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MULTIPLE SIGN-CHANGING SOLUTIONS FOR KIRCHHOFF TYPE PROBLEMS

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Abstract. This article concerns the existence of sign-changing solutions to nonlocal Kirchhoff type problems of the form

$$-\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = f(x,u) \text{ in } \Omega, \quad u=0 \text{ on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N=1,2,3)$ with smooth boundary, $a>0$, $b\geq 0$, and $f:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ is a continuous function. We first establish a new sign-changing version of the symmetric mountain pass theorem and then apply it to prove the existence of a sequence of sign-changing solutions with higher and higher energy.

1. Introduction

In this article, we study the multiplicity of sign-changing solutions to nonlocal Kirchhoff type problems of the form

$$\begin{gathered} -\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = f(x,u) \quad \text{in } \Omega,\\ u=0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N=1,2,3)$ with smooth boundary, $a>0$, $b\geq 0$, and $f:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ is a nonlinear function. We restrict $N\leq 3$ because $f(x,u)$ will behave as $|u|^p$ with $4\leq p<2^\star$, where $2^\star=2N/(N-2)$ is the critical Sobolev exponent. This will allow us to attack the problem using variational methods.

Problem (1.1) is related to the stationary analogue of the hyperbolic equation

$$u_{tt}-\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = f(x,u),$$

which is a general version of the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2dx\Big)\frac{\partial^2 u}{\partial x^2}=0 \tag{1.2}$$

proposed by Kirchhoff [12] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. This model takes into account the changing in length of the string produced by transverse vibrations. In (1.2), L is the length

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of the string, h is the area of the cross-section, E is the Young's modulus of the material, ρ is the mass density, and ρ_0 is the initial tension.

When $b > 0$, problem (1.1) is said to be nonlocal. In that case, the first equation in (1.1) is no longer a pointwise equality. This causes some mathematical difficulties which make the study of such problems particularly interesting. Some early classical studies of Kirchhoff type problems can be found in [7, 24]. However, problem (1.1) received much attention only after the paper of Lions [13], where an abstract framework to attack it was introduced. Some existence and multiplicity results can be found in [6, 10, 14, 22] without any information on the sign of the solutions. Recently, Alves et al [1], Ma and Rivera [18], and Cheng and Wu [8] obtained one positive solution. In [9], He and Zou obtained infinitely many positive solutions. The existence of sign-changing solutions to (1.1) was considered by Figuereido and Nascimento [11], Perera and Zhang [23], Mao and Zhang [20], and Mao and Luan [19]. But only one sign-changing solution was found in these papers. In case f is a pure power type nonlinearity, Alves et al [1] related the number of solutions of (1.1) to that of a local problem by using a scaling argument. As a consequence, one can obtain in that particular case infinitely many sign-changing solutions (see [27]). However, the scaling approach does not provide high energy solutions even in the simple case of power type nonlinearity.

In this article, we develop a variational approach to study high-energy sign-changing solutions to some classes of nonlocal problems.

Our result on (1.1) relies on the following standard conditions on the nonlinear term f:

- (H1) $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a constant $c > 0$ such that
$$|f(x,u)| \le c\big(1 + |u|^{p-1}\big),$$
where $p > 4$ for $N = 1, 2$ and $4 < p < 6$ for $N = 3$.
- (H2) $f(x,u) = \circ(|u|)$, uniformly in $x \in \overline{\Omega}$, as $u \to 0$.
- (H3) there exists $\mu > 4$ such that $0 < \mu F(x,u) \le u f(x,u)$ for all $u \neq 0$ and for a.e $x \in \overline{\Omega}$, where $F(x,u) = \int_0^u f(x,s)ds$.
- (H4) $f(x,-u) = -f(x,u)$ for all $(x,u) \in \overline{\Omega} \times \mathbb{R}$.

One can verify easily that the function $f(x,u) = |u|^p$, with p as in condition (H1), satisfies the above conditions.

Our result reads as follows:

Theorem 1.1. *Let $a > 0$ and $b \ge 0$. Assume that f satisfies the conditions* (H1)–(H4)*. Then* (1.1) *possesses a sequence (u_k) of sign-changing solutions such that*
$$\frac{a}{2}\int_\Omega |\nabla u_k|^2 dx + \frac{b}{4}\Big(\int_\Omega |\nabla u_k|^2 dx\Big)^2 - \int_\Omega F(x,u_k)dx \to +\infty, \quad \text{as } k \to \infty.$$

If $b = 0$, we obtain the following consequence of the above result.

Corollary 1.2. *Under assumptions* (H1)–(H4)*, the semilinear problem*
$$\begin{gathered} -\Delta u = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$
possesses a sequence (u_k) of sign-changing solutions such that
$$\frac{1}{2}\int_\Omega |\nabla u_k|^2 dx - \int_\Omega F(x,u_k)dx \to +\infty, \quad \text{as } k \to \infty.$$

Note that Corollary 1.2 was obtained by Qian and Li [25] by means of the method of invariant sets of descending flow. Earlier proofs were also given in [3, 16] under the stronger assumption that f is smooth. The arguments of [25, 3, 16] rely on sign-changing critical point theorems built only for functionals of the form

$$u \in H_0^1(\Omega) \mapsto \frac{1}{2}\|u\|^2 - \Psi(u),$$

where Ψ' is completely continuous, and cannot then be applied to (1.1) when $b > 0$. Hence our result in Theorem 1.1 can be regarded as an extension of the classical result for the semilinear problem (1.3) to the case of the nonlinear Kirchhoff type problem (1.1). We also mention here that the result of Theorem 1.1 was more or less expected. However, it seems that this paper is the first to provide a formal proof. Moreover, we believe that the critical point theorem we will establish in the next section is of independant interest and can be applied to many other nonlocal problems (indeed, some applications by the author and collaborators will appear in other journals).

The study of sign-changing solutions is related to several long-standing questions concerning the multiplicity of solutions for elliptic boundary value problems. Compared with positive and negative solutions, sign-changing solutions have more complicated qualitative properties and are more difficult to find. During the last thirty years, several sophisticated techniques in calculus of variations and in critical point theory were developed to study the multiplicity of sign-changing solutions to nonlinear elliptic partial differential equations. In [3] and [16], the authors established some multiplicity sign-changing critical point theorems in partially ordered Hilbert spaces by using Morse theory and the method of invariant sets of descending flow respectively. In [28], a parameter-depending sign-changing fountain theorem was established without any Palais-Smale type assumption. More recently, a symmetric mountain pass theorem in the presence of invariant sets of the gradient flows was introduced in [15]. However, it seems that all these powerful approaches are not directly applicable to find multiple sign-changing solutions to (1.1).

Our approach in proving Theorem 1.1 relies on a new sign-changing critical point theorem, also established in this paper, which is modelled on the fountain theorem of Bartsch (see [26, Theorem 3.6]). An essential tool in the proof of this theorem is a deformation lemma, which allows to lower sub-level sets of a functional, away from its critical set. The main ingredient in the proof of the deformation lemma is a suitable negative pseudo-gradient flow, a notion introduced by Palais [21]. Since we are interesting in sign-changing critical points, the pseudo-gradient flow must be constructed in such the way that it keeps the positive and negative cones invariant. This invariance property makes the construction of the flow very complicated when the problem contains nonlocal terms. In this paper, we borrow some ideas from recent work by Liu, Liu and Wang [15] on the nonlinear Schödinger systems and by Liu, Wang and Zhang [17] on the nonlinear Schödinger-Poisson system, where the pseudo-gradient flows were constructed by using an auxiliary operator. However, the critical point theorem used in [15, 17] cannot be applied to prove Theorem 1.1 because the corresponding auxiliary operator in the case of (1.1) is not compact.

The rest of this article is organized as follows. In Section 2, we state and prove the new sign-changing critical point theorem. In Section 3, we provide the proof of Theorem 1.1.

Throughout this article, we denote by "$\to$" the strong converge and by "$\rightharpoonup$" the weak convergence.

2. An abstract sign-changing critical point theorem for even functionals

In this section, we present a variant of the symmetric mountain pass type theorem which produces a sequence of sign-changing critical points with arbitrary large energy.

Let Φ be a C^1-functional defined on a Hilbert space X of the form

$$X := \overline{\oplus_{j=0}^{\infty} X_j}, \quad \text{with } \dim X_j < \infty. \tag{2.1}$$

We introduce for $k \geq 2$ and $m > k+2$ the following notation:

$$Y_k := \oplus_{j=0}^{k} X_j, \quad Z_k = \overline{\oplus_{j=k}^{\infty} X_j}, \quad Z_k^m = \oplus_{j=k}^{m} X_j, \quad B_k := \big\{u \in Y_k : \|u\| \leq \rho_k\big\},$$
$$N_k := \big\{u \in Z_k : \|u\| = r_k\big\}, \quad N_k^m := \big\{u \in Z_k^m : \|u\| = r_k\big\}, \text{ where } 0 < r_k < \rho_k,$$
$$\Phi_m := \Phi|_{Y_m}, \quad K_m := \big\{u \in Y_m : \Phi_m'(u) = 0\big\}, \quad E_m := Y_m \backslash K_m.$$

Let P_m be a closed convex cone of Y_m. For $\mu_m > 0$ we set

$$\pm D_m^0 := \big\{u \in Y_m : \operatorname{dist}\big(u, \pm P_m\big) < \mu_m\big\}, \quad D_m = D_m^0 \cup (-D_m^0), \quad S_m := Y_m \backslash D_m.$$

We will also denote the α-neighborhood of $S \subset Y_m$ by

$$V_\alpha(S) := \big\{u \in Y_m \,|\, \operatorname{dist}(u, S) \leq \alpha\big\}, \quad \forall \alpha > 0.$$

Let us now state our critical point theorem. It is a version of the symmetric mountain pass theorem of Ambrosetti and Rabinowitz [2], and we model it on the fountain theorem of Bartsch [4].

Theorem 2.1 (Sign-changing fountain theorem)**.** *Let $\Phi \in C^1(X, \mathbb{R})$ be an even functional which maps bounded sets to bounded sets. If, for $k \geq 2$ and $m > k+2$, there exist $0 < r_k < \rho_k$ and $\mu_m > 0$ such that*

- (H5) *$a_k := \max_{u \in \partial B_k} \Phi(u) \leq 0$ and $b_k := \inf_{u \in N_k} \Phi(u) \to +\infty$, as $k \to \infty$.*
- (H6) *$N_k^m \subset S_m$.*
- (H7) *There exists an odd locally Lipschitz continuous vector field $B : E_m \to Y_m$ such that:*
  - (i) *$B\big((\pm D_m^0) \cap E_m\big) \subset \pm D_m^0$;*
  - (ii) *there exists a constant $\alpha_1 > 0$ such that $\langle \Phi_m'(u), u - B(u)\rangle \geq \alpha_1 \|u - B(u)\|^2$, for any $u \in E_m$;*
  - (iii) *for $a < b$ and $\alpha > 0$, there exists $\beta > 0$ such that $\|u - B(u)\| \geq \beta$ if $u \in Y_m$ is such that $\Phi_m(u) \in [a, b]$ and $\|\Phi_m'(u)\| \geq \alpha$.*
- (H8) *Φ satisfies the $(PS)_{nod}^{\star}$ condition, that is:*
  - (i) *any Palais-Smale sequence of Φ_m is bounded;*
  - (ii) *any sequence $(u_{m_j}) \subset X$ such that $m_j \to \infty$, $u_{m_j} \in V_{\mu_{m_j}}(S_{m_j})$, $\sup \Phi(u_{m_j}) < \infty$, and $\Phi_{m_j}'(u_{m_j}) = 0$, has a subsequence converging to a sign-changing critical point of Φ.*

Then Φ has a sequence $(u_k)_k$ of sign-changing critical points in X such that $\Phi(u_k) \to \infty$, as $k \to \infty$.

Condition (H8) is a version of the usual compactness condition in critical point theory, namely the Palais-Smale condition. We recall that a sequence $(u_n) \subset E$ is a Palais-Smale sequence of a smooth functional J defined on a Banach space

E if the sequence $\big(J(u_n)\big)$ is bounded and $J'(u_n) \to 0$, as $n \to \infty$. If every such sequence possesses a convergent subsequence, then J is said to satisfy the Palais-Smale condition.

We need a special deformation lemma to prove the above result. We first recall the following helpful lemma.

Lemma 2.2 ([28, Lemma 2.2]). *Let $\mathcal{M}$ be a closed convex subset of a Banach space E. If $H : \mathcal{M} \to E$ is a locally Lipschitz continuous map such that*

$$\lim_{\beta\to 0^+} \frac{\operatorname{dist}\big(u+\beta H(u),\mathcal{M}\big)}{\beta} = 0, \quad \forall u \in \mathcal{M},$$

then for any $u_0 \in \mathcal{M}$, there exists $\delta > 0$ such that the initial value problem

$$\frac{d\sigma(t,u_0)}{dt} = H\big(\sigma(t,u_0)\big), \quad \sigma(0,u) = u_0,$$

has a unique solution defined on $[0,\delta)$. Moreover, $\sigma(t,u_0) \in \mathcal{M}$ for all $t \in [0,\delta)$.

Now we state a quantitative deformation lemma.

Lemma 2.3 (Deformation lemma). *Let $\Phi \in C^1(X,\mathbb{R})$ be an even functional which maps bounded sets to bounded sets. Fix m sufficiently large and assume that the condition* (H7) *holds. Let $c \in \mathbb{R}$ and $\varepsilon_0 > 0$ such that*

$$\forall u \in \Phi_m^{-1}\big([c-2\varepsilon_0, c+2\varepsilon_0]\big) \cap V_{\frac{\mu_m}{2}}(S_m) \,:\, \|\Phi_m'(u)\| \ge \varepsilon_0. \tag{2.2}$$

Then for some $\varepsilon \in]0,\varepsilon_0[$ there exists $\eta \in C\big([0,1]\times Y_m, Y_m\big)$ such that

(i) *$\eta(t,u) = u$ for $t = 0$ or $u \notin \Phi_m^{-1}\big([c-2\varepsilon, c+2\varepsilon]\big)$;*
(ii) *$\eta\big(1, \Phi_m^{-1}(]-\infty, c+\varepsilon]) \cap S_m\big) \subset \Phi_m^{-1}\big(]-\infty, c-\varepsilon]\big)$;*
(iii) *$\Phi_m\big(\eta(\cdot,u)\big)$ is not increasing, for any u;*
(iv) *$\eta([0,1]\times D_m) \subset D_m$;*
(v) *$\eta(t,\cdot)$ is odd, for any $t \in [0,1]$.*

Proof. Define $V : E_m \to Y_m$ by $V(u) = u - B(u)$, where B is given by (H7). Then there is $\delta > 0$ such that $V(u) \ge \delta$ for any $u \in \Phi_m^{-1}\big([c-2\varepsilon_0, c+2\varepsilon_0]\big) \cap V_{\frac{\mu_m}{2}}(S_m)$ $\big($in view (H7)-(iii)$\big)$. We take $\varepsilon \in]0, \min(\varepsilon_0, \frac{\delta\alpha_1\mu_m}{8})[$ and we define

$$A_1 := \Phi_m^{-1}\big([c-2\varepsilon, c+2\varepsilon]\big) \cap V_{\frac{\mu_m}{2}}(S_m), \quad A_2 := \Phi_m^{-1}\big([c-\varepsilon, c+\varepsilon]\big) \cap V_{\frac{\mu_m}{4}}(S_m),$$

$$\chi(u) := \frac{\operatorname{dist}(u, Y_m\backslash A_1)}{\operatorname{dist}(u, Y_m\backslash A_1) + \operatorname{dist}(u, A_2)}, \quad u \in Y_m$$

so that $\chi = 0$ on $Y_m\backslash A_1$, $\chi = 1$ on A_2, and $0 \le \chi \le 1$.

We consider the vector field

$$W(u) := \begin{cases} \chi(u)\|V(u)\|^{-2}V(u), & \text{for } u \in A_1 \\ 0, & \text{for } u \in Y_m\backslash A_1. \end{cases}$$

Clearly W is odd and locally Lipschitz continuous. Moreover, by our choice of ε above we have

$$\|W(u)\| \le \frac{1}{\delta} \le \frac{\alpha_1\mu_m}{8\varepsilon}, \quad \forall u \in Y_m. \tag{2.3}$$

It follows that the Cauchy problem

$$\frac{d}{dt}\sigma(t,u) = -W(\sigma(t,u)), \quad \sigma(0,u) = u \in Y_m$$

has a unique solution $\sigma(\cdot, u)$ defined on $\mathbb{R}_+$. Moreover, σ is continuous on $\mathbb{R}_+ \times Y_m$ and the map $\sigma(t, \cdot) : Y_m \to Y_m$ is a homeomorphism for each $t \geq 0$ (see, for instance [26]). In view of (2.3), we have

$$\|\sigma(t,u) - u\| \leq \int_0^t \|W(\sigma(s,u))\| ds \leq \frac{\alpha_1 \mu_m}{8\varepsilon} t, \tag{2.4}$$

and by (H7)-(ii)

$$\begin{aligned} \frac{d}{dt}\Phi_m(\sigma(t,u)) &= -\langle \Phi_m'(\sigma(t,u)), \chi(\sigma(t,u))\|V(\sigma(t,u))\|^{-2} V(\sigma(t,u))\rangle \\ &\leq -\alpha_1 \chi(\sigma(t,u)). \end{aligned} \tag{2.5}$$

Define

$$\eta : [0,1] \times Y_m \to Y_m, \quad \eta(t,u) := \sigma\big(\frac{2\varepsilon}{\alpha_1} t, u\big).$$

Conclusion (i) of the lemma is clearly satisfied and by (2.5) above (iii) is also satisfied. Since W is odd, (v) is a consequence of the uniqueness of the solution to the above Cauchy problem.

We now verify (ii). Let $v \in \eta(1, \Phi_m^{-1}(]-\infty, c+\varepsilon]) \cap S_m)$. Then $v = \eta(1,u) = \sigma(\frac{2\varepsilon}{\alpha_1}, u)$, where $u \in \Phi_m^{-1}(]-\infty, c+\varepsilon]) \cap S_m$.

If there exists $t \in [0, \frac{2\varepsilon}{\alpha_1}]$ such that $\Phi_m(\sigma(t,u)) < c - \varepsilon$, then by (iii) we have $\Phi_m(v) < c - \varepsilon$.

Assume now that $\sigma(t,u) \in \Phi_m^{-1}([c-\varepsilon, c+\varepsilon])$ for all $t \in [0, \frac{2\varepsilon}{\alpha_1}]$. By (2.4) we have $\|\sigma(t,u) - u\| \leq \frac{\mu_m}{4}$, which means, since $u \in S_m$, that $\sigma(t,u) \in V_{\frac{\mu_m}{4}}(S_m)$. Hence $\sigma(t,u) \in A_2$ and since $\chi = 1$ on A_2, we deduce from (2.5) that

$$\Phi_m\big(\sigma(\frac{2\varepsilon}{\alpha_1}, u)\big) \leq \Phi_m(u) - \alpha_1 \int_0^{\frac{2\varepsilon}{\alpha_1}} \chi(\sigma(t,u)) dt = \Phi_m(u) - 2\varepsilon.$$

Since $\Phi_m(u) \leq c + \varepsilon$, this implies $\Phi_m(v) = \Phi_m(\sigma(\frac{2\varepsilon}{\alpha_1}, u)) \leq c - \varepsilon$. Hence (ii) is satisfied.

It remains to verify (iv). Since σ is odd in u, it suffices to show that

$$\sigma\big([0,+\infty) \times D_m^0\big) \subset D_m^0. \tag{2.6}$$

We follow [28].

Claim: We have

$$\sigma([0,+\infty) \times \overline{D_m^0}) \subset \overline{D_m^0}. \tag{2.7}$$

Assume by contradiction that (2.6) does not hold. Then there exist $u_0 \in D_m^0$ and $t_0 > 0$ such that $\sigma(t_0, u_0) \notin D_m^0$. Choose a neighborhood N_{u_0} of u_0 such that $N_{u_0} \subset D_m^0$. Then there is a neighborhood N_0 of $\sigma(t_0, u_0)$ such that $\sigma(t_0, \cdot) : N_{u_0} \to N_0$ is a homeomorphism (because $\sigma(t_0, \cdot) : Y_m \to Y_m$ is a homeomorphism). Since $\sigma(t_0, u_0) \notin D_m^0$, the set $N_0 \backslash \overline{D_m^0}$ is not empty. Hence there is $w \in N_{u_0}$ such that $\sigma(t_0, w) \in N_0 \backslash \overline{D_m^0}$, contradicting (2.7).

We now terminate by giving the proof of our above claim. By (H7)-(i) we have $B(D_m^0 \cap E_m) \subset D_m^0$, which implies that $B(\overline{D_m^0} \cap E_m) \subset \overline{D_m^0}$. Since $K_m \cap A_1 = \emptyset$, we have $\sigma(t,u) = u$ for all $t \in [0,1]$ and $u \in \overline{D_m^0} \cap K_m$.

Assume that $u \in \overline{D_m^0} \cap E_m$. If there is $t_1 \in (0,1]$ such that $\sigma(t_1, u) \notin \overline{D_m^0}$, then there would be $s_1 \in [0, t_1)$ such that $\sigma(s_1, u) \in \partial \overline{D_m^0}$ and $\sigma(t,u) \notin \overline{D_m^0}$ for all

$t \in (s_1, t_1]$. The Cauchy problem

$$\frac{d}{dt}\mu(t,\sigma(s_1,u)) = -W\big(\mu(t,\sigma(s_1,u))\big), \quad \mu(0,\sigma(s_1,u)) = \sigma(s_1,u) \in Y_m$$

has $\sigma(t,\sigma(s_1,u))$ as unique solution. Recalling that $W = 0$ on $Y_m \backslash A_1$, we have $v - W(v) \in \overline{D_m^0} \cap (Y_m \backslash A_1)$ for any $v \in \overline{D_m^0} \cap (Y_m \backslash A_1)$.

Assume that $v \in A_1 \cap \overline{D_m^0}$. Since $\|V(u)\| \geq \delta$, we deduce that $1 - \beta\chi(v)\|V(v)\|^{-2} \geq 0$ for all β such that $0 < \beta \leq \delta^2$. Recalling that $v \in \overline{D_m^0}$ implies $\operatorname{dist}(v, P_m) \leq \mu_m$, that $V(v) = v - B(v)$, and that $aP_m + bP_m \subset P_m$ for all $a, b \geq 0$ (because P_m is a cone), we obtain for any $\beta \in]0, \delta^2]$

$$\begin{aligned}
&\operatorname{dist}\big(v - \beta W(v), P_m\big)\\
&= \operatorname{dist}\big(v - \beta\chi(v)\|V(v)\|^{-2}V(v), P_m\big)\\
&= \operatorname{dist}\big(v - \beta\chi\|V(v)\|^{-2}(v - B(v)), P_m\big)\\
&= \operatorname{dist}\Big((1 - \beta\chi(v)\|V(v)\|^{-2})v + \beta\chi(v)\|V(v)\|^{-2}B(v), P_m\Big)\\
&\leq \operatorname{dist}\Big((1 - \beta\chi(v)\|V(v)\|^{-2})v + \beta\chi(v)\|V(v)\|^{-2}B(v),\\
&\quad \beta\chi(u)\|V(v)\|^{-2}P_m + (1 - \beta\chi(v)\|V(v)\|^{-2})P_m\Big)\\
&\leq (1 - \beta\chi(v)\|V(v)\|^{-2})\operatorname{dist}(v, P_m) + \beta\chi(v)\|V(v)\|^{-2}\operatorname{dist}(B(v), P_m)\\
&\leq (1 - \beta\chi(v)\|V(v)\|^{-2})\mu_m + \beta\chi(v)\|V(v)\|^{-2}\mu_m = \mu_m.
\end{aligned}$$

It follows that $v - \beta W(v) \in \overline{D_m^0}$ for $0 < \beta \leq \delta^2$. This implies that

$$\lim_{\beta \to 0^+} \frac{\operatorname{dist}\big(v + \beta(-W(v)), \overline{D_m^0}\big)}{\beta} = 0, \quad \forall u \in \overline{D_m^0}.$$

By Lemma 2.2 there exists $\delta_0 > 0$ such that $\sigma\big(t, \sigma(s_1, u)\big) \in \overline{D_m^0}$ for all $t \in [0, \delta_0)$. This implies that $\sigma\big(t, \sigma(s_1, u)\big) = \sigma(t + s_1, u) \in \overline{D_m^0}$ for all $t \in [0, \delta_0)$, which contradicts the definition of s_1. This last contradiction assures that $\sigma([0, +\infty) \times \overline{D_m^0}) \subset \overline{D_m^0}$. $\square$

Proof of Theorem 2.1. (H5) and (H6) imply that $a_k < b_k \leq \inf_{u \in N_k^m} \Phi_m(u)$, for k big enough. Let

$$\Gamma_k^m := \big\{\gamma \in C(B_k, Y_m) : \gamma \text{ is odd, } \gamma|_{\partial B_k} = id \text{ and } \ gamma(D_m) \subset D_m\big\}.$$

The set Γ_k^m is clearly non empty and for any $\gamma \in \Gamma_k^m$ the set $U := \big\{u \in B_k : \|\gamma(u)\| < r_k\big\}$ is a bounded symmetric (i.e. $-U = U$) neighborhood of the origin in Y_k. Moreover, U is open. Indeed, U is an open subset of B_k (with respect to the metric topology of B_k) and the conditions $\gamma|_{\partial B_k} = id$ and $r_k < \rho_k$ imply that an element of ∂B_k cannot belong to U. By the Borsuk-Ulam theorem the continuous odd map $\Pi_k \circ \gamma : \partial U \subset Y_k \to Y_{k-1}$ has a zero u_0, where $\Pi_k : X \to Y_{k-1}$ is the orthogonal projection. Hence $\gamma(u_0) \in \gamma(B_k) \cap N_k^m$ which implies that $\gamma(B_k) \cap N_k^m \neq \emptyset$. Since $N_k^m \subset S_m$, we deduce that $\gamma(B_k) \cap S_m \neq \emptyset$. This intersection property implies that

$$c_{k,m} := \inf_{\gamma \in \Gamma_k^m} \max_{u \in \gamma(B_k) \cap S_m} \Phi_m(u) \geq \inf_{u \in N_k^m} \Phi(u) \geq b_k.$$

We would like to show that for any $\varepsilon_0 \in]0, \frac{c_{k,m}-a_k}{2}[$, there exists $u \in \Phi_m^{-1}\big([c_{k,m} - 2\varepsilon_0, c_{k,m} + 2\varepsilon_0]\big) \cap V_{\frac{\mu_m}{2}}(S_m)$ such that $\|\Phi'_m(u)\| < \varepsilon_0$. Arguing by contradiction, we assume that we can find $\varepsilon_0 \in]0, \frac{c_{k,m}-a_k}{2}[$ such that

$$\|\Phi'_m(u)\| \geq \varepsilon_0, \quad \forall u \in \Phi_m^{-1}\big([c_{k,m} - 2\varepsilon_0, c_{k,m} + 2\varepsilon_0]\big) \cap V_{\frac{\mu_m}{2}}(S_m).$$

We apply Lemma 2.3 with $c = c_{k,m}$ and define, using the deformation η obtained, the map

$$\theta : B_k \to Y_m, \quad \theta(u) := \eta(1, \gamma(u)),$$

where $\gamma \in \Gamma_k^m$ satisfies

$$\max_{u \in \gamma(B_k) \cap S_m} \Phi_m(u) \leq c_{k,m} + \varepsilon, \tag{2.8}$$

with ε also given by Lemma 2.3.

Using the properties of η (see Lemma 2.3), one can easily verify that $\theta \in \Gamma_k^m$. On the other hand, we have

$$\eta\big(1, \gamma(B_k)\big) \cap S_m \subset \eta\big(1, \Phi_m^{-1}(] - \infty, c_{k,m} + \varepsilon]) \cap S_m\big). \tag{2.9}$$

In fact, if $u \in \eta\big(1, \gamma(B_k)\big) \cap S_m$ then $u = \eta\big(1, \gamma(v)\big) \in S_m$ for some $v \in B_k$. Observe that $\gamma(v) \in S_m$. Indeed, if this is not true then $\gamma(v) \in D_m$, and by (iv) of Lemma 2.3 we obtain $u = \eta(1, \gamma(v)) \in D_m$ which contradicts the fact that $u \in S_m$. Now (2.8) implies that $\gamma(v) \in \Phi_m^{-1}(] - \infty, c_{k,m} + \varepsilon])$. It then follows, using (ii) of Lemma 2.3, that $u = \eta(1, \gamma(v)) \in \eta\big(1,] - \infty, c_{k,m} + \varepsilon] \cap S_m\big)$. Hence (2.9) holds. Using (2.9) and (ii) of Lemma 2.3, we obtain

$$\begin{aligned}
\max_{u \in \theta(B_k) \cap S_m} \Phi_m(u) &= \max_{u \in \eta\left(1, \gamma(B_k)\right) \cap S_m} \Phi_m(u) \\
&\leq \max_{u \in \eta\left(1, \Phi_m^{-1}(] - \infty, c_{k,m} + \varepsilon]) \cap S_m\right)} \Phi_m(u) \\
&\leq c_{k,m} - \varepsilon,
\end{aligned}$$

contradicting the definition of $c_{k,m}$.

The above contradiction assures that for any $\varepsilon_0 \in]0, \frac{c_{k,m}-a_k}{2}[$, there exists

$$u \in \Phi_m^{-1}\big([c_{k,m} - 2\varepsilon_0, c_{k,m} + 2\varepsilon_0]\big) \cap V_{\frac{\mu_m}{2}}(S_m)$$

such that $\|\Phi'_m(u)\| < \varepsilon_0$.
It follows that there is a sequence $(u_{k,m}^n)_n \subset V_{\frac{\mu_m}{2}}(S_m)$ such that

$$\Phi'_m(u_{k,m}^n) \to 0 \text{ and } \Phi_m(u_{k,m}^n) \to c_{k,m}, \text{ as } n \to \infty.$$

We deduce from (H8)-(i), using the fact Y_m is finite-dimensional, that there exists $u_{k,m} \in V_{\frac{\mu_m}{2}}(S_m)$ such that

$$\Phi'_m(u_{k,m}) = 0 \text{ and } \Phi_m(u_{k,m}) = c_{k,m}.$$

Noting that $c_{k,m} \leq \max_{u \in B_k} \Phi(u)$, we deduce using (H8)-(ii) that Φ has a sign-changing critical point u_k such that $b_k \leq \Phi(u_k) \leq \max_{u \in B_k} \Phi(u)$. Since $b_k \to \infty$, as $k \to \infty$, the conclusion follows. $\square$

3. Proof of the main result

Throughout this section, we assume that (H1)–(H4) are satisfied. We denote by $|\cdot|_q$ the usual norm of the Lebesgue space $L^q(\Omega)$. Let $X := H_0^1(\Omega)$ be the usual Sobolev space endowed with the inner product

$$\langle u, v\rangle = \int_\Omega \nabla u \nabla v \, dx$$

and norm $\|u\|^2 = \langle u, u\rangle$, for $u, v \in H_0^1(\Omega)$.

It is well known that solutions of (1.1) are critical points of the functional

$$\Phi(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega F(x,u)dx, \quad u \in X := H_0^1(\Omega). \tag{3.1}$$

By a standard argument, one can easily verify that Φ is of class C^1 and

$$\langle \Phi'(u), v\rangle = \big(a + b\|u\|^2\big)\int_\Omega \nabla u \nabla v dx - \int_\Omega v f(x,u)dx \tag{3.2}$$

Let $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ be the distinct eigenvalues of the problem

$$-\Delta u = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Then each λ_j has finite multiplicity. It is well known that the principal eigenvalue λ_1 is simple with a positive eigenfunction e_1, and the eigenfunctions e_j corresponding to λ_j ($j \ge 2$) are sign-changing. Let X_j be the eigenspace associated to λ_j. We set for $k \ge 2$

$$Y_k := \oplus_{j=1}^k X_j \quad \text{and} \quad Z_k = \overline{\oplus_{j=k}^\infty X_j}.$$

Lemma 3.1.

(1) *For any $u \in Y_k$ we have $\Phi(u) \to -\infty$, as $\|u\| \to \infty$.*

(2) *There exists $r_k > 0$ such that*

$$\inf_{u \in Z_k \|u\| = r_k} \Phi(u) \to \infty, \ \textit{as } k \to \infty.$$

Proof. (1) It is well known that integrating (H3) yields the existence of two constants $c_1, c_2 > 0$ such that $F(x,u) \ge c_1|u|^\mu - c_2$. This together with the fact that all norms are equivalent in the finite-dimensional subspace Y_k imply that

$$\Phi(u) \le \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - c_3\|u\|^\mu + c_4, \quad \forall u \in Y_k,$$

where c_3, c_4 are positive constant. Since $\mu > 4$, it follows that $\Phi(u) \to -\infty$, as $\|u\| \to \infty$.

(2) Using (H1), we obtain

$$\Phi(u) \ge \frac{a}{2}\|u\|^2 - c_5|u|_p^p - c_6, \quad \forall u \in X,$$

where c_5, c_6 are poisitive constant. Set

$$\beta_k := \sup_{v \in Z_k,\, |v\| = 1} |v|_p.$$

Then we obtain

$$\Phi(u) \ge a\big(\frac{1}{2} - \frac{1}{p}\big)\big(\frac{c_5}{a}p\beta_k^p\big)^{\frac{2}{2-p}} - c_6$$

for every $u \in Z_k$ such that

$$\|u\| = r_k := \big(\frac{c_5}{a}p\beta_k^p\big)^{\frac{1}{2-p}}.$$

We know from [26, Lemma 3.8] that $\beta_k \to 0$, as $k \to \infty$. This implies that $r_k \to \infty$, as $k \to \infty$. $\square$

Now we fix k large enough, and for $m > k+2$, we set

$$\Phi_m := \Phi|_{Y_m}, \quad K_m := \{u \in Y_m : \Phi_m'(u) = 0\}, \quad E_m := Y_m \backslash K_m,$$
$$P_m := \{u \in Y_m \,;\, u(x) \geq 0\}, \quad Z_k^m := \oplus_{j=k}^m X_j, \quad N_k^m := \{u \in Z_k^m \mid \|u\| = r_k\}.$$

We remark that for all $u \in P_m \backslash \{0\}$ we have $\int_\Omega ue_1 dx > 0$, while for all $u \in Z_k$, $\int_\Omega ue_1 dx = 0$, where e_1 is the principal eigenfunction of the Laplacian. This implies that $P_m \cap Z_k = \{0\}$. It then follows, since N_k^m is compact, that

$$\delta_m := \operatorname{dist}\big(N_k^m, -P_m \cup P_m\big) > 0. \tag{3.3}$$

For $u \in Y_m$ fixed, we consider the functional

$$I_u(v) = \frac{1}{2}\big(a + b\|u\|^2\big)\int_\Omega |\nabla v|^2 dx - \int_\Omega v f(x,u) dx, \quad v \in Y_m. \tag{3.4}$$

It is not difficult to see that I_u is of class C^1, coercive, bounded below, weakly lower semicontinuous, and strictly convex. Therefore I_u admits a unique minimizer $v = Au \in Y_m$, which is the unique solution to the problem

$$-\big(a + b\|u\|^2\big)\Delta v = f(x,u), \quad v \in Y_m.$$

Clearly, the set of fixed points of A coincide with K_m. Moreover, the operator $A : Y_m \to Y_m$ has the following important properties.

Lemma 3.2.
(1) *A is continuous and maps bounded sets to bounded sets.*
(2) *For any $u \in Y_m$ we have*

$$\langle \Phi_m'(u), u - Au\rangle \geq a\|u - Au\|^2, \tag{3.5}$$
$$\|\Phi_m'(u)\| \leq (a+b)\big(1 + \|u\|^2\big)\|u - Au\|. \tag{3.6}$$

(3) *There exists $\mu_m \in]0, \delta_m[$ such that $A(\pm D_m^0) \subset \pm D_m^0$, where δ_m is defined by* (3.3).

Proof. (1) Let $(u_n) \subset Y_m$ such that $u_n \to u$. We set $v_n = Au_n$ and $v = Au$. By the definition of A we have for any $w \in Y_m$,

$$\big(a + b\|u_n\|^2\big)\int_\Omega \nabla v_n \nabla w \, dx = \int_\Omega w f(x,u_n) dx \tag{3.7}$$
$$\big(a + b\|u\|^2\big)\int_\Omega \nabla v \nabla w \, dx = \int_\Omega w f(x,u) dx. \tag{3.8}$$

Taking $w = v_n - v$ in (3.7) and in (3.8), and using the Hölder inequality and the Sobolev embedding theorem, we obtain

$$\begin{aligned}
&\big(a + b\|u_n\|^2\big)\|v_n - v\|^2 \\
&= b\big(\|u_n\|^2 - \|u\|^2\big)\int_\Omega \nabla v \nabla (v_n - v) dx + \int_\Omega (v - v_n)\big(f(u_n) - f(u)\big) dx \\
&\leq c_1\big|\|u_n\|^2 - \|u\|^2\big|\|v\|\|v_n - v\| + c_2\|v_n - v\||f(u_n) - f(u)|_{\frac{p}{p-1}},
\end{aligned}$$

where $c_1, c_2 > 0$ are constant. By (H1) and [26, Theorem A.2], we have $f(u_n) - f(u) \to 0$ in $L^{\frac{p}{p-1}}(\Omega)$. Hence $\|Au_n - Au\| = \|v_n - v\| \to 0$, that is, A is continuous.

On the other hand, for any $u \in Y_m$ we have, taking $v = w = Au$ in (3.8)

$$\big(a + b\|u\|^2\big)\|Au\|^2 = \int_\Omega Auf(x,u)dx.$$

By using (H1), the Hölder inequality, and the Sobolev embedding theorem, we obtain

$$a\|Au\| \le C\big(1 + \|u\|^{p-1}\big),$$

where $C > 0$ a constant. This shows that Au is bounded whenever u is bounded.

(2) Taking $w = u - Au$ in (3.8), we obtain

$$\big(a + b\|u\|^2\big)\int_\Omega \nabla(Au)\nabla(u - Au)dx = \int_\Omega (u - Au)f(x,u)dx,$$

which implies

$$\langle \Phi_m'(u), u - Au\rangle = \big(a + b\|u\|^2\big)\|u - Au\|^2 \ge a\|u - Au\|^2.$$

On the other hand, using (3.8), we obtain

$$\begin{aligned}
\langle \Phi_m'(u), w\rangle &= \big(a + b\|u\|^2\big)\int_\Omega \nabla u\nabla w dx - \int_\Omega wf(x,u)dx \\
&= \big(a + b\|u\|^2\big)\int_\Omega \nabla(u - Au)\nabla w dx, \quad \forall w \in Y_m.
\end{aligned}$$

This implies

$$\|\Phi_m'(u)\| \le \big(a + b\|u\|^2\big)\|u - Au\|.$$

(3) It follows from (H1) and (H2) that for each $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$|f(x,t)| \le \varepsilon|t| + c_\varepsilon|t|^{p-1}, \quad \forall t \in \mathbb{R}. \tag{3.9}$$

Let $u \in Y_m$ and let $v = Au$. As usual we denote $w^\pm = \max\{0, \pm w\}$, for any $w \in X$. Taking $w = v^+$ in (3.8) and using the Hölder inequality, we obtain

$$\big(a + b\|u\|^2\big)\|v^+\|^2 = \int_\Omega v^+f(x,u)dx \le \varepsilon|u^+|_2|v^+|_2 + c_\varepsilon|u^+|_p^{p-1}|v^+|_p,$$

which implies

$$\|v^+\|^2 \le \frac{1}{a}\Big(\varepsilon|u^+|_2|v^+|_2 + c_\varepsilon|u^+|_p^{p-1}|v^+|_p\Big). \tag{3.10}$$

On the other hand it is not difficult to see that $|u^+|_q \le |u - w|_q$ for all $w \in -P_m$ and $1 \le q \le 2^\star$. Hence there is a constant $c_1 = c_1(q) > 0$ such that $|u^+|_q \le c_1 \operatorname{dist}(u, -P_m)$. It is obvious that $\operatorname{dist}(v, -P_m) \le \|v^+\|$. So we deduce from (3.10) and the Sobolev embedding theorem that

$$\begin{aligned}
&\operatorname{dist}(v, -P_m)\|v^+\| \le \|v^+\|^2 \\
&\le c_2\Big(\varepsilon \operatorname{dist}(u, -P_m) + c_\varepsilon \operatorname{dist}(u, -P_m)^{p-1}\Big)\|v^+\|,
\end{aligned}$$

where $c_2 > 0$ is constant. This implies

$$\operatorname{dist}(v, -P_m) \le c_2\Big(\varepsilon \operatorname{dist}(u, -P_m) + c_\varepsilon \operatorname{dist}(u, -P_m)^{p-1}\Big).$$

Similarly one can show that

$$\operatorname{dist}(v, P_m) \le c_3\Big(\varepsilon \operatorname{dist}(u, P_m) + c_\varepsilon \operatorname{dist}(u, P_m)^{p-1}\Big),$$

for some constant $c_3 > 0$.

Choosing ε small enough, we can then find $\mu_m \in]0,\delta_m[$ such that

$$\operatorname{dist}(v,\pm P_m) \le \frac{1}{2}\operatorname{dist}(u,\pm P_m)$$

whenever $\operatorname{dist}(u,\pm P_m) < \mu_m$. $\square$

Using the μ_m obtained above, we define

$$\pm D_m^0 := \big\{u \in Y_m : \operatorname{dist}\big(u,\pm P_m\big) < \mu_m\big\},$$
$$D_m = D_m^0 \cup (-D_m^0), \quad S_m := Y_m \backslash D_m.$$

Remark 3.3. Note that $\mu_m < \delta_m$ implies $N_k^m \subset S_m$.

The vector field $A : Y_m \to Y_m$ does not satisfy the assumption (H7) of Theorem 2.1 as it is not locally Liptschitz continuous. However, it will be used in the spirit of [5] to construct a vector field which will satisfy the above mentioned condition.

Lemma 3.4. *There exists an odd locally Lipschitz continuous operator $B : E_m \to Y_m$ such that*

(1) $\langle \Phi'(u), u - B(u)\rangle \ge \frac{1}{2}\|u - A(u)\|^2$, *for any $u \in E_m$.*
(2) $\frac{1}{2}\|u - B(u)\| \le \|u - A(u)\| \le 2\|u - B(u)\|$, *for any $u \in E_m$.*
(3) $B\big((\pm D_m^0) \cap E_m\big) \subset \pm D_m^0$.

The proof of this lemma follows the lines of [5]. We provide a sketch of the proof here for completeness.

Proof. We define $\Delta_1, \Delta_2 : E_m \to \mathbb{R}$ as

$$\Delta_1(u) = \frac{1}{2}\|u - Au\| \quad \text{and} \quad \Delta_2(u) = \frac{a}{2(a+b)}(1 + \|u\|^2)^{-1}\|u - Au\|. \tag{3.11}$$

For any $u \in E_m$ we choose $\gamma(u) > 0$ such that

$$\|A(v) - A(w)\| < \min\big\{\Delta_1(v), \Delta_1(w), \Delta_2(v), \Delta_2(w)\big\} \tag{3.12}$$

holds for every $v, w \in N(u) := \big\{z \in Y_m\,;\, \|z - u\| < \gamma(u)\big\}$.

Let $\mathcal{V}$ be a locally finite open refinement of $\big\{N(u)\,;\, u \in E_m\big\}$ and define

$$\mathcal{V}^\star := \big\{V \in \mathcal{V} : D_m^0 \cap V \ne \emptyset,\ -D_m^0 \cap V \ne \emptyset,\ -D_m^0 \cap D_m^0 \cap V \ne \emptyset\big\},$$
$$\mathcal{U} := \bigcup_{V \in \mathcal{V}\backslash\mathcal{V}^\star} \big\{V\big\} \cup \bigcup_{V \in \mathcal{V}^\star} \big\{V\backslash D_m^0, V\backslash(-D_m^0)\big\}.$$

By construction $\mathcal{U}$ is a locally finite open refinement of $\big\{N(u) : u \in E_m\big\}$ and has a property that any $U \in \mathcal{U}$ is such that

$$U \cap D_m^0 \ne \emptyset \text{ and } U \cap (-D_m^0) \ne \emptyset \Longrightarrow U \cap D_m^0 \cap (-D_m^0) \ne \emptyset. \tag{3.13}$$

Let $\big\{\Pi_U : U \in \mathcal{U}\big\}$ be the partition of unity subordinated to $\mathcal{U}$ defined by

$$\Pi_U(u) := \frac{\alpha_U(u)}{\sum_{v\in\mathcal{U}} \alpha_U(v)}, \text{ where } \alpha_U(u) = \operatorname{dist}\big(u, E_m\backslash U\big).$$

For any $u \in \mathcal{U}$ choose a_U such that if $U \cap (\pm D_m^0) \ne \emptyset$ then $a_U \in U \cap (\pm D_m^0)$ $\big($such an element exists in view of (3.13)$\big)$. Define $B : E_m \to Y_m$ by

$$B(u) := \frac{1}{2}\big(H(u) - H(-u)\big), \quad \text{where } H(u) = \sum_{U\in\mathcal{U}} \Pi_U(u)A(a_U).$$

We then conclude as in [5] by using Lemma 3.2-(3), (3.11), (3.12), and (3.5). $\square$

Remark 3.5. Lemmas 3.2 and 3.4 imply that

$$\begin{gathered}\langle \Phi_m'(u), u - B(u)\rangle \geq \frac{1}{8}\|u - B(u)\|^2 \text{ and} \\ \|\Phi_m'(u)\| \leq 2(a+b)(1+\|u\|^2)\|u - B(u)\|,\end{gathered}$$

for all $u \in E_m$.

Lemma 3.6. *Let $c < d$ and $\alpha > 0$. For all $u \in Y_m$ such that $\Phi_m(u) \in [c,d]$ and $\|\Phi_m'(u)\| \geq \alpha$, there exists $\beta > 0$ such that $\|u - B(u)\| \geq \beta$.*

Proof. By the definition of the operator A, we have for any $u \in Y_m$,

$$\big(a + b\|u\|^2\big)\int_\Omega \nabla(Au)\nabla u dx = \int_\Omega uf(x,u)dx.$$

It follows that

$$\begin{aligned}&\Phi_m(u) - \frac{1}{\mu}\big(a+b\|u\|^2\big)\int_\Omega \nabla u\nabla(u - Au)dx \\ &= a\big(\frac{1}{2} - \frac{1}{\mu}\big)\|u\|^2 + b\big(\frac{1}{4} - \frac{1}{\mu}\big)\|u\|^4 + \int_\Omega \big(\frac{1}{\mu}uf(x,u) - F(x,u)\big)dx\end{aligned}$$

which implies, using (H3) and Lemma 3.4-(2), that

$$\begin{aligned}b\big(\frac{1}{4} - \frac{1}{\mu}\big)\|u\|^4 &\leq |\Phi_m(u)| + \frac{1}{\mu}\big(a+b\|u\|^2\big)\|u\|\|u - Au\| \\ &\leq |\Phi_m(u)| + \frac{2}{\mu}\big(a+b\|u\|^2\big)\|u\|\|u - Bu\|.\end{aligned} \tag{3.14}$$

Suppose that there exists a sequence $(u_n) \subset Y_m$ such that: $\Phi_m(u_n) \in [c,d]$, $\|\Phi_m'(u_n)\| \geq \alpha$ and $\|u_n - Bu_n\| \to 0$. By (3.14) we see that $(\|u_n\|)$ is bounded. It follows from Remark 3.5 above that $\Phi_m'(u_n) \to 0$, which is a contradiction. $\square$

Now we verify the compactness condition for Φ.

Lemma 3.7. *Φ satisfies the $(PS)^\star_{nod}$ condition, that is:*

- *any Palais-Smale sequence of Φ_m is bounded,*
- *any sequence $(u_{m_j}) \subset X$ such that: $m_j \to \infty$, $u_{m_j} \in V_{\frac{\mu_{m_j}}{2}}(S_{m_j})$, $\sup \Phi(u_{m_j}) < \infty$, and $\Phi_{m_j}'(u_{m_j}) = 0$, has a subsequence converging to a sign-changing critical point of Φ.*

Proof. For any $u \in Y_m$ we have, in view of (H3),

$$\begin{aligned}&\Phi_m(u) - \frac{1}{\mu}\langle \Phi_m'(u), u\rangle \\ &= a\big(\frac{1}{2} - \frac{1}{\mu}\big)\|u\|^2 + b\big(\frac{1}{4} - \frac{1}{\mu}\big)\|u\|^4 + \int_\Omega \big(\frac{1}{\mu}uf(x,u) - F(x,u)\big)dx \\ &\geq a\big(\frac{1}{2} - \frac{1}{\mu}\big)\|u\|^2 + b\big(\frac{1}{4} - \frac{1}{\mu}\big)\|u\|^4.\end{aligned} \tag{3.15}$$

It then follows that any sequence $(u_n) \subset Y_m$ such that $\sup_n \Phi_m(u_n) < \infty$ and $\Phi_m'(u_n) \to 0$ is bounded.

Now let $(u_{m_j}) \subset X$ be such that

$$m_j \to \infty, \quad u_{m_j} \in V_{\frac{\mu_{m_j}}{2}}(S_{m_j}), \quad \sup \Phi(u_{m_j}) < \infty, \quad \Phi_{m_j}'(u_{m_j}) = 0.$$

In view of (3.15) the sequence (u_{m_j}) is bounded. Hence, up to a subsequence, $u_{m_j} \rightharpoonup u$ in X and $u_{m_j} \to u$ in $L^p(\Omega)$. Observe that the condition $\Phi'_{m_j}(u_{m_j}) = 0$ is weaker than $\Phi'(u_{m_j}) = 0$. Therefore, the fact that (u_{m_j}) converges strongly, up to a subsequence, to u in X does not follow from the usual standard argument.

Let us denote by $\Pi_{m_j} : X \to Y_{m_j}$ the orthogonal projection. Then it is clear that $\Pi_{m_j} u \to u$ in X, as $m_j \to \infty$. We have

$$\begin{aligned} &\langle \Phi'_{m_j}(u_{m_j}), u_{m_j} - \Pi_{m_j} u\rangle \\ &= \big(a + b\|u_{m_j}\|^2\big)\langle u_{m_j}, u_{m_j} - \Pi_{m_j} u\rangle - \int_\Omega \big(u_{m_j} - \Pi_{m_j} u\big) f(x, u_{m_j}) dx. \end{aligned} \tag{3.16}$$

Since (u_{m_j}) is bounded, we deduce from (H1) that $\big(|f(x, u_{m_j})|_{p/p-1}\big)$ is bounded. We then obtain by using the Hölder inequality

$$\Big|\int_\Omega \big(u_{m_j} - \Pi_{m_j} u\big) f(x, u_{m_j}) dx\Big| \le |u_{m_j} - \Pi_{m_j} u|_p |f(x, u_{m_j})|_{\frac{p}{p-1}} \to 0.$$

Recalling that $\Phi'_{m_j}(u_{m_j}) = 0$, we deduce from (3.16) that

$$\langle u_{m_j}, u_{m_j} - \Pi_{m_j} u\rangle = \|u_{m_j}\|^2 - \langle u_{m_j}, u\rangle + \langle u_{m_j}, u - \Pi_{m_j} u\rangle = \circ(1).$$

It then follows that $\|u_{m_j}\| \to \|u\|$ which implies, since X is uniformly convex, that $u_{m_j} \to u$ in X. It is readily seen that u is a critical point of Φ.

To show that the limit u is sign-changing, we first observe that

$$\begin{aligned} \langle \Phi'_{m_j}(u_{m_j}), u^\pm_{m_j}\rangle = 0 \;&\Leftrightarrow\; \big(a + b\|u_{m_j}\|^2\big)\|u^\pm_{m_j}\|^2 = \int_\Omega u^\pm_{m_j} f(x, u_{m_j}) dx \\ &\Rightarrow\; a\|u^\pm_{m_j}\|^2 \le \int_\Omega u^\pm_{m_j} f(x, u^\pm_{m_j}) dx. \end{aligned}$$

By using (3.9) and the Sobolev embedding theorem, we obtain

$$a\|u^\pm_{m_j}\|^2 \le \int_\Omega u^\pm_{m_j} f(x, u^\pm_{m_j}) dx \le c\big(\varepsilon\|u^\pm_{m_j}\|^2 + c_\varepsilon\|u^\pm_{m_j}\|^p\big),$$

where $c > 0$ is a constant. Since u_{m_j} is sign-changing, $u^\pm_{m_j}$ are not equal to 0. Choosing ε small enough $\big($for instance $\varepsilon < \frac{a}{2c}\big)$, we see that $(\|u^\pm_{m_j}\|)$ are bounded below by strictly positive constants which do not depend on m_j. This implies that the limit u of the sequence (u_{m_j}) is also sign-changing. □

We are now in a position for proving our main result.

Proof Theorem 1.1. By Lemmas 3.1, 3.4, 3.6, and 3.7, and Remarks 3.3 and 3.5, conditions (H5), (H6), (H7) and (H8) of Theorem 2.1 are satisfied. It then suffices to apply Theorem 2.1 to conclude. □

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