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INVERSE PROBLEM OF A HYPERBOLIC EQUATION WITH AN INTEGRAL OVERDETERMINATION CONDITION

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Abstract. In this article we study the inverse problem of a hyperbolic equation with an integral overdetermination condition. The existence, uniqueness and continuous dependence of the solution of the solution upon the data are established.

1. Introduction

In this article we study the unique solvability of the inverse problem of determining a pair of functions $\{u, f\}$ satisfying the equation

$$u_{tt} - \Delta u + \beta u_t = f(t)g(x,t), \quad x \in \Omega,\ t \in [0,T], \tag{1.1}$$

with the initial conditions

$$u(x,0) = \varphi(x), \quad x \in \Omega, \tag{1.2}$$

$$u_t(x,0) = \psi(x), \quad x \in \Omega, \tag{1.3}$$

the boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{1.4}$$

and the nonlocal condition

$$\int_\Omega v(x)u(x,t)dx = \theta(t), \quad t \in [0,T]. \tag{1.5}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. The functions g, φ, ψ, θ are known functions and β is a positive constant.

Inverse problems for hyperbolic PDEs arise naturally in geophysics, oil prospecting, in the design of optical devices, and in many other areas where the interior of an object is to be imaged using the response of the object to acoustic waves (satisfying hyperbolic PDEs). Additional information about the solution to the inverse problem is given in the form of integral observation condition (1.5).

The parameter identification in a partial differential equation from the data of integral overdetermination condition plays an important role in engineering and physics [4, 5, 6, 8, 9, 13]. From the physical point of view, these conditions may be interpreted as measurements of the temperature $u(x,t)$ by a device averaging over the domain of spatial variables.

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Note that inverse problems with integral overdetermination are closely related to nonlocal problems [2, 11]. Studies have shown that classical methods often do not work when we deal with nonlocal problems [1, 7]. To date, several methods have been proposed for overcoming the difficulties arising from nonlocal conditions. The choice of method depends on the kind of nonlocal conditions.

We note that the inverse problem for a parabolic equation with integral condition (1.5) and its unique solubility have been studied by many authors (see for example [3, 10, 14, 15]).

Also, there are some papers devoted to the study of existence and uniqueness of solutions of inverse problems for various parabolic equations with unknown source functions. Inverse problems of determining the right-hand side of a parabolic equation under a final overdetermination condition were studied in papers [11, 12, 15, 16].

In the present work, new studies are presented for the inverse problem for a hyperbolic equation. The existence and uniqueness of the classical solution of the problem (1.1)-(1.5) is reduced to a fixed point problem.

2. Functional space

Let us introduce certain notation used below. We set

$$g^*(t) = \int_\Omega v(x)g(x,t)dx, \quad Q_T = \Omega \times [0,T]. \tag{2.1}$$

The spaces $W_2^1(\Omega)$, $C((0,T), L_2(\Omega))$ and $W_2^{2,1}(Q_T)$ with corresponding norms are understood as follows: The Banach space $W_2^1(\Omega)$ consists of all functions from $L_2(\Omega)$ having all weak derivatives of the first order that are square integrable over with norm

$$\|u\|_{W_2^1(\Omega)} = (\|u\|^2_{L_2(\Omega)} + \|u_x\|^2_{L_2(\Omega)})^{1/2},$$

We denote by $C((0,T), L_2(\Omega))$ the space comprises of all continuous functions on $(0,T)$ with values in $L_2(\Omega)$. The corresponding norm is given by

$$\|u\|_{C((0,T),L_2(\Omega))} = \max_{(0,T)} \|u\|_{L_2(\Omega)} < \infty.$$

Let us also introduce the Sobolev space $W_2^{2,1}(Q_T)$ of functions $u(x,t)$ with finite norm

$$\|u\|_{W_2^{2,1}(Q_T)} = \Big(\|u\|^2_{L_2(Q_T)} + \|u_x\|^2_{L_2(Q_T)} + \|D_t u\|^2_{L_2(Q_T)}\Big)^{1/2}$$

where

$$\|u\| \equiv \|u\|_{L_2(\Omega)},$$

We note that the weighted arithmetic-geometric mean inequality (Cauchy's ε-inequality) is:

$$2|ab| \leqslant \varepsilon a^2 + \frac{1}{\varepsilon} b^2, \quad \text{for } \varepsilon > 0.$$

Also, we use the notation

$$\|\nabla u\| = \Big(\int_\Omega \sum_{i=1}^n u_{x_i}^2 dx\Big)^{1/2} \quad \text{and} \quad \|\Delta u\| = \Big(\int_\Omega \sum_{i,j=1}^n u_{x_i,x_j}^2 dx\Big)^{1/2}.$$

3. Existence and uniqueness of the solution to the inverse problem

We seek a solution of the original inverse problem in the form $\{u, f\} = \{z, f\} + \{y, 0\}$ where y is the solution of the direct problem

$$y_{tt} - \Delta y + \beta y_t = 0, \quad (x,t) \in Q_T, \tag{3.1}$$

$$y(x,0) = \varphi(x), \quad x \in \Omega, \tag{3.2}$$

$$y_t(x,0) = \psi(x), \quad x \in \Omega, \tag{3.3}$$

$$y(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{3.4}$$

while the pair $\{z, f\}$ is the solution of the inverse problem

$$z_{tt} - \Delta z + \beta z_t = f(t)g(x,t), \quad (x,t) \in Q, \tag{3.5}$$

$$z(x,0) = 0, \quad x \in \Omega, \tag{3.6}$$

$$z_t(x,0) = 0, \quad x \in \Omega, \tag{3.7}$$

$$z(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{3.8}$$

$$\int_\Omega v(x)z(x,t)\,dx = E(t), \quad t \in [0,T], \tag{3.9}$$

where

$$E(t) = \theta(t) - \int_\Omega v(x)y(x,t)\,dx.$$

We shall assume that the functions appearing in the data for the problem are measurable and satisfy the following conditions:

- (H1) $g \in C((0,T), L_2(\Omega))$, $v \in W_2^1(\Omega)$, $E \in W_2^2(0,T)$, $\|g(x,t)\| \leqslant m$, $|g^*(t)| \geqslant p > 0$, for $p \in \mathbb{R}$, $(x,t) \in Q_T$, $\varphi(x), \psi(x) \in W_2^1(\Omega)$

The correspondence between f and z may be viewed as one possible way of specifying the linear operator

$$A : L_2(0,T) \to L_2(0,T, L_2(\Omega)). \tag{3.10}$$

with the values

$$(Af)(t) = \frac{1}{g^*}\Big\{\int_\Omega \nabla z \nabla v dx\Big\}. \tag{3.11}$$

In this view, it is reasonable to refer to the linear equation of the second kind for the function f over the space $L_2(0,T)$:

$$f = Af + W, \tag{3.12}$$

where

$$W = \frac{E'' + \beta E}{g^*}.$$

Remark 3.1. As $\{u, f\} = \{z, f\} + \{y, 0\}$ where y is the solution of the direct problem (3.1)–(3.4). Obviously, y exists and is unique, so instead of proving the solvability of the original problem (1.1)–(1.5), we prove the existence and uniqueness of the solution of the inverse problem (3.5)–(3.9).

Theorem 3.2. *Suppose the input data of the inverse problem* (3.5)-(3.9) *satisfies (H1). Then the following assertions are valid: (i) if the inverse problem* (3.5)–(3.9) *is solvable, then so is equation* (3.12)*. (ii) if equation* (3.12) *possesses a solution and the compatibility condition*

$$E(0) = 0, \quad E'(0) = 0, \tag{3.13}$$

holds, then there exists a solution of the inverse problem (3.5)-(3.9).

Proof. (i) Assume that the inverse problem (3.5)–(3.9) is solvable. We denote its solution by $\{z, f\}$. Multiplying both sides of (3.5) by the function v scalarly in $L_2(\Omega)$, we obtain the relation

$$\frac{d}{dt}\int_\Omega z_t v dx + \int_\Omega \nabla z \nabla v dx + \beta \int_\Omega z_t v dx = f(t) g^*(x,t). \tag{3.14}$$

With (3.9) and (3.11), it follows from (3.14) that $f = Af + \frac{E''+\beta E}{g^*}$. This means that f solves equation (3.12).

(ii) By the assumption, equation (3.12) has a solution in the space $L_2(0,T)$, say f.

When inserting this function in (3.5), the resulting relations (3.5)–(3.8) can be treated as a direct problem having a unique solution $z \in W^{2,1}_{2,0}(Q_T)$. Let us show that the function z satisfies also the integral overdetermination condition (3.9).

Equation (3.14) yields

$$\frac{d}{dt}\int_\Omega z_t v dx + \int_\Omega \nabla z \nabla v dx + \beta \int_\Omega z_t v dx = f(t) g^*(x,t). \tag{3.15}$$

On the other hand, being a solution of (3.12), the function z is subject to relation

$$E'' + \beta E' + \int_\Omega \nabla z \nabla v \, dx = f(t) g^*(x,t). \tag{3.16}$$

Subtracting (3.15) from (3.16), we obtain

$$\frac{d}{dt}\int_\Omega z_t v dx + \beta \int_\Omega z_t v dx = E'' + \beta E'.$$

Integrating the preceding differential equation and taking into account the compatibility condition (3.13), we find that the function z satisfies the overdetermination condition (3.9) and the pair of functions $\{z, f\}$ is a solution of the inverse problem (3.5)–(3.9). This completes the proof. $\square$

Now, we state some properties of the operator A.

Lemma 3.3. *Let condition* (H1) *hold. Then there exist a positive* ε *for which* A *is a contracting operator in* $L_2(0,T)$.

Proof. Obviously, (3.11) yields the estimate

$$\|Af\|_{L_2(0,t)} \le \frac{k}{p}\Big(\int_0^t \|\nabla z(.,\tau)\|^2_{L_2(\Omega)} d\tau\Big)^{1/2}, \tag{3.17}$$

where $k = \|\nabla v\|_{L_2(\Omega)}$. Multiplying both sides of (3.5) by z_t scalarly in $L_2(\Omega)$ and integrating the resulting expressions by parts, we obtain the identity

$$\frac{1}{2}\frac{d}{dt}\|z_t(\cdot,t)\|^2_{L_2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|\nabla z(\cdot,t)\|^2_{L_2(\Omega)} + \beta\|z_t(\cdot,t)\|^2_{L_2(\Omega)} = f(t)\int_\Omega g(x,t) z_t dx.$$

So, by using the Cauchy's ε-inequality, we obtain the relation

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|z_t(\cdot,t)\|^2_{L_2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|\nabla z(\cdot,t)\|^2_{L_2(\Omega)} + \beta\|z_t(\cdot,t)\|^2_{L_2(\Omega)} \\
&\le \frac{m^2}{2\varepsilon}|f(t)|^2 + \frac{\varepsilon}{2}\|z_t(\cdot,t)\|^2_{L_2(\Omega)},
\end{aligned} \tag{3.18}$$

Choosing $0 < \varepsilon < 2\beta$, and integrating (3.18) over $(0,t)$, and using (3.6) and (3.7)), we obtain

$$\begin{aligned}
&\frac{1}{2}\|z_t(\cdot,t)\|^2_{L_2(\Omega)} + \frac{1}{2}\|\nabla z(\cdot,t)\|^2_{L_2(\Omega)} + (\beta - \frac{\varepsilon}{2})\int_0^t \|z_t(.,\tau)\|^2_{L_2(\Omega)} \\
&\leq \frac{m^2}{2\varepsilon}\int_0^t |f(\tau)|^2.
\end{aligned} \tag{3.19}$$

Omitting some terms on the left-hand side (3.19) and integrating over $(0,t)$, and using (3.6), leads to

$$\int_0^t \|\nabla z(.,\tau)\|^2_{L_2(\Omega)} d\tau \leq \frac{m^2}{\varepsilon}\int_0^t \|f(\tau)\|^2_{L_2(0,T)} d\tau. \tag{3.20}$$

So, according to (3.17) and (3.20), we obtain the estimate

$$\|Af\|_{L_2(0,t)} \leq \delta\Big(\int_0^t \|f(\tau)\|^2_{L_2(0,T)} d\tau\Big)^{1/2}, \quad 0 \leqslant t \leqslant T, \tag{3.21}$$

where

$$\delta = \frac{km}{p\sqrt{\varepsilon}}.$$

So, we obtain

$$\|Af\|_{L_2(0,T)} \leq \delta\|f\|_{L_2(0,T,L_2(0,T))}. \tag{3.22}$$

It follows from the foregoing that there exists a positive ε such that

$$\delta < 1. \tag{3.23}$$

Inequality (3.23) shows that the linear operator A is a contracting mapping on $L_2(0,T,L_2(0,T))$ and completes the proof. □

Regarding the unique solvability of the inverse problem, the following result will be useful.

Theorem 3.4. *Let condition* (H1) *and the compatibility condition* (3.13) *hold. Then the following assertions are valid: (i) a solution* $\{z, f\}$ *of the inverse problem* (3.5)-(3.9) *exists and is unique, and (ii) with any initial iteration* $f_0 \in L_2(0,T,L_2(0,T))$ *the successive approximations*

$$f_{n+1} = \widetilde{A} f_n\,. \tag{3.24}$$

converge to f *in the* $L_2(0,T,L_2(0,T))$*-norm (for* $\widetilde{A}$ *see below).*

Proof. (ii) We use the nonlinear operator $\widetilde{A} : L_2(0,T) \to L_2(0,T,L_2(0,T))$ acting in accordance to the rule

$$\widetilde{A}f = Af + \frac{E'' + \beta E}{g^*}, \tag{3.25}$$

where the operator A and the function g^* arise from (3.11). From (3.24) it follows that (3.12) can be written as

$$f = \widetilde{A}f. \tag{3.26}$$

Hence it is sufficient to show that operator $\widetilde{A}$ has a fixed point in the space $L_2(0,T,L_2(0,T))$. By the relations

$$\widetilde{A}f_1 - \widetilde{A}f_2 = Af_1 - Af_2 = A(f_1 - f_2),$$

from estimate (3.22) we deduce that

$$\begin{aligned} \|\widetilde{A}f_1 - \widetilde{A}f_2\|_{L_2(0,T)} &= \|A(f_1 - f_2)\|_{L_2(0,T)} \\ &\le \delta\|(f_1 - f_2)\|_{L_2(0,T,L_2(0,T))}. \end{aligned} \tag{3.27}$$

From (3.23) and (3.26), we find that $\widetilde{A}$ is a contracting mapping on $L_2(0,T,L_2(0,T))$.

Therefore $\widetilde{A}$ has a unique fixed point f in $L_2(0,T,L_2(0,T))$ and the successive approximations (3.24) converge to f in the $L_2(0,T,L_2(0,T))$-norm irrespective of the initial iteration $f_0 \in L_2(0,T,L_2(0,T))$.

(i) This shows that, equation (3.26) and, in turn, equation (3.12) have a unique solution f in $L_2(0,T,L_2(0,T))$.

According to Theorem 3.2, this confirms the existence of solution to the inverse problem (3.5)–(3.9).

It remains to prove the uniqueness of this solution. Assume to the contrary that there were two distinct solutions $\{z_1, f_1\}$ and $\{z_2, f_2\}$ of the inverse problem under consideration.

We claim that in that case $f_1 \neq f_2$ almost everywhere on $(0,T)$. If $f_1 = f_2$, then applying the uniqueness theorem to the corresponding direct problem (3.1)–(3.4) we would have $z_1 = z_2$ almost everywhere in Q_T.

Since both pairs satisfy identity (3.14), the functions f_1 and f_2 give two distinct solutions to equation (3.26). But this contradicts the uniqueness of the solution to equation (3.26) just established and proves the theorem. □

Corollary 3.5. *Under the conditions of Theorem 3.4, the solution f to equation* (3.12) *depends continuously upon the data W.*

Proof. Let W and V be two sets of data, which satisfy the assumptions of Theorem 3.4. Let f and g be solutions of the equation (3.12) corresponding to the data W and V, respectively. According to (3.12), we have

$$\begin{aligned} f &= Af + W, \\ g &= Ag + V. \end{aligned}$$

First, let us estimate the difference $f - g$. It is easy to see by using (3.22), that

$$\begin{aligned} \|f - g\|_{L_2(0,T,L_2(0,T))} &= \|(Af + W) - (Ag + V)\|_{L_2(0,T)} \\ &= \|A(f - g) + (W - V)\|_{L_2(0,T)} \\ &\leqslant \delta\|f - g\|_{L_2(0,T,L_2(0,T))} + \|(W - V)\|_{L_2(0,T)}; \end{aligned}$$

so, we obtain

$$\|f - g\|_{L_2(0,T,L_2(0,T))} \leqslant \frac{1}{(1-\delta)}\|(W - V)\|_{L_2(0,T)}.$$

The proof is complete. □

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