate the previous relation on $[t,\infty)$, implying

$$\begin{aligned} -x'(t)\sim{}& \frac{m_2^{\frac{2(\alpha+1)}{\alpha(2\alpha+1)}}}{(-(\sigma+\beta\rho+m_2)(\sigma+\beta\rho+2m_2))^{1/\alpha}}\\ &\times\int_t^{\infty} R(s)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}} l_p(s)^{\frac{1-2\alpha}{\alpha(2\alpha+1)}} l_q(s)^{1/\alpha} l_x(s)^{\beta/\alpha} ds, \quad t\to\infty, \end{aligned} \tag{3.40}$$

where $\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2\le 0$, because of the convergence of the last integral. We distinguish two cases:
(2.b.1) $\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2=0$ and (2.b.2) $\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2<0$.

The case (2.b.1) is impossible because the left-hand side of (3.40) is integrable on $[t_0,\infty)$, while the right-hand side is not, because it is in this case slowly varying with respect to R.

Assume now that (2.b.2) holds. Then, application of Lemma (3.1) in (3.40) and integration of resulting relation on $[t,\infty)$ leads to

$$\begin{aligned} x(t)\sim{}& -\frac{m_2^{\frac{(\alpha+1)(\alpha+2)}{\alpha(2\alpha+1)}}}{(-(\sigma+\beta\rho+m_2)(\sigma+\beta\rho+2m_2))^{1/\alpha}(\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2)}\\ &\times\int_t^{\infty} R(s)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2} l_p(s)^{\frac{1-\alpha}{\alpha(2\alpha+1)}} l_q(s)^{1/\alpha} l_x(s)^{\beta/\alpha} ds, \end{aligned} \tag{3.41}$$

as $t \to \infty$, which brings us to the observation of two possible cases: (2.b.2.1) $\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha} + 2m_2 = 0$ and (2.b.2.2) $\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha} + 2m_2 < 0$.

In the case (2.b.2.1) the integral in the right-hand side of relation (3.41) is slowly varying with respect to R by Proposition 2.4 and so $x \in \mathrm{SV}_R$ too.

In the case (2.b.2.2) an application of Lemma 3.1 gives

$$\begin{aligned} x(t) \sim m_2^{\frac{2(\alpha+1)^2}{\alpha(2\alpha+1)}} \div \Big(&(-(\sigma+\beta\rho+m_2)(\sigma+\beta\rho+2m_2))^{1/\alpha} \\ &\times \Big(\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha} + m_2\Big)\Big(\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha} + 2m_2\Big)\Big) \\ &\times R(t)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+2m_2} l_p(t)^{\frac{1}{\alpha(2\alpha+1)}} l_q(t)^{1/\alpha} l_x(t)^{\beta/\alpha}, \quad t \to \infty, \end{aligned} \tag{3.42}$$

implying that $x \in \mathrm{RV}_R\big(\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha} + 2m_2\big)$.

Suppose that x is a type (I1) solution of (1.1) belonging to $\mathrm{ntr}-\mathrm{RV}_R(m_1)$. From the above observations this is possible only when (2.a) holds, in which case (3.39) is satisfied by $x(t)$. Thus, $\rho = m_1$, $\sigma = -m_1\beta - 2m_2$. Using $x(t) = R(t)^{m_1} l_x(t)$, (3.39) can be expressed as

$$l_x(t) \sim K_1 l_p(t)^{-\frac{1}{\alpha(2\alpha+1)}} S_1(t)^{1/\alpha}, \quad t \to \infty, \tag{3.43}$$

where

$$K_1 = \frac{m_2}{m_1(m_1 - m_2)},$$

and S_1 is defined by (3.37). Then (3.43) is transformed into the differential asymptotic relation for S_1:

$$S_1(t)^{-\frac{\beta}{\alpha}}\, S_1'(t) \sim K_1^{\beta} R(t)^{-m_2} l_p(t)^{\frac{\alpha-\beta}{\alpha(2\alpha+1)}} l_q(t), \quad t \to \infty. \tag{3.44}$$

From (3.39), since $\lim_{t\to\infty} x(t)/\varphi_1(t) = \infty$, we have $\lim_{t\to\infty} S_1(t) = \infty$. Integrating (3.44) on $[t_0, t]$, since $\lim_{t\to\infty} S_1(t)^{\frac{\alpha-\beta}{\alpha}} = \infty$, in view of notation (3.24) and (3.23), we find that the second condition in (3.13) is satisfied and

$$S_1(t)^{1/\alpha} \sim \Big(\frac{\alpha-\beta}{\alpha} K_1^{\beta} J_1(t)\Big)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty,$$

implying with (3.43) that

$$x(t) \sim R(t)^{m_1} l_p(t)^{-\frac{1}{\alpha(2\alpha+1)}} \Big(\frac{\alpha-\beta}{\alpha} K_1^{\alpha} J_1(t)\Big)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty. \tag{3.45}$$

Noting that in the proof of Lemma 3.5, using (3.2), (3.5) and (3.10), we have obtained expression (3.25) for X_1 given by (3.14), (3.45) in fact proves that $x(t) \sim X_1(t)$, $t \to \infty$, completing the "only if" part of the proof of Theorem 3.2.

Next, suppose that x is a solution of (1.1) belonging to $\mathrm{RV}_R(\rho), \rho \in (m_1, -1)$. This is possible only when (2.b.2.2) holds, in which case x satisfies the asymptotic relation (3.42). Therefore,

$$\rho = \frac{\sigma+\beta\rho+2m_2-\eta}{\alpha} + 2m_2 \;\Rightarrow\; \rho = \frac{\sigma+m_2-\alpha}{\alpha-\beta}, \tag{3.46}$$

which justifies (3.16). An elementary calculation shows that

$$m_1 < \rho < -1 \implies -\beta m_1 - 2m_2 < \sigma < \beta - m_2,$$

which determines the range (3.15) of σ. In view of (3.26) and (3.46), we conclude from (3.42) that x enjoys the asymptotic behavior $x(t) \sim X_2(t)$, $t \to \infty$, where X_2 is given by (3.17). This proves the "only if" part of the Theorem 3.3.

Finally, suppose that x is a type-(I1) intermediate solution of (1.1) belonging to $\mathrm{ntr} - \mathrm{RV}_R(-1)$. Then, the case (1) is the only possibility for x, which means that $\sigma = \beta - m_2$ and (3.35) is satisfied by x, with S_3 defined by (3.33). Using $x(t) = R(t)^{-1} l_x(t)$, (3.35) can be expressed as

$$l_x(t) \sim K_3\, S_3(t)^{1/\alpha},\ t \to \infty, \quad \text{where } K_3 = \frac{m_2}{m_2+1}\,, \tag{3.47}$$

implying the differential asymptotic relation

$$-\, S_3(t)^{-\frac{\beta}{\alpha}}\, S_3'(t) \sim K_3^{\beta} R(t)^{-m_2} l_q(t), \quad t \to \infty. \tag{3.48}$$

From (3.35), since $\lim_{t\to\infty} x(t)/R(t)^{-1} = 0$, we have $\lim_{t\to\infty} S_3(t) = 0$, implying that the left-hand side of (3.48) is integrable over $[t_0, \infty)$. This, in view of (3.28) and the notation (3.29), implies the second condition in (3.18). Integrating (3.48) on $[t, \infty)$ and combining result with (3.47), we find that

$$x(t) \sim R(t)^{-1} \Big(\frac{\alpha - \beta}{\alpha} K_3^{\alpha} J_3(t) \Big)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty,$$

which due to the expression (3.30) gives $x(t) \sim X_3(t)$ as $t \to \infty$. This proves the "only if" part of Theorem 3.4.

Proof of the part "if" of Theorems 3.2, 3.3 and 3.4. Suppose that (3.13) or (3.15) or (3.18) holds. From Lemmas 3.5, 3.6 and 3.7 it is known that X_i, $i = 1, 2, 3$, defined by (3.14), (3.17) and (3.19) satisfy the asymptotic relation (3.22) for any $b \geq a$. We perform the simultaneous proof for X_i, $i = 1, 2, 3$ so the subscripts $i = 1, 2, 3$ will be deleted in the rest of the proof. By (3.22) there exists $T_0 > a$ such that

$$\frac{X(t)}{2} \leq \int_t^{\infty} \frac{s-t}{p(s)^{1/\alpha}} \Big(\int_{T_0}^{s} \int_r^{\infty} q(u)\, X(u)^{\beta}\, du\, dr \Big)^{1/\alpha} ds \leq 2X(t), \quad t \geq T_0. \tag{3.49}$$

Let such a T_0 be fixed. Choose positive constants m and M such that

$$m^{1-\frac{\beta}{\alpha}} \leq \frac{1}{2}, \quad M^{1-\frac{\beta}{\alpha}} \geq 2. \tag{3.50}$$

Define the integral operator

$$\mathcal{G}x(t) = \int_t^{\infty} (s-t) \Big(\frac{1}{p(s)} \int_{T_0}^{s} \int_r^{\infty} q(u)\, x(u)^{\beta}\, du\, dr \Big)^{1/\alpha} ds, \quad t \geq T_0, \tag{3.51}$$

and let it act on the set

$$\mathcal{X} = \{x \in C[T_0, \infty) : mX(t) \leq x(t) \leq M\, X(t),\ t \geq T_0\}. \tag{3.52}$$

It is clear that $\mathcal{X}$ is a closed, convex subset of the locally convex space $C[T_0, \infty)$ equipped with the topology of uniform convergence on compact subintervals of $[T_0, \infty)$.

It can be shown that $\mathcal{G}$ is a continuous self-map on $\mathcal{X}$ and that the set $\mathcal{G}(\mathcal{X})$ is relatively compact in $C[T_0, \infty)$.

(i) $\mathcal{G}(\mathcal{X}) \subset \mathcal{X}$: Let $x(t) \in \mathcal{X}$. Using (3.49), (3.50) and (3.52) we obtain

$$\mathcal{G}x(t) \leq M^{\beta/\alpha} \int_t^{\infty} (s-t) \Big(\frac{1}{p(s)} \int_{T_0}^{s} \int_r^{\infty} q(u)\, X(u)^{\beta}\, du\, dr \Big)^{1/\alpha} ds$$

$$\le 2M^{\beta/\alpha}\, X(t) \le M\, X(t), \quad t \ge T_0,$$

and

$$\begin{aligned}\mathcal{G}x(t) &\ge m^{\beta/\alpha}\int_t^\infty (s-t)\Big(\frac{1}{p(s)}\int_{T_0}^s\int_r^\infty q(u)\,X(u)^\beta\,du\,dr\Big)^{1/\alpha}ds\\ &\ge m^{\beta/\alpha}\frac{X(t)}{2} \ge m\,X(t), \quad t \ge T_0.\end{aligned}$$

This shows that $\mathcal{G}x(t) \in \mathcal{X}$; that is, $\mathcal{G}$ maps $\mathcal{X}$ into itself.
(ii) $\mathcal{G}(\mathcal{X})$ is relatively compact. The inclusion $\mathcal{G}(\mathcal{X}) \subset \mathcal{X}$ ensures that $\mathcal{G}(\mathcal{X})$ is locally uniformly bounded on $[T_0, T_1]$, for any $T_1 > T_0$. From

$$\mathcal{G}x(t) = \int_t^\infty\int_s^\infty \Big(\frac{1}{p(r)}\int_{T_0}^r\int_u^\infty q(\omega)\,x(\omega)^\beta\,d\omega du\Big)^{1/\alpha} dr\,ds,$$

we have

$$(\mathcal{G}x)'(t) = -\int_t^\infty \Big(\frac{1}{p(s)}\int_{T_0}^s\int_r^\infty q(u)\,x(u)^\beta\,du\,dr\Big)^{1/\alpha} ds, \quad t \in [T_0, T_1].$$

From the inequality

$$-M^{\beta/\alpha}\int_t^\infty \Big(\frac{1}{p(s)}\int_{T_0}^s\int_r^\infty q(u)\,X(u)^\beta\,du\,dr\Big)^{1/\alpha} ds \le (\mathcal{G}x)'(t) \le 0, \quad t \in [T_0, T_1],$$

holding for all $x \in \mathcal{X}$ it follows that $\mathcal{G}(\mathcal{X})$ is locally equicontinuous on $[T_0, T_1] \subset [T_0, \infty)$. Then, the relative compactness of $\mathcal{G}(\mathcal{X})$ follows from the Arzela-Ascoli lemma.
(iii) $\mathcal{G}$ *is continuous on* $\mathcal{X}$. Let $\{x_n(t)\}$ be a sequence in $\mathcal{X}$ converging to $x(t)$ in $\mathcal{X}$ uniformly on any compact subinterval of $[T_0, \infty)$. Let $T_1 > T_0$ any fixed real number. From (3.51) we have

$$|\mathcal{G}x_n(t) - \mathcal{G}x(t)| \le \int_t^\infty \frac{s-t}{p(s)^{1/\alpha}} G_n(s)\,ds, \quad t \in [T_0, T_1],$$

where

$$G_n(t) = \Big|\Big(\int_{T_0}^t\int_t^\infty q(s)\,x_n(s)^\beta\,ds\Big)^{1/\alpha} - \Big(\int_{T_0}^t\int_s^\infty q(s)\,x(s)^\beta\,ds\Big)^{1/\alpha}\Big|.$$

Using the inequality $|x^\lambda - y^\lambda| \le |x-y|^\lambda$, $x, y \in \mathbb{R}^+$ holding for $\lambda \in (0,1)$, we see that if $\alpha \ge 1$, then

$$G_n(t) \le \Big(\int_t^\infty (s-t)q(s)|x_n(s)^\beta - x(s)^\beta|ds\Big)^{1/\alpha}.$$

On the other hand, using the mean value theorem, if $\alpha < 1$ we obtain

$$G_n(t) \le \frac{1}{\alpha}\Big(M^\beta\int_t^\infty (s-t)q(s)X(s)^\beta ds\Big)^{\frac{\alpha-1}{\alpha}}\int_t^\infty (s-t)q(s)|x_n(s)^\beta - x(s)^\beta|ds.$$

Thus, using that $q(t)\big|x_n(t)^\beta - x(t)^\beta| \to 0$ as $n \to \infty$ at each point $t \in [T_0, \infty)$ and $q(t)\big|x_n(t)^\beta - x(t)^\beta| \le 2M^\beta q(t)X(t)^\beta$ for $t \ge T_0$, while $q(t)X(t)^\beta$ is integrable on $[T_0, \infty)$, the uniform convergence $G_n(t) \to 0$ on $[T_0, \infty)$ follows by the application of the Lebesgue dominated convergence theorem. We conclude that $\mathcal{G}x_n(t) \to \mathcal{G}x(t)$ uniformly on any compact subinterval of $[T_0, \infty)$ as $n \to \infty$, which proves the continuity of $\mathcal{G}$.

Thus, all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled and so there exists a fixed point $x \in \mathcal{X}$ of $\mathcal{G}$, which satisfies integral equation

$$x(t) = \int_t^\infty (s-t)\Big(\frac{1}{p(s)}\int_{T_0}^s\int_r^\infty q(u)\,x(u)^\beta\,du\,dr\Big)^{1/\alpha} ds, \quad t \ge T_0.$$

Differentiating the above expression four times shows that $x(t)$ is a solution of (1.1) on $[T_0,\infty)$, which due to (3.52) is an intermediate solution of type $(I1)$. Therefore, the proof of our main results will be completed with the verification that the intermediate solutions of (1.1) constructed above are actually regularly varying functions with respect to R. We define the function

$$\chi(t) = \int_t^\infty (s-t)\Big(\frac{1}{p(s)}\int_{T_0}^s\int_r^\infty q(u)\,X(u)^\beta\,du\,dr\Big)^{1/\alpha} ds, \quad t \ge T_0,$$

and put

$$l = \liminf_{t\to\infty}\frac{x(t)}{\chi(t)}, \quad L = \limsup_{t\to\infty}\frac{x(t)}{\chi(t)}.$$

By Lemmas 3.5, 3.6 and 3.7 we have $X(t) \sim \chi(t)$, $t\to\infty$. Since, $x \in \mathcal{X}$, it is clear that $0 < l \le L < \infty$. We first consider L. Applying Lemma 3.8 four times, we obtain

$$\begin{aligned}
L &\le \limsup_{t\to\infty}\frac{x'(t)}{\chi'(t)} = \limsup_{t\to\infty}\frac{\int_t^\infty\Big(\frac{1}{p(s)}\int_{T_0}^s\int_r^\infty q(u)\,x(u)^\beta\,du\,dr\Big)^{1/\alpha} ds}{\int_t^\infty\Big(\frac{1}{p(s)}\int_{T_0}^s\int_r^\infty q(u)\,X(u)^\beta\,du\,dr\Big)^{1/\alpha} ds}\\
&\le \limsup_{t\to\infty}\Big(\frac{\int_t^\infty (s-t)q(s)x(s)^\beta\,ds}{\int_t^\infty (s-t)q(s)X(s)^\beta\,ds}\Big)^{1/\alpha} \le \Big(\limsup_{t\to\infty}\frac{\int_t^\infty q(s)x(s)^\beta\,ds}{\int_t^\infty q(s)X(s)^\beta\,ds}\Big)^{1/\alpha}\\
&\le \Big(\limsup_{t\to\infty}\frac{q(t)x(t)^\beta}{q(t)X(t)^\beta}\Big)^{1/\alpha} = \Big(\limsup_{t\to\infty}\frac{x(t)}{X(t)}\Big)^{\beta/\alpha}\\
&= \Big(\limsup_{t\to\infty}\frac{x(t)}{\chi(t)}\Big)^{\beta/\alpha} = L^{\beta/\alpha},
\end{aligned}$$

where we have used $X(t) \sim \chi(t)$, $t\to\infty$, in the last step. Since $\beta/\alpha < 1$, the inequality $L \le L^{\beta/\alpha}$ implies that $L \le 1$. Similarly, repeated application of Lemma 3.8 to l leads to $l \ge 1$, from which it follows that $L = l = 1$, that is,

$$\lim_{t\to\infty}\frac{x(t)}{\chi(t)} = 1 \implies x(t) \sim \chi(t) \sim X(t),\ t\to\infty.$$

Therefore it is concluded that if $p \in \mathrm{RV}_R(\eta)$ and $q \in \mathrm{RV}_R(\sigma)$, then the type-$(I1)$ solution x under consideration is a member of $\mathrm{RV}_R(\rho)$, where

$$\rho = m_1 \quad \text{or} \quad \rho = \frac{\sigma + m_2 - \alpha}{\alpha - \beta} \in (m_1, -1) \quad \text{or} \quad \rho = -1,$$

according to whether the pair (η,σ) satisfies (3.13), (3.15) or (3.18), respectively. Needless to say, any such solution $x \in \mathrm{RV}_R(\rho)$ enjoys one and the same asymptotic behavior (3.14), (3.17) or (3.19), respectively. This completes the "if" parts of Theorems 3.2, 3.3 and 3.4.

3.2. **Regularly varying solutions of type (I2).** Let us turn our attention to the study of intermediate solutions of type (I2) of equation (1.1); that is, those solutions x such that $1 \prec x(t) \prec t$ as $t \to \infty$. As in the preceding section use is made of the expressions (3.2) and (3.12) for the coefficients p, q and solutions x. Since $\psi_1 \in \mathrm{SV}_R$, $\psi_1(t) = 1$ and $\psi_2 \in \mathrm{RV}_R(m_2(\alpha,\eta))$, $\psi_2(t) = t$ (cf. (3.7) and (3.5)), the regularity index ρ of x must satisfy $0 \le \rho \le m_2(\alpha,\eta)$. If $\rho = 0$, then since $x(t) = l_x(t) \to \infty$, $t \to \infty$, x is a member of $\mathrm{ntr}-\mathrm{SV}_R$, while if $\rho = m_2(\alpha,\eta)$, then $x(t)/R(t)^{m_2(\alpha,\eta)} \to 0$, $t \to \infty$, and so x is a member of $\mathrm{ntr}-\mathrm{RV}_R(m_2(\alpha,\eta))$. If $0 < \rho < m_2(\alpha,\eta)$, then x belongs to $\mathrm{RV}_R(\rho)$ and clearly satisfies $x(t) \to \infty$ and $x(t)/R(t)^{m_2(\alpha,\eta)} \to 0$ as $t \to \infty$. Therefore, it is natural to divide the totality of type-($I2$) intermediate solutions of (1.1) into the following three classes

$$\mathrm{ntr}-\mathrm{SV}_R, \quad \mathrm{RV}_R(\rho),\ \rho \in (0, m_2(\alpha,\eta)), \quad \mathrm{ntr}-\mathrm{RV}_R(m_2(\alpha,\eta)).$$

Our purpose is to show that, for each of the above classes, necessary and sufficient conditions for the membership are establish and that the asymptotic behavior at infinity of all members of each class is determined precisely by a unique explicit formula.

Main results.

Theorem 3.9. *Let $p \in \mathrm{RV}_R(\eta), q \in \mathrm{RV}_R(\sigma)$. Then* (1.1) *has intermediate solutions $x \in \mathrm{ntr}-\mathrm{SV}_R$ satisfying* (I2) *if and only if*

$$\sigma = \alpha - m_2(\alpha,\eta) \quad \text{and} \quad \int_a^\infty t\Big(\frac{1}{p(t)}\int_a^t (t-s)\,q(s)\,ds\Big)^{1/\alpha} dt = \infty. \tag{3.53}$$

The asymptotic behavior of any such solution x is governed by the unique formula $x(t) \sim Y_1(t)$, $t \to \infty$, where

$$Y_1(t) = \Big(\frac{\alpha-\beta}{\alpha}\int_a^t s\left(\frac{1}{p(s)}\int_a^s (s-r)q(r)\,dr\right)^{1/\alpha} ds\Big)^{\frac{\alpha}{\alpha-\beta}}. \tag{3.54}$$

Theorem 3.10. *Let $p \in \mathrm{RV}_R(\eta), q \in \mathrm{RV}_R(\sigma)$. Then* (1.1) *has intermediate solutions $x \in \mathrm{RV}_R(\rho)$ with $\rho \in (0,2)$ if and only if*

$$\alpha - m_2(\alpha,\eta) < \sigma < \eta - (\alpha+\beta+2)m_2(\alpha,\eta) \tag{3.55}$$

in which case ρ is given by (3.16) *and the asymptotic behavior of any such solution x is governed by the unique formula $x(t) \sim Y_2(t)$ $t \to \infty$, where*

$$Y_2(t) = \Big(\Big(\frac{m_2(\alpha,\eta)^{\frac{(\alpha+1)^2}{2\alpha+1}}}{\alpha}\Big)^2 \frac{p(t)^{\frac{1}{2\alpha+1}}q(t)R(t)^{-2\frac{\alpha(\alpha+1)}{2\alpha+1}}}{(\rho^\alpha(m_2(\alpha,\eta)-\rho))^\alpha\,(\rho-m_1(\alpha,\eta))\,(\rho+1)}\Big)^{\frac{1}{\alpha-\beta}}. \tag{3.56}$$

Theorem 3.11. *Let $p \in \mathrm{RV}_R(\eta), q \in \mathrm{RV}_R(\sigma)$. Then* (1.1) *has intermediate solutions $x \in \mathrm{ntr}-\mathrm{RV}_R\,(m_2(\alpha,\eta))$ satisfying* (I2) *if and only if*

$$\sigma = \eta - (\alpha+\beta+2)m_2(\alpha,\eta), \qquad \int_a^\infty \Big(\frac{1}{p(t)}\int_a^t (t-s)\,s^\beta\,q(s)\,ds\Big)^{1/\alpha} dt < \infty. \tag{3.57}$$

The asymptotic behavior of any such solution x is governed by the unique formula $x(t) \sim Y_3(t)$, $t \to \infty$, where

$$Y_3(t) = t\Big(\frac{\alpha-\beta}{\alpha}\int_t^\infty \Big(\frac{1}{p(s)}\int_a^s (s-r)r^\beta q(r)\,dr\Big)^{1/\alpha} ds\Big)^{\frac{\alpha}{\alpha-\beta}}. \tag{3.58}$$

Preparatory results. Let x be a type-(I2) intermediate solution of (1.1) defined on $[t_0,\infty)$. It is known that

$$\begin{gathered}\lim_{t\to\infty} x'(t)=0,\\ \lim_{t\to\infty}(p(t)|x''(t)|^{\alpha-1}x''(t))'=\lim_{t\to\infty} p(t)|x''(t)|^{\alpha-1}x''(t)=\lim_{t\to\infty} x(t)=\infty\,.\end{gathered}\tag{3.59}$$

Integrating (1.1) twice on $[t_0,t]$, then on $[t_0,\infty)$ and finally on $[t_0,t]$, we obtain, for $t\ge t_0\ge a$,

$$x(t)=c_0+\int_{t_0}^t\int_s^\infty \frac{1}{p(r)^{1/\alpha}}\Big(c_2+c_3(r-t_0)+\int_{t_0}^r (r-u)q(u)x(u)^\beta\,du\Big)^{1/\alpha}\,dr\,ds,\tag{3.60}$$

where $c_0=x(t_0)$, $c_2=p(t_0)(-x''(t_0))^\alpha$, and $c_3=(p(t_0)(-x''(t_0))^\alpha)'$. From (3.60) we easily see that $x(t)$ satisfies the integral asymptotic relation

$$x(t)\sim\int_b^t\int_s^\infty\Big(\frac{1}{p(r)}\int_b^r (r-u)q(u)x(u)^\beta\,du\Big)^{1/\alpha}\,dr\,ds,\quad t\to\infty,\tag{3.61}$$

for some $b\ge a$, which will play a central role in constructing generalized RV-intermediate solutions of type (I2).

Lemma 3.12. *Suppose that* (3.53) *holds. Then the function* Y_1 *given by* (3.54) *satisfies the asymptotic relation* (3.61) *for any* $b\ge a$ *and belongs to* $\mathrm{ntr}-\mathrm{SV}_R$.

Proof. First we give an expression for $Y_1(t)$ in terms of $R(t)$, $l_p(t)$ and $l_q(t)$. Applying Lemma 3.1 twice we have

$$\begin{aligned}&\int_a^t\int_a^s q(u)\,du\,ds\\ &=\int_a^t\int_a^s R(u)^{\alpha-m_2}l_q(u)\,du\,ds\sim\frac{m_2^{2\frac{\alpha+1}{2\alpha+1}}}{\alpha(\alpha+m_2)}R(t)^{\alpha+m_2}l_p(t)^{\frac{2}{2\alpha+1}}l_q(t),\quad t\to\infty.\end{aligned}$$

Using (3.2), (3.5) and (3.9)-(ii), we have

$$t\Big(\frac{1}{p(t)}\int_a^t (t-s)q(s)\,ds\Big)^{1/\alpha}\sim\frac{m_2^{\frac{2\alpha+2-\alpha^2}{\alpha(2\alpha+1)}}}{(\alpha(\alpha+m_2))^{1/\alpha}}R(t)^{-m_2}l_p(t)^{\frac{1-\alpha}{\alpha(2\alpha+1)}}l_q(t)^{1/\alpha}.\tag{3.62}$$

Integrating the above on $[b,t]$ for any $b\ge a$, we show that

$$Y_1(t)\sim W_1^{\frac{1}{\alpha-\beta}}\Big(\frac{\alpha-\beta}{\alpha}\,Q_1(t)\Big)^{\frac{\alpha}{\alpha-\beta}}\tag{3.63}$$

where

$$\begin{gathered}Q_1(t)=\int_b^t R(s)^{-m_2}l_p(s)^{\frac{1-\alpha}{\alpha(2\alpha+1)}}l_q(s)^{1/\alpha}\,ds\in\mathrm{SV}_R,\\ W_1=\frac{m_2^{\frac{2\alpha+2-\alpha^2}{2\alpha+1}}}{\alpha(\alpha+m_2)}\,.\end{gathered}\tag{3.64}$$

From (3.63) we conclude that $Y_1\in\mathrm{ntr}-\mathrm{SV}_R$.

To verify the asymptotic relation (3.61) for Y_1, we integrate $q(t)Y_1(t)^\beta$ twice on $[b,t]$ and use $Y_1\in\mathrm{ntr}-\mathrm{SV}_R$ to obtain

$$\int_b^t\int_b^s q(r)Y_1(r)^\beta\,dr\,ds\sim\frac{m_2^{2\frac{\alpha+1}{2\alpha+1}}}{(\sigma+m_2)(\sigma+2m_2)}R(t)^{\sigma+2m_2}l_p(t)^{\frac{2}{2\alpha+1}}l_q(t)Y_1(t)^\beta$$

as $t \to \infty$, which together with (3.63), by assumption (3.53) and (3.9)-(ii), yields

$$\begin{aligned} &\Big(\frac{1}{p(t)}\int_b^t (t-s)q(s)Y_1(s)^\beta ds\Big)^{1/\alpha} \\ &\sim \Big(\frac{m_2^{\frac{2\alpha+2-\alpha\beta}{2\alpha+1}}}{\alpha(\alpha+m_2)}\Big)^{\frac{1}{\alpha-\beta}} R(t)^{-2m_2} l_p(t)^{\frac{1-2\alpha}{\alpha(2\alpha+1)}} l_q(t)^{1/\alpha}\Big(\frac{\alpha-\beta}{\alpha}\,Q_1(t)\Big)^{\frac{\beta}{\alpha-\beta}}, \end{aligned} \tag{3.65}$$

as $t \to \infty$. Integration of (3.65) on $[t,\infty)$ gives

$$\begin{aligned} &\int_t^\infty \Big(\frac{1}{p(r)}\int_b^r (r-u)q(u)Y_1(u)^\beta\,du\Big)^{1/\alpha} dr \\ &\sim \Big(\frac{m_2^{\frac{2\alpha+2-\alpha\beta}{2\alpha+1}}}{\alpha(\alpha+m_2)}\Big)^{\frac{1}{\alpha-\beta}} \Big(\frac{\alpha-\beta}{\alpha}\Big)^{\frac{\beta}{\alpha-\beta}} m_2^{-\frac{\alpha}{2\alpha+1}} R(t)^{-m_2} l_p(t)^{\frac{1-\alpha}{\alpha(2\alpha+1)}} l_q(t)^{1/\alpha} Q_1(t)^{\frac{\beta}{\alpha-\beta}}, \end{aligned}$$

as $t \to \infty$, implying, by integration on $[b,t]$,

$$\begin{aligned} &\int_b^t\int_s^\infty \Big(\frac{1}{p(r)}\int_b^r (r-u)q(u)Y_1(u)^\beta\,du\Big)^{1/\alpha} dr\,ds \\ &\sim W_1^{\frac{1}{\alpha-\beta}}\Big(\frac{\alpha-\beta}{\alpha}\Big)^{\frac{\beta}{\alpha-\beta}} \int_b^t R(s)^{-m_2} l_p(s)^{\frac{1-\alpha}{\alpha(2\alpha+1)}} l_q(s)^{1/\alpha} Q_1(s)^{\frac{\beta}{\alpha-\beta}}\,ds \\ &\sim W_1^{\frac{1}{\alpha-\beta}}\Big(\frac{\alpha-\beta}{\alpha}\Big)^{\frac{\beta}{\alpha-\beta}} \int_b^t Q_1(s)^{\frac{\beta}{\alpha-\beta}}\,dQ_1(s) \\ &= W_1^{\frac{1}{\alpha-\beta}}\Big(\frac{\alpha-\beta}{\alpha}\Big)^{\frac{\alpha}{\alpha-\beta}} Q_1(t)^{\frac{\alpha}{\alpha-\beta}}, \quad t\to\infty, \end{aligned}$$

establishing, in view of (3.63), that Y_1 satisfies the asymptotic relation (3.61). □

Lemma 3.13. *Suppose that* (3.55) *holds and let* ρ *be defined by* (3.16)*. Then the function* Y_2 *given by* (3.56) *satisfies the asymptotic relation* (3.61) *for any* $b \ge a$ *and belongs to* $\mathrm{RV}_R(\rho)$.

Proof. Using (3.2) and (3.8), since $\frac{\eta-2\alpha(\alpha+1)}{2\alpha+1} = m_2-\alpha$, we can express $Y_2(t)$ in the form

$$Y_2(t) \sim W_2 R(t)^\rho \,\Big(l_p(t)^{\frac{1}{2\alpha+1}} l_q(t)\Big)^{\frac{1}{\alpha-\beta}}, \tag{3.66}$$

where

$$C = m_2^{\frac{(\alpha+1)^2}{2\alpha+1}}, \quad \nu = \big(\rho(m_2-\rho)\big)^\alpha(\rho-m_1)(\rho+1), \quad W_2 = \Big(\frac{C^2}{\alpha^2\nu}\Big)^{\frac{1}{\alpha-\beta}}. \tag{3.67}$$

Therefore, $Y_2 \in \mathrm{RV}_R(\rho)$. Next we prove that Y_2 satisfies the asymptotic relation (3.61) and to that end we first integrate $q(t)Y_2(t)^\beta$ twice on $[b,t]$ for some $b \ge a$ with application of Lemma 3.1 and equalities (3.9), (3.26):

$$\begin{aligned} &\int_b^t\int_b^s q(r)Y_2(r)^\beta\,dr\,ds \\ &\sim W_2^\beta \int_b^t\int_b^s R(r)^{\sigma+\rho\beta}\Big(l_p(t)^{\frac{\beta}{2\alpha+1}} l_q(t)^\alpha\Big)^{\frac{1}{\alpha-\beta}}\,drds \\ &\sim \frac{W_2^\beta}{(\sigma+\rho\beta+m_2)(\sigma+\rho\beta+2m_2)} m_2^{2\frac{\alpha+1}{2\alpha+1}} R(t)^{\sigma+\rho\beta+2m_2}\Big(l_p(t)^{\frac{2\alpha-\beta}{2\alpha+1}} l_q(t)^\alpha\Big)^{\frac{1}{\alpha-\beta}} \\ &= \frac{W_2^\beta}{\alpha^2(\rho+1)(\rho-m_1)} m_2^{2\frac{\alpha+1}{2\alpha+1}} R(t)^{\alpha(\rho-m_1)}\,\Big(l_p(t)^{\frac{2\alpha-\beta}{2\alpha+1}} l_q(t)^\alpha\Big)^{\frac{1}{\alpha-\beta}} \end{aligned}$$

$$= \frac{W_2^\beta}{\alpha^2(\rho+1)(\rho-m_1)} m_2^{2\frac{\alpha+1}{2\alpha+1}} R(t)^{\alpha(\rho-2m_2-\frac{\eta}{\alpha})} \Big(l_p(t)^{\frac{2\alpha-\beta}{2\alpha+1}} l_q(t)^\alpha\Big)^{\frac{1}{\alpha-\beta}}, \quad t\to\infty,$$

implying further that

$$\begin{aligned}
&\int_b^t \int_s^\infty \Big(\frac{1}{p(r)}\int_b^r (r-u)q(u)Y_2(u)^\beta\,du\Big)^{1/\alpha} dr\,ds \\
&\sim \Big(\frac{W_2^\beta}{\alpha^2(\rho+1)(\rho-m_1)} m_2^{2\frac{\alpha+1}{2\alpha+1}}\Big)^{1/\alpha} \int_b^t \int_s^\infty R(r)^{\rho-2m_2}\Big(l_p(r)^{\frac{2\beta-2\alpha+1}{2\alpha+1}} l_q(r)\Big)^{\frac{1}{\alpha-\beta}} dr\,ds \\
&\sim \frac{W_2^{\beta/\alpha} m_2^{2\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}}{(\alpha^2(\rho+1)(\rho-m_1))^{1/\alpha}(m_2-\rho)\rho} R(t)^\rho \Big(l_p(t)^{\frac{1}{2\alpha+1}} l_q(t)\Big)^{\frac{1}{\alpha(\alpha-\beta)}} \\
&= W_2^{\beta/\alpha}\Big(\frac{C^2}{\nu\alpha^2}\Big)^{1/\alpha} R(t)^\rho \Big(l_p(t)^{\frac{1}{2\alpha+1}} l_q(t)\Big)^{\frac{1}{\alpha(\alpha-\beta)}}, \quad t\to\infty,
\end{aligned}$$

which by (3.66) and (3.67) proves that Y_2 satisfies the asymptotic relation (3.61). □

Lemma 3.14. *Suppose that* (3.57) *holds. Then the function* Y_3 *given by* (3.58) *satisfies the asymptotic relation* (3.61) *for any* $b\ge a$ *and belongs to* $\mathrm{RV}_R(1)$.

Proof. Using (3.5) and (3.57), application of Lemma 3.1 we have

$$\begin{aligned}
\int_b^t\int_b^s r^\beta q(r)\,dr\,ds &\sim m_2^{-\frac{\alpha\beta}{2\alpha+1}} \int_b^t\int_b^s R(r)^{\eta-(\alpha+2)m_2} l_p(s)^{\frac{\beta}{2\alpha+1}} l_q(s) ds \\
&\sim \frac{m_2^{\frac{2(\alpha+1)-\alpha\beta}{2\alpha+1}}}{(\eta-(\alpha+1)m_2)(\eta-\alpha m_2)} R(t)^{\eta-\alpha m_2} l_p(t)^{\frac{\beta+2}{2\alpha+1}} l_q(t),
\end{aligned}$$

as $t\to\infty$. Since by (3.9)-(ii) we have that

$$\eta-(\alpha+1)m_2 = \alpha(m_2+1), \tag{3.68}$$

from the last relation, we conclude that

$$\begin{aligned}
&\int_t^\infty \Big(\frac{1}{p(s)}\int_b^s (s-r)r^\beta q(r)dr\Big)^{1/\alpha} ds \\
&\sim \Big(\frac{m_2^{\frac{2(\alpha+1)-\alpha\beta}{2\alpha+1}}}{\alpha(m_2+1)(\alpha+\alpha m_2+m_2)}\Big)^{1/\alpha} \int_t^\infty R(s)^{-m_2} l_p(s)^{\frac{\beta-2\alpha+1}{\alpha(2\alpha+1)}} l_q(s)^{1/\alpha} ds,
\end{aligned} \tag{3.69}$$

as $t\to\infty$. We denote by

$$Q_3(t) = \int_t^\infty R(s)^{-m_2} l_p(s)^{\frac{\beta-2\alpha+1}{\alpha(2\alpha+1)}} l_q(s)^{1/\alpha} ds \in \mathrm{SV}_R \tag{3.70}$$

and combining (3.69) with (3.58) and (3.5), we obtain the following asymptotic representation for $Y_3(t)$ in terms of $R(t)$, $l_p(t)$ and $l_q(t)$:

$$Y_3(t) \sim W_3^{\frac{1}{\alpha-\beta}} R(t)^{m_2} l_p(t)^{\frac{1}{2\alpha+1}} \Big(\frac{\alpha-\beta}{\alpha} Q_3(t)\Big)^{\frac{\alpha}{\alpha-\beta}}, \quad t\to\infty, \tag{3.71}$$

where

$$W_3 = \frac{m_2^{\frac{-\alpha^2+2\alpha+2}{2\alpha+1}}}{\alpha(m_2+1)(\alpha m_2+m_2+\alpha)}. \tag{3.72}$$

From (3.71) we conclude that $Y_3 \in \mathrm{RV}_R(m_2)$ and compute with the help of Lemma 3.1,

$$\begin{aligned} &\int_b^t \int_b^s q(r) Y_3(r)^\beta \, dr \, ds \\ &\sim \Big(\frac{\alpha-\beta}{\alpha} Q_3(t)\Big)^{\frac{\alpha\beta}{\alpha-\beta}} W_3^{\frac{\beta}{\alpha-\beta}} \frac{m_2^{\frac{2(\alpha+1)}{2\alpha+1}} R(t)^{\sigma+m_2\beta+2m_2}}{(\sigma+m_2\beta+2m_2)(\sigma+m_2\beta+m_2)} l_p(t)^{\frac{\beta+2}{2\alpha+1}} l_q(t), \end{aligned}$$

as $t \to \infty$. Next, using (3.57) and (3.68) we obtain

$$\begin{aligned} &\int_t^\infty \Big(\frac{1}{p(s)} \int_b^s (s-r) q(r) Y_3(r)^\beta \, dr\Big)^{1/\alpha} ds \\ &\sim \Big(\frac{\alpha-\beta}{\alpha}\Big)^{\frac{\beta}{\alpha-\beta}} \frac{W_3^{\frac{\beta}{\alpha(\alpha-\beta)}} m_2^{\frac{2(\alpha+1)}{\alpha(2\alpha+1)}}}{(\alpha(m_2+1)(\alpha m_2+m_2+\alpha))^{1/\alpha}} \\ &\quad \times \int_t^\infty R(s)^{-m_2} l_p(s)^{\frac{\beta-2\alpha+1}{\alpha(2\alpha+1)}} l_q(s)^{1/\alpha} Q_3(s)^{\frac{\beta}{\alpha-\beta}} ds \\ &\sim \Big(\frac{\alpha-\beta}{\alpha}\Big)^{\frac{\beta}{\alpha-\beta}} \frac{W_3^{\frac{\beta}{\alpha(\alpha-\beta)}} m_2^{\frac{2(\alpha+1)}{\alpha(2\alpha+1)}}}{(\alpha\,(m_2+1)(\alpha m_2+m_2+\alpha))^{1/\alpha}} \int_t^\infty Q_3(s)^{\frac{\beta}{\alpha-\beta}} d(-Q_3(s)) \\ &= \Big(\frac{\alpha-\beta}{\alpha} Q_3(t)\Big)^{\frac{\alpha}{\alpha-\beta}} \frac{W_3^{\frac{\beta}{\alpha(\alpha-\beta)}} m_2^{\frac{2(\alpha+1)}{\alpha(2\alpha+1)}}}{(\alpha\,(m_2+1)(\alpha m_2+m_2+\alpha))^{1/\alpha}}, \quad t \to \infty. \end{aligned}$$

Noting that the last expression in the previous relation is slowly varying with respect to R, integration of this relation over $[b,t]$ leads to

$$\begin{aligned} &\int_b^t \int_s^\infty \Big(\frac{1}{p(r)} \int_b^r (r-u) q(u) Y_3(u)^\beta \, du\Big)^{1/\alpha} dr \, ds \\ &\sim \Big(\frac{\alpha-\beta}{\alpha} Q_3(t)\Big)^{\frac{\alpha}{\alpha-\beta}} \frac{W_3^{\frac{\beta}{\alpha(\alpha-\beta)}} m_2^{\frac{(\alpha+2)(\alpha+1)}{\alpha(2\alpha+1)}}}{(\alpha(m_2+1)(\alpha m_2+m_2+\alpha))^{1/\alpha}} \frac{R(t)^{m_2}}{m_2} l_p(t)^{\frac{1}{2\alpha+1}} \\ &= \Big(\frac{\alpha-\beta}{\alpha} Q_3(t)\Big)^{\frac{\alpha}{\alpha-\beta}} W_3^{\frac{\beta}{\alpha(\alpha-\beta)}} W_3^{1/\alpha} R(t)^{m_2} l_p(t)^{\frac{1}{2\alpha+1}}, \quad t \to \infty, \end{aligned}$$

and in view of (3.71) proves that the desired integral asymptotic relation (3.61) is satisfied by Y_3. □

Proof of main results.

Proof of the "only if" part of Theorems 3.9, 3.10 and 3.11. Suppose that (1.1) has a type-(I2) intermediate solution $x \in \mathrm{RV}_R(\rho), \rho \in [0, m_2]$, defined on $[t_0, \infty)$. We begin by integrating (1.1) on $[t_0, t]$. Using (3.2), (3.12), we have

$$(p(t)(-x''(t))^\alpha)' \sim \int_{t_0}^t q(s) x(s)^\beta ds = \int_{t_0}^t R(s)^{\sigma+\beta\rho} l_q(s) l_x(s)^\beta \, ds, \tag{3.73}$$

and conclude by (3.59) that $\sigma + \beta\rho + m_2 \geq 0$. Thus, we distinguish the two cases: (1) $\sigma + \beta\rho + m_2 = 0$ and (2) $\sigma + \beta\rho + m_2 > 0$.

Let case (1) hold, so that

$$H_4(t) = \int_{t_0}^t R(s)^{\sigma+\beta\rho} l_q(s) l_x(s)^\beta \, ds = \int_{t_0}^t R(s)^{-m_2} l_q(s) l_x(s)^\beta \, ds, \tag{3.74}$$

and $H_4 \in \mathrm{SV}_R$. Integration of (3.73) on $[t_0, t]$ with (3.9)-(ii) yields

$$\begin{aligned} -x''(t) &\sim m_2^{-\frac{1}{2\alpha+1}} R(t)^{\frac{m_2-\eta}{\alpha}} l_p(t)^{-\frac{2}{2\alpha+1}} H_4(t)^{\frac{1}{\alpha}} \\ &= m_2^{-\frac{1}{2\alpha+1}} R(t)^{-2m_2-1} l_p(t)^{-\frac{2}{2\alpha+1}} H_4(t)^{1/\alpha}, \quad t \to \infty, \end{aligned}$$

Since $-m_2 - 1 < 0$ we may integrate previous relation on $[t, \infty)$ and obtain via Lemma 3.1 that

$$x'(t) \sim \frac{m_2^{\frac{\alpha}{2\alpha+1}}}{m_2+1} R(t)^{-m_2-1} H_4(t)^{1/\alpha}, \quad t \to \infty.$$

The right hand side in the last relation is integrable on $[t, \infty)$, because $-m_2 - 1 < -m_2$, but on the other hand in view of (3.59) the left hand side of last relation isn't integrable on $[t, \infty)$, so we conclude that this case is impossible.

Let case (2) hold. Then, from (3.73) it follows that

$$(p(t)(-x''(t))^\alpha)' \sim \frac{m_2^{\frac{\alpha+1}{2\alpha+1}}}{\sigma+\beta\rho+m_2} R(t)^{\sigma+\beta\rho+m_2} l_p(t)^{\frac{1}{2\alpha+1}} l_q(t) l_x(t)^\beta$$

which, integrated on $[t_0, t]$ and the fact that $\sigma + \beta\rho + 2m_2 > 0$, gives

$$\begin{aligned} -x''(t) \sim \Big(&\frac{m_2^{\frac{2(\alpha+1)}{2\alpha+1}}}{(\sigma+\beta\rho+m_2)(\sigma+\beta\rho+2m_2)}\Big)^{1/\alpha} \\ &\times R(t)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}} l_p(t)^{\frac{1-2\alpha}{\alpha(2\alpha+1)}} l_q(t)^{1/\alpha} l_x(t)^{\beta/\alpha}, \end{aligned}$$

as $t \to \infty$, implying in view of (3.59) by integration on $[t, \infty)$,

$$\begin{aligned} x'(t) \sim \Big(&\frac{m_2^{\frac{2(\alpha+1)}{2\alpha+1}}}{(\sigma+\beta\rho+m_2)(\sigma+\beta\rho+2m_2)}\Big)^{1/\alpha} \\ &\times \int_t^\infty R(s)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}} \Big(l_p(s)^{\frac{1-2\alpha}{2\alpha+1}} l_q(s) l_x(s)^\beta\Big)^{1/\alpha} ds. \end{aligned} \tag{3.75}$$

Thus, we further consider the following two possible cases:
(2.a) $\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha} + m_2 = 0$ and (2.b) $\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha} + m_2 < 0$.

Suppose that (2.a) holds, and let

$$H_3(t) = \int_t^\infty R(s)^{-m_2} l_p(s)^{\frac{1-2\alpha}{\alpha(2\alpha+1)}} l_q(s)^{1/\alpha} l_x(s)^{\beta/\alpha} ds. \tag{3.76}$$

Using (3.68) and (3.9)-(ii), since we have $\sigma + \rho\beta + m_2 = \alpha(m_2 + 1)$, integration of (3.75) on $[t_0, t]$ implies

$$x(t) \sim \Big(\frac{m_2^{\frac{-\alpha^2+2(\alpha+1)}{2\alpha+1}}}{\alpha(m_2+1)(\alpha(m_2+1)+m_2)}\Big)^{1/\alpha} R(t)^{m_2} l_p(t)^{\frac{1}{2\alpha+1}} H_3(t), \quad t \to \infty. \tag{3.77}$$

Since $H_3 \in \mathrm{SV}_R$, we conclude that $x \in \mathrm{RV}_R(m_2)$.

Suppose that (2.b) holds. Application of Lemma 3.1 in (3.75) implies

$$\begin{aligned} x'(t) \sim -&\frac{m_2^{\frac{(\alpha+1)(\alpha+2)}{\alpha(2\alpha+1)}}}{((\sigma+\beta\rho+m_2)(\sigma+\beta\rho+2m_2))^{1/\alpha}(\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2)} \\ &\times R(t)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2} l_p(t)^{\frac{1-\alpha}{\alpha(2\alpha+1)}} l_q(t)^{1/\alpha} l_x(t)^{\beta/\alpha}, \quad t \to \infty. \end{aligned} \tag{3.78}$$

Integrating (3.78) on $[t_0, t]$ using (3.59) we obtain

$$\begin{aligned} x(t) \sim & \frac{m_2^{\frac{(\alpha+1)(\alpha+2)}{\alpha(2\alpha+1)}}}{((\sigma+\beta\rho+m_2)(\sigma+\beta\rho+2m_2))^{1/\alpha}(-(\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2))} \\ & \times \int_{t_0}^{t} R(s)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2} l_p(s)^{\frac{1-\alpha}{\alpha(2\alpha+1)}} l_q(s)^{1/\alpha} l_x(s)^{\beta/\alpha} ds, \quad t\to\infty. \end{aligned} \tag{3.79}$$

Thus, since $x(t)\to\infty$ as $t\to\infty$, from the previous relation we conclude that two possibilities may hold:
(2.b.1) $\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+2m_2=0$ and (2.b.2) $\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+2m_2>0$.

In the case (2.b.1), using (3.9)-(ii) we obtain $\sigma+\beta\rho+m_2=\alpha$. Application of Lemma 3.1 in (3.79) leads us to

$$x(t) \sim \Big(\frac{m_2^{\frac{-\alpha^2+2(\alpha+1)}{2\alpha+1}}}{\alpha(\alpha+m_2)}\Big)^{1/\alpha} H_1(t), \quad t\to\infty, \tag{3.80}$$

where

$$H_1(t) = \int_{t_0}^{t} R(s)^{-m_2} l_p(s)^{\frac{1-\alpha}{\alpha(2\alpha+1)}} l_q(s)^{1/\alpha} l_x(s)^{\beta/\alpha} ds, \quad H_1\in \mathrm{SV}_R\,. \tag{3.81}$$

Thus, $x\in\mathrm{SV}_R$.

Application of Lemma 3.1 in (3.79) in the case (2.b.2) gives

$$\begin{aligned} x(t) \sim & \Big(m_2^{\frac{2(\alpha+1)^2}{\alpha(2\alpha+1)}}\Big) \div \Big(((\sigma+\beta\rho+m_2)(\sigma+\beta\rho+2m_2))^{1/\alpha} \\ & \times \Big(-(\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+m_2)\Big)\Big(\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+2m_2\Big)\Big) \\ & \times R(t)^{\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+2m_2} l_p(t)^{\frac{1}{\alpha(2\alpha+1)}} l_q(t)^{1/\alpha} l_x(t)^{\beta/\alpha} ds, \quad t\to\infty. \end{aligned} \tag{3.82}$$

This implies that $x\in\mathrm{RV}\big(\frac{\sigma+\beta\rho+2m_2-\eta}{\alpha}+2m_2\big)$.

Now, let x be a type-$(I2)$ intermediate solution of (1.1) belonging to $\mathrm{ntr}-\mathrm{SV}_R$. Then, from the above observations it is clear that only the case (2.b.1) is admissible, in which case $\sigma=\alpha-m_2$, and (3.80) is satisfied by $x(t)$. Using $x(t)=l_x(t)$, from (3.80) we have

$$l_x(t) \sim W_1^{1/\alpha} H_1(t), \quad t\to\infty, \tag{3.83}$$

where W_1 is given by (3.64) and H_1 is defined by (3.81). Then, (3.83) is transformed into the following differential asymptotic relation for H_1,

$$H_1(t)^{-\frac{\beta}{\alpha}}\, H_1'(t) \sim W_1^{\beta/\alpha} R(t)^{-m_2} l_p(t)^{\frac{1-\alpha}{\alpha(2\alpha+1)}}\, l_q(t)^{1/\alpha}, \quad t\to\infty. \tag{3.84}$$

From (3.59), since $\lim_{t\to\infty} x(t)=\infty$, we have $\lim_{t\to\infty} H_1(t)=\infty$. Integrating (3.84) on $[t_0,t]$, using that $\lim_{t\to\infty} H_1(t)^{\frac{\alpha-\beta}{\alpha}}=\infty$, in view of notation (3.64) and (3.62), we find that the second condition in (3.53) is satisfied and

$$H_1(t) \sim \Big(W_1^{\beta/\alpha}\frac{\alpha-\beta}{\alpha} Q_1(t)\Big)^{\frac{\alpha}{\alpha-\beta}}, \quad t\to\infty$$

which with (3.83) implies

$$x(t) \sim W_1^{\frac{1}{\alpha-\beta}}\Big(\frac{\alpha-\beta}{\alpha} Q_1(t)\Big)^{\frac{\alpha}{\alpha-\beta}}, \quad t\to\infty. \tag{3.85}$$

Note that in Lemma 3.12 we have obtained expression (3.63) for $Y_1(t)$ given by (3.54). Therefore, (3.85) in fact proves that $x(t) \sim Y_1(t)$, $t \to \infty$, completing the "only if" part of Theorem 3.9.

Next, let x be a type-(I2) intermediate solution of (1.1) belonging to $\mathrm{RV}_R(\rho)$ for some $\rho \in (0, m_2)$. Clearly, only case $(2.b.2)$ can hold and hence x satisfies the asymptotic relation (3.82). This means that

$$\rho = \frac{\sigma + \beta\rho + 2m_2 - \eta}{\alpha} + 2m_2 \;\Leftrightarrow\; \rho = \frac{\sigma + m_2 - \alpha}{\alpha - \beta}, \tag{3.86}$$

verifying that the regularity index ρ is given by (3.16). An elementary computation shows that

$$0 < \rho < m_2 \;\Rightarrow\; \alpha - m_2 < \sigma < \alpha + m_2(\alpha - \beta - 1),$$

showing that the range of σ is given by (3.55). In view of (3.26) and (3.86), we conclude from (3.82) that x enjoys the asymptotic behavior $x(t) \sim Y_2(t)$, $t \to \infty$, where Y_2 is given by (3.56). This proves the "only if" part of the Theorem 3.10.

Finally, let x is a type-$(I2)$ intermediate solution of (1.1) belonging to $\mathrm{RV}_R(m_2)$. Since only the case (2.a) is possible for x, it satisfies (3.77), where H_3 is defined by (3.76), implying $\rho = m_2$ and $\sigma = \alpha + m_2(\alpha - \beta - 1)$. Using $x(t) = R(t)^{m_2} l_x(t)$, (3.77) can be expressed as

$$l_x(t) \sim W_3^{1/\alpha} l_p(t)^{\frac{1}{2\alpha+1}} H_3(t),\; t \to \infty, \tag{3.87}$$

where W_3 is defined by (3.72), implying the differential asymptotic relation

$$- H_3(t)^{-\frac{\beta}{\alpha}} H_3'(t) \sim W_3^{\frac{\beta}{\alpha^2}} R(t)^{-m_2} l_p(t)^{\frac{\beta+1-2\alpha}{\alpha(2\alpha+1)}} l_q(t)^{1/\alpha}, \quad t \to \infty. \tag{3.88}$$

From (3.77), since $\lim_{t\to\infty} R(t)^{-m_2} x(t) = 0$, we have that $\lim_{t\to\infty} H_3(t) = 0$, implying that the left-hand side od (3.88) is integrable over $[t, \infty)$. This, in view of (3.69) and notation (3.70) implies the second condition in (3.57). Integrating (3.88) on $[t, \infty)$ and combining result with (3.87), using the expression (3.71), we find that

$$x(t) \sim W_3^{\frac{1}{\alpha-\beta}} R(t)^{m_2} l_p(t)^{\frac{1}{2\alpha+1}} \Big(\frac{\alpha}{\alpha-\beta} Q_3(t)\Big)^{\frac{\alpha}{\alpha-\beta}} \sim Y_3(t), \quad t \to \infty,$$

where Q_3 is defined with (3.70). Thus the "only if" part of the Theorem 3.11 has been proved.

Proof of the "if" part of Theorem 3.9, 3.10 and 3.11. Suppose that (3.53) or (3.55) or (3.57) holds. From Lemmas 3.12, 3.13 and 3.14 it is known that Y_i, $i = 1, 2, 3$, defined by (3.54), (3.56) and (3.58) satisfy the asymptotic relation (3.61). We perform the simultaneous proof for Y_i, $i = 1, 2, 3$ so the subscripts $i = 1, 2, 3$ will be deleted in the rest of the proof. By (3.61) there exists $T_0 > a$ such that

$$\int_{T_0}^t \int_s^\infty \Big(\frac{1}{p(r)} \int_{T_0}^r (r - u) q(u) Y(u)^\beta \, du\Big)^{1/\alpha} dr\, ds \le 2Y(t),\; t \ge T_0.$$

Let such a T_0 be fixed. We may assume that Y is increasing on $[T_0, \infty)$. Since (3.61) holds with $b = T_0$, there exists $T_1 > T_0$ such that

$$\int_{T_0}^t \int_s^\infty \Big(\frac{1}{p(r)} \int_{T_0}^r (r - u) q(u) Y(u)^\beta \, du\Big)^{1/\alpha} dr\, ds \ge \frac{Y(t)}{2}, \quad t \ge T_1.$$

Choose positive constants k and K such that

$$k^{1-\frac{\beta}{\alpha}} \le \frac{1}{2}, \quad K^{1-\frac{\beta}{\alpha}} \ge 4, \quad 2kY(T_1) \le KY(T_0).$$

Considering the integral operator

$$\mathcal{H}y(t) = y_0 + \int_{T_0}^{t}\int_{s}^{\infty} \Big(\frac{1}{p(r)}\int_{T_0}^{r}(r-u)q(u)\,y(u)^{\beta}\,du\Big)^{1/\alpha}\,dr\,ds, \quad t \ge T_0,$$

where y_0 is a constant such that $kY(T_1) \le y_0 \le \frac{K}{2}Y(T_0)$, we may verify that $\mathcal{H}$ is continuous self-map on the set

$$\mathcal{Y} = \{y \in C[T_0,\infty) : kY(t) \le y(t) \le KY(t),\ t \ge T_0\},$$

and that $\mathcal{H}$ sends $\mathcal{Y}$ into relatively compact subset of $C[T_0,\infty)$. Thus, $\mathcal{H}$ has a fixed point $y \in \mathcal{Y}$, which generates a solution of equation (1.1) of type $(I2)$ satisfying the above inequalities and thus yields that

$$0 < \liminf_{t\to\infty}\frac{y(t)}{Y(t)} \le \limsup_{t\to\infty}\frac{y(t)}{Y(t)} < \infty.$$

Denoting

$$L(t) = \int_{a}^{t}\int_{s}^{\infty}\Big(\frac{1}{p(r)}\int_{a}^{r}(r-u)q(u)Y(u)^{\beta}\,du\Big)^{1/\alpha}\,dr\,ds$$

and using $Y(t) \sim L(t)$ as $t \to \infty$ we obtain

$$0 < \liminf_{t\to\infty}\frac{y(t)}{L(t)} \le \limsup_{t\to\infty}\frac{y(t)}{L(t)} < \infty.$$

Then, proceeding exactly as in the proof of the "if" part of Theorems 3.2–3.4, with application of Lemma 3.8, we conclude that $y(t) \sim L(t) \sim Y(t)$, $t \to \infty$. Therefore, y is a generalized regularly varying solution with respect to R with requested regularity index and the asymptotic behavior (3.54), (3.56), (3.58) depending on if $q \in \mathrm{RV}_R(\sigma)$ satisfies, respectively, (3.53) or (3.55) or (3.57). Thus, the "if part" of Theorems 3.9, 3.10 and 3.11 has been proved.

4. Corollaries

The final section is concerned with equation (1.1) whose coefficients $p(t)$ and $q(t)$ are regularly varying functions (in the sense of Karamata). It is natural to expect that such equation may possess. Our purpose here is to show that the problem of getting necessary and sufficient conditions for the existence of intermediate solutions which are regularly varying in the sense of Karamata, can be embedded in the framework of generalized regularly varying functions, so that the results of the preceding section provide full information about the existence and the precise asymptotic behavior of intermediate regularly varying solutions of (1.1).

We assume that $p(t)$ and $q(t)$ are regularly varying functions of indices η and σ, respectively, i.e.,

$$p(t) = t^{\eta}l_p(t), \quad q(t) = t^{\sigma}l_q(t), \quad l_p, l_q \in \mathrm{SV}, \tag{4.1}$$

and seek regularly varying solutions $x(t)$ of (1.1) expressed in the from

$$x(t) = t^{\rho}l_x(t), \quad l_x \in \mathrm{SV}. \tag{4.2}$$

We begin by noticing that in order that the condition (1.2) be satisfied we have to assume that $\eta \geq 1+2\alpha$. Since $R(t)$ defined by (3.1) due to (4.1) takes the form

$$R(t) = \Big(\int_t^\infty s^{1+\frac{1}{\alpha}-\frac{\eta}{\alpha}} l_p(s)^{-1/\alpha} ds \Big)^{-1},$$

it is easy to see that

$$R \in \mathrm{RV}\Big(\frac{\eta - 1 - 2\alpha}{\alpha}\Big). \tag{4.3}$$

An important remark is that the possibility $\eta = 2\alpha + 1$ should be excluded. If this equality holds, then $R(t)$ is slowly varying by (4.3), and this fact prevents $p(t)$ from being a generalized regularly varying function with respect to R. In fact, if $p \in \mathrm{RV}_R(\eta^*)$ for some η^*, then there exists $f \in \mathrm{RV}(\eta^*)$ such that $p(t) = f(R(t))$, which implies that $p \in \mathrm{SV}$. But this contradicts the hypothesis that $p \in \mathrm{RV}(\eta) = \mathrm{RV}(2\alpha+1)$. Thus, the case $\eta = 2\alpha+1$ is impossible, and so η must be restricted to

$$\eta > 1 + 2\alpha, \tag{4.4}$$

in which case R satisfies

$$R(t) \sim \frac{\eta - 2\alpha - 1}{\alpha} t^{\frac{\eta-2\alpha-1}{\alpha}} l_p(t)^{1/\alpha}, \quad t \to \infty, \tag{4.5}$$

implying that $R \in \mathrm{RV}\left(\frac{\eta-2\alpha-1}{\alpha}\right)$. Since R is monotone increasing, its inverse function $R^{-1}(t)$ is a regularly varying of index $\alpha/(\eta-2\alpha-1)$. Therefore, any regularly varying function of index λ is considered as a generalized regularly varying function with respect to R which regularity index is $\alpha\lambda/(\eta-2\alpha-1)$, and conversely any generalized regularly varying function with respect to R of index λ^* is regarded as a regularly varying function in the sense of Karamata of index $\lambda = \lambda^*(\eta-2\alpha-1)/\alpha$. It follows form (4.1) and (4.2) that

$$p \in \mathrm{RV}_R\Big(\frac{\alpha\,\eta}{\eta-2\alpha-1}\Big), \quad q \in \mathrm{RV}_R\Big(\frac{\alpha\,\sigma}{\eta-2\alpha-1}\Big), \quad x \in \mathrm{RV}_R\Big(\frac{\alpha\,\rho}{\eta-2\alpha-1}\Big).$$

Put

$$\eta^* = \frac{\alpha\eta}{\eta-2\alpha-1}, \quad \sigma^* = \frac{\alpha\sigma}{\eta-2\alpha-1}, \quad \rho^* = \frac{\alpha\rho}{\eta-2\alpha-1}.$$

Note that (4.4) implies $\eta > \alpha$ because $\alpha > 0$ and that the two constants given by (3.8) are reduced to

$$m_1(\alpha,\eta^*) = \frac{2\alpha-\eta}{\eta-2\alpha-1}, \quad m_2(\alpha,\eta^*) = \frac{\alpha}{\eta-2\alpha-1}.$$

It turns out therefore that any type-(I1) intermediate regularly varying solution of (1.1) is a member of one of the three classes

$$\mathrm{ntr}-\mathrm{RV}\Big(\frac{2\alpha-\eta}{\alpha}\Big), \quad \mathrm{RV}(\rho),\ \rho \in \Big(\frac{2\alpha-\eta}{\alpha}, \frac{1+2\alpha-\eta}{\alpha}\Big), \quad \mathrm{ntr}-\mathrm{RV}\Big(\frac{1+2\alpha-\eta}{\alpha}\Big),$$

while any type-(I2) intermediate regularly varying solution belongs to one of the three classes

$$\mathrm{ntr}-\mathrm{SV}, \quad \mathrm{RV}(\rho),\ \rho \in (0,1), \quad \mathrm{ntr}-\mathrm{RV}(1).$$

Based on the above observations we are able to apply our main results in Section 3, establishing necessary and sufficient conditions for the existence of intermediate regularly varying solutions of (1.1) and determining the asymptotic behavior of all such solutions explicitly.

First, we state the results on type-(I1) intermediate solutions that can be derived as corollaries of Theorems 3.2, 3.3 and 3.4.

Theorem 4.1. *Assume that $p \in \mathrm{RV}(\eta)$ and $q \in \mathrm{RV}(\sigma)$. Equation (1.1) possess intermediate solutions belonging to $\mathrm{ntr}-\mathrm{RV}(\frac{2\alpha-\eta}{\alpha})$ if and only if*

$$\sigma = \frac{\beta}{\alpha}\eta - 2\beta - 2 \quad \textit{and} \quad \int_a^\infty tq(t)\varphi_1(t)^\beta\,dt = \infty.$$

Any such solution x enjoys one and the same asymptotic behavior $x(t) \sim X_1(t)$ as $t \to \infty$, where $X_1(t)$ is given by (3.14).

Theorem 4.2. *Assume that $p \in \mathrm{RV}(\eta)$ and $q \in \mathrm{RV}(\sigma)$. Equation (1.1) possess intermediate regularly varying solutions of index ρ with $\rho \in \big(\frac{2\alpha-\eta}{\alpha}, \frac{1+2\alpha-\eta}{\alpha}\big)$ if and only if*

$$\frac{\beta}{\alpha}\eta - 2\beta - 2 < \sigma < \frac{\beta}{\alpha}(\eta-1) - 2\beta - 1,$$

in which case ρ is given by

$$\rho = \frac{2\alpha - \eta + \sigma + 2}{\alpha - \beta} \tag{4.6}$$

and any such solution x enjoys one and the same asymptotic behavior

$$x(t) \sim \Big(\frac{t^2\,p(t)^{-1}\,q(t)}{(\rho(\rho-1))^\alpha\ (2\alpha-\eta)\ (\rho\alpha+\eta-1-2\alpha)}\Big)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty.$$

Theorem 4.3. *Assume that $p \in \mathrm{RV}(\eta)$ and $q \in \mathrm{RV}(\sigma)$. Equation (1.1) possess intermediate solutions belonging to $\mathrm{ntr}-\mathrm{RV}\big(\frac{1+2\alpha-\eta}{\alpha}\big)$ if and only if*

$$\sigma = \frac{\beta}{\alpha}(\eta-1) - 2\beta - 1 \quad \textit{and} \quad \int_a^\infty q(t)\,\varphi_2(t)^\beta\,dt < \infty.$$

Any such solution x enjoys one and the same asymptotic behavior $x(t) \sim X_3(t)$ as $t \to \infty$, where $X_3(t)$ is given by (3.19).

Proof. To prove Theorem 4.1 and 4.3 we need only to check that

$$\sigma^* = -m_1(\alpha,\eta^*)\beta - 2m_2(\alpha,\eta^*) \ \Leftrightarrow\ \sigma = \frac{\beta}{\alpha}\eta - 2\beta - 2,$$
$$\sigma^* = \beta - m_2(\alpha,\eta^*) \ \Leftrightarrow\ \sigma = \frac{\beta}{\alpha}(\eta-1) - 2\beta - 1,$$

and to prove Theorem 4.2 it suffices to note that

$$\rho^* = \frac{\sigma^* + m_2(\alpha,\eta^*) - \alpha}{\alpha-\beta} \ \Leftrightarrow\ \rho = \frac{2\alpha + \sigma - \eta + 2}{\alpha - \beta},$$

and to combine the relation (4.5) with the equality

$$\begin{aligned}&\alpha^2 m_2(\alpha,\eta^*)^{-\frac{2(\alpha+1)^2}{2\alpha+1}}\Big[(m_1(\alpha,\eta^*)-\rho^*)(\rho^*+1)\big((\rho^* - m_2(\alpha,\eta^*)\rho^*\big)^\alpha\Big]\\ &= (2\alpha-\eta)(\rho\alpha+\eta-1-2\alpha)(\rho(\rho-1))^\alpha.\end{aligned}$$

∎

Similarly, we are able to gain a through knowledge of type-($I2$) intermediate regularly varying solutions of (1.1) from Theorems 3.9, 3.10 and 3.11.

Theorem 4.4. *Assume that $p \in \mathrm{RV}(\eta)$ and $q \in \mathrm{RV}(\sigma)$. Equation (1.1) possess intermediate nontrivial slowly varying solutions if and only if*

$$\sigma = \eta - 2\alpha - 2 \quad \text{and} \quad \int_a^\infty t\Big(\frac{1}{p(t)}\int_a^t (t-s)\,q(s)\,ds\Big)^{1/\alpha} dt = \infty.$$

The asymptotic behavior of any such solution x is governed by the unique formula $x(t) \sim Y_1(t)$, $t \to \infty$, where $Y_1(t)$ is given by (3.54).

Theorem 4.5. *Assume that $p \in \mathrm{RV}(\eta)$ and $q \in \mathrm{RV}(\sigma)$. Equation (1.1) possess intermediate regularly varying solutions of index ρ with $\rho \in (0,1)$ if and only if*

$$\eta - 2\alpha - 2 < \sigma < \eta - \alpha - \beta - 2,$$

in which case ρ is given by (4.6) and the asymptotic behavior of any such solution x is governed by the unique formula

$$x(t) \sim \Big(\frac{t^2\,p(t)^{-1}\,q(t)}{\big(\rho(1-\rho)\big)^\alpha\;(\eta-2\alpha)\;(\rho\alpha+\eta-1-2\alpha)}\Big)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty.$$

Theorem 4.6. *Assume that $p(t) \in \mathrm{RV}(\eta)$ and $q(t) \in \mathrm{RV}(\sigma)$. Equation (1.1) possess intermediate nontrivial regularly varying solutions of index 1 if and only if*

$$\sigma = \eta - \alpha - \beta - 2 \quad \text{and} \quad \int_a^\infty \Big(\frac{1}{p(t)}\int_a^t (t-s)s^\beta q(s)ds\Big)^{1/\alpha} dt < \infty.$$

The asymptotic behavior of any such solution x is governed by the unique formula $x(t) \sim Y_3(t)$, $t \to \infty$, where $Y_3(t)$ is given by (3.58).

The above corollaries combined with Theorems 1.1–1.4 enable us to describe in full details the structure of RV-solutions of equation (1.1) with RV-coefficients. Denote with $\mathcal{R}$ the set of all regularly varying solutions of (1.1) and define the subsets

$$\mathcal{R}(\rho) = \mathcal{R} \cap \mathrm{RV}(\rho), \quad \mathrm{tr}-\mathcal{R}(\rho) = \mathcal{R} \cap \mathrm{tr}-\mathrm{RV}(\rho), \quad \mathrm{ntr}-\mathcal{R}(\rho) = \mathcal{R} \cap \mathrm{ntr}-\mathrm{RV}(\rho).$$

Corollary 4.7. *Let $p \in \mathrm{RV}(\eta)$, $q \in \mathrm{RV}(\sigma)$.*

(i) *If $\sigma < \frac{\beta}{\alpha}\eta - 2\beta - 2$, or $\sigma = \frac{\beta}{\alpha}\eta - 2\beta - 2$ and $\mathcal{J}_3 < \infty$, then*

$$\mathcal{R} = \mathrm{tr}-\mathcal{R}\Big(\frac{2\alpha-\eta}{\alpha}\Big) \cup \mathrm{tr}-\mathcal{R}\Big(\frac{1+2\alpha-\eta}{\alpha}\Big) \cup \mathrm{tr}-\mathcal{R}(0) \cup \mathrm{tr}-\mathcal{R}(1).$$

(ii) *If $\sigma = \frac{\beta}{\alpha}\eta - 2\beta - 2$ and $\mathcal{J}_3 = \infty$, then*

$$\mathcal{R} = \mathrm{ntr}-\mathcal{R}\Big(\frac{2\alpha-\eta}{\alpha}\Big) \cup \mathrm{tr}-\mathcal{R}\Big(\frac{1+2\alpha-\eta}{\alpha}\Big) \cup \mathrm{tr}-\mathcal{R}(0) \cup \mathrm{tr}-\mathcal{R}(1).$$

(iii) *If $\sigma \in \big(\frac{\beta}{\alpha}\eta - 2\beta - 2, \frac{\beta}{\alpha}(\eta-1) - 2\beta - 1\big)$, then*

$$\mathcal{R} = \mathcal{R}\Big(\frac{\sigma+2\alpha+2-\eta}{\alpha-\beta}\Big) \cup \mathrm{tr}-\mathcal{R}\Big(\frac{1+2\alpha-\eta}{\alpha}\Big) \cup \mathrm{tr}-\mathcal{R}(0) \cup \mathrm{tr}-\mathcal{R}(1).$$

(iv) *If $\sigma = \frac{\beta}{\alpha}(\eta-1) - 2\beta - 1$ and $\mathcal{J}_4 < \infty$, then*

$$\mathcal{R} = \mathrm{tr}-\mathcal{R}\Big(\frac{1+2\alpha-\eta}{\alpha}\Big) \cup \mathrm{ntr}-\mathcal{R}\Big(\frac{1+2\alpha-\eta}{\alpha}\Big) \cup \mathrm{tr}-\mathcal{R}(0) \cup \mathrm{tr}-\mathcal{R}(1).$$

(v) *If $\sigma = \frac{\beta}{\alpha}(\eta-1) - 2\beta - 1$ and $\mathcal{J}_4 = \infty$, or $\sigma \in \big(\frac{\beta}{\alpha}(\eta-1) - 2\beta - 1, \eta - 2\alpha - 2\big)$, or $\sigma = \eta - 2\alpha - 2$ and $\mathcal{J}_1 < \infty$, then*

$$\mathcal{R} = \mathrm{tr}-\mathcal{R}(0) \cup \mathrm{tr}-\mathcal{R}(1).$$

(vi) *If* $\sigma = \eta - 2\alpha - 2$ *and* $\mathcal{J}_1 = \infty$*, then*
$$\mathcal{R} = \mathrm{ntr} - \mathcal{R}(0) \cup \mathrm{tr} - \mathcal{R}(1).$$

(vii) *If* $\sigma \in (\eta - 2\alpha - 2, \eta - \alpha - \beta - 2)$*, then*
$$\mathcal{R} = \mathcal{R}\big(\frac{\sigma + 2\alpha + 2 - \eta}{\alpha - \beta}\big) \cup \mathrm{tr} - \mathcal{R}(1).$$

(viii) *If* $\sigma = \eta - \alpha - \beta - 2$ *and* $\mathcal{J}_2 < \infty$*, then*
$$\mathcal{R} = \mathrm{tr} - \mathcal{R}(1) \cup \mathrm{ntr} - \mathcal{R}(1).$$

(ix) *If* $\sigma = \eta - \alpha - \beta - 2$ *and* $\mathcal{J}_2 = \infty$*, or* $\sigma > \eta - \alpha - \beta - 2$*, then* $\mathcal{R} = \emptyset$*.*

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