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LIOUVILLE-TYPE THEOREMS FOR ELLIPTIC INEQUALITIES WITH POWER NONLINEARITIES INVOLVING VARIABLE EXPONENTS FOR A FRACTIONAL GRUSHIN OPERATOR

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Abstract. We establish Liouville-type theorems for the elliptic inequality

$$u \geq 0, \quad G_{\alpha,\beta,\theta}\big(u^{p(x,y)}, u^{q(x,y)}\big) \geq u^{r(x,y)}, \quad (x,y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},$$

where $G_{\alpha,\beta,\theta}$, $0 < \alpha, \beta < 2$, $\theta \geq 0$, is the fractional Grushin operator of mixed orders α, β, defined by

$$G_{\alpha,\beta,\theta}(u,v) = (-\Delta_x)^{\alpha/2} u + |x|^{2\theta} (-\Delta_y)^{\beta/2} v,$$

where, $(-\Delta_x)^{\alpha/2}$ is the fractional Laplacian operator of order α with respect to the variable $x \in \mathbb{R}^{N_1}$, and $(-\Delta_y)^{\beta/2}$ is the fractional Laplacian operator of order β with respect to the variable $y \in \mathbb{R}^{N_2}$. Here, $p, q, r : \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \to [1, \infty)$ are measurable functions satisfying certain conditions.

1. Introduction

The standard Liouville theorem [20] states that any bounded complex function which is harmonic (or holomorphic) on the entire space is constant. The first proof of this theorem was published by Cauchy [4]. In the recent literature, Gidas and Spruck [12] extended this result to the case of non-negative solutions of semilinear elliptic equations in the whole space $\mathbb{R}^N$ or in half-spaces. In the case of the whole space $\mathbb{R}^N$, they established that if $1 \leq r < \frac{N+2}{N-2}$, then the unique non-negative solution of

$$-\Delta u = C u^r \quad \text{in } \mathbb{R}^N,$$

where C is a strictly positive constant, is the trivial solution. A simple proof based on the moving planes method was suggested by Chen and Li [5] in the whole range of r, i.e., $0 < r < \frac{N+2}{N-2}$. This result is optimal in the sense that for any $r \geq \frac{N+2}{N-2}$, we have infinitely many positive solutions. The same result holds for the elliptic inequality

$$-\Delta u \geq C u^r \quad \text{in } \mathbb{R}^N,$$

see [13]. Berestycki *et al.* [3], established Liouville-type theorems for semilinear elliptic inequalities of the form

$$u \geq 0, \quad -\Delta u \geq h(x) u^r \quad \text{in } \Sigma,$$

where Σ is a cone in $\mathbb{R}^N$ and h is a positive function.

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Recently, several Liouville-type theorems were established for various classes of degenerate elliptic equations. Serrin and Zou [26] generalized the standard Liouville theorem for p-harmonic functions on the whole space $\mathbb{R}^N$ and on exterior domains. In [17, 18], Liouville-type theorems for some linear degenerate elliptic operators such as X-elliptic operators, Kohn-Laplacian and Ornstein-Uhlenbeck operators were proved. Dolcetta and Cutri [7] established a Liouville-type theorem for an elliptic inequality involving the Grushin operator. More precisely, they considered the problem

$$u \geq 0, \quad G_\theta u \geq u^r \quad \text{in } \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, \tag{1.1}$$

where $\theta > 1$ and G_θ is the Grushin operator defined by

$$G_\theta u = (-\Delta_x)u + |x|^{2\theta}(-\Delta_y)u, \quad (x,y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}. \tag{1.2}$$

They proved that if $1 < r < \frac{Q}{Q-2}$, then the only solution of (1.1) is the trivial solution. Here, Q is the homogeneous dimension of the space, given by $Q = N_1 + (\theta+1)N_2$. For other related results, we refer to [1, 22, 23, 28].

Recently, a lot of attention has been paid to the study of linear and nonlinear integral operators, involving the fractional Laplacian. In [21], using the moving plane method, Ma and Chen established a Liouville-type result for the system of equations

$$\begin{aligned} (-\Delta)^{\mu/2} u &= v^q, \\ (-\Delta)^{\mu/2} v &= u^p, \end{aligned}$$

where $\mu \in (0,2)$, $1 < p, q \leq \frac{N+\mu}{N-\mu}$, and $N \geq 2$. Here, $(-\Delta)^{\mu/2}$ is the fractional Laplacian operator of order $\mu/2$. Using the test function method [24], Dahmani *et al.* [6] extended the result in [21] to various classes of systems involving fractional Laplacian operators with different orders. Quaas and Xia [25] established Liouville-type results for a class of fractional elliptic equations and systems in the half space. For other related works, we refer to [8, 9, 10, 14, 16], and the references therein.

This article is devoted to the study of nonexistence results of solutions for the elliptic inequality

$$u \geq 0, \quad G_{\alpha,\beta,\theta}\big(u^{p(x,y)}, u^{q(x,y)}\big) \geq u^{r(x,y)}, \quad (x,y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, \tag{1.3}$$

where $G_{\alpha,\beta,\theta}$, $0 < \alpha, \beta < 2$, $\theta \geq 0$, is the fractional Grushin operator of mixed orders α, β, defined by

$$G_{\alpha,\beta,\theta}(u,v) = (-\Delta_x)^{\alpha/2} u + |x|^{2\theta}(-\Delta_y)^{\beta/2} v,$$

where, $(-\Delta_x)^{\alpha/2}$ is the fractional Laplacian operator of order α with respect to the variable $x \in \mathbb{R}^{N_1}$, and $(-\Delta_y)^{\beta/2}$ is the fractional Laplacian operator of order β with respect to the variable $y \in \mathbb{R}^{N_2}$. Here, $p, q, r : \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \to [1, \infty)$ are supposed to be measurable functions satisfying certain conditions. Observe that the standard Grushin operator defined by (1.2) can be written in the form

$$G_\theta u = G_{2,2,\theta}(u,u).$$

Up to our knowledge, there are not many works dealing with Liouville-type properties involving elliptic inequalities with variable exponents non-linearity. In this direction, we refer to the recent paper [11].

Before stating and proving the main results of this work, let us present some basic definitions and some lemmas that will be used later.

The nonlocal operator $(-\Delta)^s$, $0 < s < 1$, is defined for any function h in the Schwartz class through the Fourier transform
$$(-\Delta)^s h(x) = \mathcal{F}^{-1}\left(|\xi|^{2s}\mathcal{F}(h)(\xi)\right)(x),$$
where $\mathcal{F}$ stands for the Fourier transform and $\mathcal{F}^{-1}$ for its inverse. It can be also defined via the Riesz potential
$$(-\Delta)^s h(x) = c_{N,s}\ \mathrm{PV}\int_{\mathbb{R}^N}\frac{h(x)-h(\overline{x})}{|x-\overline{x}|^{N+2s}}\,d\overline{x},$$
where $c_{N,s}$ is a normalisation constant and PV is the Cauchy principal value (see [19, 27]).

Lemma 1.1 ([15])**.** *Suppose that $\delta \in (0,2)$, $\beta + 1 \geq 0$, and $\psi \in C_0^\infty(\mathbb{R}^N)$, $\psi \geq 0$. Then the following point-wise inequality holds:*
$$(-\Delta)^{\delta/2}\psi^{\beta+2}(x) \leq (\beta+2)\psi^{\beta+1}(x)(-\Delta)^{\delta/2}\psi(x).$$

Lemma 1.2 (ε-Young's inequality)**.** *Let $1 < p, q < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then*
$$ab \leq \varepsilon a^p + C(\varepsilon) b^q, \quad (a, b > 0, \varepsilon > 0),$$
where $C(\varepsilon) = (\varepsilon p)^{-q/p} q^{-1}$.

For a measurable function $p : \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \to [1, \infty)$, we denote by $L^{p(\cdot,\cdot)}(\mathbb{R}^{N_1} \times \mathbb{R}^{N_2})$ the Lebesgue space with variable exponent, defined by
$$\begin{aligned} &L^{p(\cdot,\cdot)}(\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}) \\ &= \Big\{u : \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \to \mathbb{R} : u \text{ measurable}, \int_{\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}} |u|^{p(x,y)}\,dx\,dy < \infty\Big\}. \end{aligned}$$
We denote by $L^{p(\cdot,\cdot)}_{\rm loc}(\mathbb{R}^{N_1} \times \mathbb{R}^{N_2})$ the set defined by
$$\begin{aligned} &L^{p(\cdot,\cdot)}_{\rm loc}(\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}) \\ &= \big\{u : \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \to \mathbb{R} : u \text{ measurable}, \int_K |u|^{p(x,y)}\,dx\,dy < \infty,\ K \text{ compact}\big\}. \end{aligned}$$
For more details on Lebesgue spaces with variable exponents, we refer to [2].

2. Main results

We consider the elliptic inequality (1.3) under the assumptions:

$\theta \geq 0$, $0 < \alpha, \beta < 2$,
$p, q, r \in L^\infty(\mathbb{R}^N)$, $N = N_1 + N_2$,
$r(x,y) > \max\{p(x,y), q(x,y)\} \geq 1$,
$\lambda := \inf_{(x,y)\in\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}} \{r(x,y) - p(x,y)\} > 0$,
$\mu := \inf_{(x,y)\in\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}} \{r(x,y) - q(x,y)\} > 0$.

The definition of solutions we adopt for (1.3) is the following.

Definition 2.1. We say that u is a weak solution of (1.3), if $u \in L^{i(\cdot,\cdot)}_{\rm loc}(\mathbb{R}^{N_1} \times \mathbb{R}^{N_2})$, $i \in \{p, q, r\}$, $u \geq 0$, and
$$\begin{aligned} &\int_{\mathbb{R}^N} u^{p(x,y)}(-\Delta_x)^{\alpha/2}\varphi\,dx\,dy + \int_{\mathbb{R}^N} |x|^{2\theta} u^{q(x,y)}(-\Delta_y)^{\beta/2}\varphi\,dx\,dy \\ &\geq \int_{\mathbb{R}^N} u^{r(x,y)}\varphi\,dx\,dy, \end{aligned}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \geq 0$.

Given $R > 0$, we denote by $\Omega_{R,\theta}$ the subset of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ defined by

$$\Omega_{R,\theta} = \Big\{(x,y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} : 1 \leq \frac{|x|^2}{R^2} + \frac{|y|^2}{R^{2(\theta+1)}} \leq 2\Big\}.$$

We have the following Liouville-type theorem for the elliptic inequality (1.3).

Theorem 2.2. *Suppose that*

$$\lim_{R\to\infty} \Big(\int_{\Omega_{R,\theta}} R^{\frac{-\alpha r(x,y)}{r(x,y)-p(x,y)}}\, dx\, dy + \int_{\Omega_{R,\theta}} R^{\frac{[2\theta-\beta(\theta+1)]r(x,y)}{r(x,y)-q(x,y)}}\, dx\, dy\Big) = 0. \tag{2.1}$$

Then inequality (1.3) *has no nontrivial weak solution.*

Proof. Suppose that u is a nontrivial weak solution of (1.3). Let ω be a real number such that

$$\omega > \max\Big\{\frac{\|r\|_{L^\infty(\mathbb{R}^N)}}{\lambda}, \frac{\|r\|_{L^\infty(\mathbb{R}^N)}}{\mu}, 1\Big\}. \tag{2.2}$$

By the weak formulation of (1.3), we have

$$\begin{aligned} &\int_{\mathbb{R}^N} u^{p(x,y)}(-\Delta_x)^{\alpha/2}\varphi^\omega\, dx\, dy + \int_{\mathbb{R}^N} |x|^{2\theta} u^{q(x,y)}(-\Delta_y)^{\beta/2}\varphi^\omega\, dx\, dy \\ &\geq \int_{\mathbb{R}^N} u^{r(x,y)}\varphi^\omega\, dx\, dy, \end{aligned} \tag{2.3}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \geq 0$. By Lemma 1.1, we have

$$\int_{\mathbb{R}^N} u^{p(x,y)}(-\Delta_x)^{\alpha/2}\varphi^\omega\, dx\, dy \leq \omega \int_{\mathbb{R}^N} u^{p(x,y)}\varphi^{\omega-1}|(-\Delta_x)^{\alpha/2}\varphi|\, dx\, dy.$$

Using the ε-Young inequality (see Lemma 1.2) with parameters $s(x,y) = \frac{r(x,y)}{p(x,y)}$ and $s'(x,y) = \frac{r(x,y)}{r(x,y)-p(x,y)}$, for all $\varepsilon > 0$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} u^{p(x,y)}\varphi^{\omega-1}|(-\Delta_x)^{\alpha/2}\varphi|\, dx\, dy \\ &= \int_{\mathbb{R}^N} u^{p(x,y)}\varphi^{\frac{\omega}{s(x,y)}}\varphi^{\omega-1-\frac{\omega}{s(x,y)}}|(-\Delta_x)^{\alpha/2}\varphi|\, dx\, dy \\ &\leq \varepsilon \int_{\mathbb{R}^N} u^{r(x,y)}\varphi^\omega\, dx\, dy \\ &\quad + \int_{\mathbb{R}^N} C_1(x,y,\varepsilon)\varphi^{[\omega-1-\frac{\omega}{s(x,y)}]s'(x,y)}|(-\Delta_x)^{\alpha/2}\varphi|^{s'(x,y)}\, dx\, dy, \end{aligned}$$

where

$$C_1(x,y,\varepsilon) = \Big(\frac{\varepsilon r(x,y)}{p(x,y)}\Big)^{\frac{-p(x,y)}{r(x,y)-p(x,y)}}\Big(\frac{r(x,y)-p(x,y)}{r(x,y)}\Big),$$

$(x,y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, and $\varepsilon > 0$. Observe that for all $\varepsilon > 0$, we have $C_1(\cdot,\cdot,\varepsilon) \in L^\infty(\mathbb{R}^N)$. In fact, under the considered assumptions, we have

$$C_1(x,y,\varepsilon) \leq \varepsilon^{\frac{\|p\|_{L^\infty(\mathbb{R}^N)}}{\lambda}}, \quad (x,y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}.$$

Let $C_1(\varepsilon) = \|C_1(\cdot,\cdot,\varepsilon)\|_{L^\infty(\mathbb{R}^N)}$. Therefore,

$$\int_{\mathbb{R}^N} u^{p(x,y)}\varphi^{\omega-1}|(-\Delta_x)^{\alpha/2}\varphi|\, dx\, dy$$

$$\leq \varepsilon \int_{\mathbb{R}^N} u^{r(x,y)}\varphi^\omega\,dx\,dy + C_1(\varepsilon)\int_{\mathbb{R}^N} \varphi^{[\omega-1-\frac{\omega}{s(x,y)}]s'(x,y)}|(-\Delta_x)^{\alpha/2}\varphi|^{s'(x,y)}\,dx\,dy.$$

Observe that thanks to (2.2), we have

$$\int_{\mathbb{R}^N} \varphi^{[\omega-1-\frac{\omega}{s(x,y)}]s'(x,y)}|(-\Delta_x)^{\alpha/2}\varphi|^{s'(x,y)}\,dx\,dy < \infty.$$

Indeed, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \varphi^{[\omega-1-\frac{\omega}{s(x,y)}]s'(x,y)}|(-\Delta_x)^{\alpha/2}\varphi|^{s'(x,y)}\,dx\,dy\\
&= \int_{\mathbb{R}^N} \varphi^{\omega-\frac{r(x,y)}{r(x,y)-p(x,y)}}|(-\Delta_x)^{\alpha/2}\varphi|^{\frac{r(x,y)}{r(x,y)-p(x,y)}}\,dx\,dy.
\end{aligned}$$

On the other hand, from (2.2), we have

$$\frac{r(x,y)}{r(x,y)-p(x,y)} \leq \frac{\|r\|_{L^\infty(\mathbb{R}^N)}}{\lambda} < \omega, \quad (x,y)\in\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}.$$

As consequence, we have the estimate

$$\begin{aligned}
&\int_{\mathbb{R}^N} u^{p(x,y)}(-\Delta_x)^{\alpha/2}\varphi^\omega\,dx\,dy\\
&\leq \omega\varepsilon\int_{\mathbb{R}^N} u^{r(x,y)}\varphi^\omega\,dx\,dy\\
&\quad + C_1(\varepsilon)\omega\int_{\mathbb{R}^N} \varphi^{\omega-\frac{r(x,y)}{r(x,y)-p(x,y)}}|(-\Delta_x)^{\alpha/2}\varphi|^{\frac{r(x,y)}{r(x,y)-p(x,y)}}\,dx\,dy.
\end{aligned} \tag{2.4}$$

Again, using Lemma 1.1, we obtain

$$\int_{\mathbb{R}^N} |x|^{2\theta}u^{q(x,y)}(-\Delta_y)^{\beta/2}\varphi^\omega\,dx\,dy \leq \omega\int_{\mathbb{R}^N} |x|^{2\theta}u^{q(x,y)}\varphi^{\omega-1}|(-\Delta_y)^{\beta/2}\varphi|\,dx\,dy.$$

Using the ε-Young inequality with parameters $k(x,y) = \frac{r(x,y)}{q(x,y)}$ and $k'(x,y) = \frac{r(x,y)}{r(x,y)-q(x,y)}$, for all $\varepsilon>0$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} |x|^{2\theta}u^{q(x,y)}\varphi^{\omega-1}|(-\Delta_y)^{\beta/2}\varphi|\,dx\,dy\\
&= \int_{\mathbb{R}^N} u^{q(x,y)}\varphi^{\frac{\omega}{k(x,y)}}\varphi^{\omega-1-\frac{\omega}{k(x,y)}}|x|^{2\theta}|(-\Delta_y)^{\beta/2}\varphi|\,dx\,dy\\
&\leq \varepsilon\int_{\mathbb{R}^N} u^{r(x,y)}\varphi^\omega\,dx\,dy\\
&\quad + \int_{\mathbb{R}^N} C_2(x,y,\varepsilon)\varphi^{[\omega-1-\frac{\omega}{k(x,y)}]k'(x,y)}|x|^{2\theta k'(x,y)}|(-\Delta_y)^{\beta/2}\varphi|^{k'(x,y)}\,dx\,dy,
\end{aligned}$$

where

$$C_2(x,y,\varepsilon) = \Big(\frac{\varepsilon r(x,y)}{q(x,y)}\Big)^{\frac{-q(x,y)}{r(x,y)-q(x,y)}}\Big(\frac{r(x,y)-q(x,y)}{r(x,y)}\Big), \quad (x,y)\in\mathbb{R}^{N_1}\times\mathbb{R}^{N_2},\ \varepsilon>0.$$

As previously, under the considered assumptions, we have

$$C_2(x,y,\varepsilon) \leq \varepsilon^{\frac{\|q\|_{L^\infty(\mathbb{R}^N)}}{\mu}},$$

$(x,y)\in\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}$, which implies that $C_2(\cdot,\cdot,\varepsilon)\in L^\infty(\mathbb{R}^N)$, for all $\varepsilon>0$. Let $C_2(\varepsilon)=\|C_2(\cdot,\cdot,\varepsilon)\|_{L^\infty(\mathbb{R}^N)}$. Therefore, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}|x|^{2\theta}u^{q(x,y)}\varphi^{\omega-1}|(-\Delta_y)^{\beta/2}\varphi|\,dx\,dy\\&\le\varepsilon\int_{\mathbb{R}^N}u^{r(x,y)}\varphi^{\omega}\,dx\,dy\\&\quad+C_2(\varepsilon)\int_{\mathbb{R}^N}\varphi^{[\omega-1-\frac{\omega}{k(x,y)}]k'(x,y)}|x|^{2\theta k'(x,y)}|(-\Delta_y)^{\beta/2}\varphi|^{k'(x,y)}\,dx\,dy.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}\varphi^{[\omega-1-\frac{\omega}{k(x,y)}]k'(x,y)}|x|^{2\theta k'(x,y)}|(-\Delta_y)^{\beta/2}\varphi|^{k'(x,y)}\,dx\,dy\\&=\int_{\mathbb{R}^N}\varphi^{\omega-\frac{r(x,y)}{r(x,y)-q(x,y)}}|x|^{\frac{2\theta r(x,y)}{r(x,y)-q(x,y)}}|(-\Delta_y)^{\beta/2}\varphi|^{\frac{r(x,y)}{r(x,y)-q(x,y)}}\,dx\,dy.\end{aligned}$$

From (2.2), we have

$$\frac{r(x,y)}{r(x,y)-q(x,y)}\le\frac{\|r\|_{L^\infty(\mathbb{R}^N)}}{\mu}<\omega,\quad (x,y)\in\mathbb{R}^{N_1}\times\mathbb{R}^{N_2};$$

then

$$\int_{\mathbb{R}^N}\varphi^{[\omega-1-\frac{\omega}{k(x,y)}]k'(x,y)}|x|^{2\theta k'(x,y)}|(-\Delta_y)^{\beta/2}\varphi|^{k'(x,y)}\,dx\,dy<\infty.$$

As consequence, we have the estimate

$$\begin{aligned}&\int_{\mathbb{R}^N}|x|^{2\theta}u^{q(x,y)}(-\Delta_y)^{\beta/2}\varphi^{\omega}\,dx\,dy\\&\le\omega\varepsilon\int_{\mathbb{R}^N}u^{r(x,y)}\varphi^{\omega}\,dx\,dy\\&\quad+C_2(\varepsilon)\omega\int_{\mathbb{R}^N}\varphi^{\omega-\frac{r(x,y)}{r(x,y)-q(x,y)}}|x|^{\frac{2\theta r(x,y)}{r(x,y)-q(x,y)}}|(-\Delta_y)^{\beta/2}\varphi|^{\frac{r(x,y)}{r(x,y)-q(x,y)}}\,dx\,dy.\end{aligned}\tag{2.5}$$

Now, combining (2.3), (2.4) and (2.5), we obtain

$$\begin{aligned}&(1-2\omega\varepsilon)\int_{\mathbb{R}^N}u^{r(x,y)}\varphi^{\omega}\,dx\,dy\\&\le C_1(\varepsilon)\omega\int_{\mathbb{R}^N}\varphi^{\omega-\frac{r(x,y)}{r(x,y)-p(x,y)}}|(-\Delta_x)^{\alpha/2}\varphi|^{\frac{r(x,y)}{r(x,y)-p(x,y)}}\,dx\,dy\\&\quad+C_2(\varepsilon)\omega\int_{\mathbb{R}^N}\varphi^{\omega-\frac{r(x,y)}{r(x,y)-q(x,y)}}|x|^{\frac{2\theta r(x,y)}{r(x,y)-q(x,y)}}|(-\Delta_y)^{\beta/2}\varphi|^{\frac{r(x,y)}{r(x,y)-q(x,y)}}\,dx\,dy.\end{aligned}$$

Taking $\varepsilon=(4\omega)^{-1}$, we obtain

$$\int_{\mathbb{R}^N}u^{r(x,y)}\varphi^{\omega}\,dx\,dy\le C\left(A(\varphi)+B(\varphi)\right),\tag{2.6}$$

where

$$\begin{aligned}A(\varphi)&=\int_{\mathbb{R}^N}\varphi^{\omega-\frac{r(x,y)}{r(x,y)-p(x,y)}}|(-\Delta_x)^{\alpha/2}\varphi|^{\frac{r(x,y)}{r(x,y)-p(x,y)}}\,dx\,dy,\\B(\varphi)&=\int_{\mathbb{R}^N}\varphi^{\omega-\frac{r(x,y)}{r(x,y)-q(x,y)}}|x|^{\frac{2\theta r(x,y)}{r(x,y)-q(x,y)}}|(-\Delta_y)^{\beta/2}\varphi|^{\frac{r(x,y)}{r(x,y)-q(x,y)}}\,dx\,dy.\end{aligned}$$

Let φ_0 be the standard cutoff function; that is, $\varphi_0 \in C_0^\infty(0,\infty)$ is a smooth decreasing function such that

$$0 \le \varphi_0 \le 1, \quad |\varphi_0'(\sigma)| \le \frac{C}{\sigma}, \quad \varphi_0(\sigma) = \begin{cases} 1 & \text{if } 0 < \sigma \le 1, \\ 0 & \text{if } \sigma \ge 2. \end{cases}$$

As a test function, we take

$$\varphi(x,y) = \varphi_0\Big(\frac{|x|^2}{R^2} + \frac{|y|^2}{R^{2(\theta+1)}}\Big), \quad (x,y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},$$

where $R > 0$ is a real number (large enough). Let Ω be the subset of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ defined by

$$\Omega = \big\{(z,w) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} : 1 \le |z|^2 + |w|^2 \le 2\big\}.$$

Let

$$\eta(z,w) = |z|^2 + |w|^2, \quad (z,w) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}.$$

Using the change of variables

$$z = \frac{x}{R}, \quad w = \frac{y}{R^{2(\theta+1)}},$$

we obtain

$$\begin{aligned} A(\varphi) &= \int_\Omega [\varphi_0(\eta)]^{\omega - s'(Rz, R^{\theta+1}w)} |(-\Delta_z)^{\alpha/2}\varphi_0(\eta)|^{s'(Rz,R^{\theta+1}w)} \\ &\quad \times R^{N_1+N_2(\theta+1)-\alpha s'(Rz,R^{\theta+1}w)}\, dz\, dw \\ &\le C \int_\Omega R^{N_1+N_2(\theta+1)-\alpha s'(Rz,R^{\theta+1}w)}\, dz\, dw \\ &= C \int_{\Omega_R} R^{\frac{-\alpha r(x,y)}{r(x,y)-p(x,y)}}\, dx\, dy. \end{aligned}$$

Therefore, we have the estimate

$$A(\varphi) \le C \int_{\Omega_R} R^{\frac{-\alpha r(x,y)}{r(x,y)-p(x,y)}}\, dx\, dy. \tag{2.7}$$

Under the same change of variables, we obtain

$$\begin{aligned} B(\varphi) &\le C \int_\Omega R^{N_1+N_2(\theta+1)+[2\theta-\beta(\theta+1)]k'(Rz,R^{\theta+1}w)}\, dz\, dw \\ &= C \int_{\Omega_R} R^{\frac{[2\theta-\beta(\theta+1)]r(x,y)}{r(x,y)-q(x,y)}}\, dx\, dy. \end{aligned}$$

Therefore, we have the estimate

$$B(\varphi) \le C \int_{\Omega_R} R^{\frac{[2\theta-\beta(\theta+1)]r(x,y)}{r(x,y)-q(x,y)}}\, dx\, dy. \tag{2.8}$$

Combining (2.6), (2.7) and (2.8), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} u^{r(x,y)} \varphi_0^\omega\Big(\frac{|x|^2}{R^2} + \frac{|y|^2}{R^{2(\theta+1)}}\Big)\, dx\, dy \\ &\le C\Big(\int_{\Omega_R} R^{\frac{-\alpha r(x,y)}{r(x,y)-p(x,y)}}\, dx\, dy + \int_{\Omega_R} R^{\frac{[2\theta-\beta(\theta+1)]r(x,y)}{r(x,y)-q(x,y)}}\, dx\, dy\Big). \end{aligned}$$

Passing to the limit as $R \to \infty$ in the above inequality, using the monotone convergence theorem and (2.1), we obtain

$$\int_{\mathbb{R}^N} u^{r(x,y)}\,dx\,dy = 0,$$

which is a contradiction with the fact that u is a nontrivial solution. $\square$

In the case of constant exponents, we have the following Liouville-type theorem.

Theorem 2.3. *Let u be a non-negative weak solution of the elliptic inequality*

$$(-\Delta_x)^{\alpha/2}u^p + |x|^{2\theta}(-\Delta_y)^{\beta/2}u^q \geq u^r, \quad (x,y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},$$

where $0 < \alpha, \beta < 2$, $\theta \geq 0$, and $N = N_1 + N_2 \geq 2$. Suppose that

$$1 \leq \max\{p,q\} < r < Q \min\Big\{\frac{p}{Q-\alpha}, \frac{q}{\theta(2-\beta)+Q-\beta}\Big\}, \tag{2.9}$$

where $Q = N_1 + N_2(\theta+1)$. Then u is trivial.

Proof. Following the proof of Theorem 2.2 and taking

$$(p(x,y), q(x,y), r(x,y)) = (p,q,r), \quad (x,y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},$$

we obtain

$$A(\varphi) \leq C|\Omega| R^{N_1+N_2(\theta+1)-\frac{\alpha r}{r-p}}, \quad B(\varphi) \leq C|\Omega| R^{N_1+N_2(\theta+1)+\frac{[2\theta-\beta(\theta+1)]r}{r-q}}.$$

Using (2.6), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} u^{r(x,y)}\varphi_0^{\omega}\Big(\frac{|x|^2}{R^2} + \frac{|y|^2}{R^{2(\theta+1)}}\Big)\,dx\,dy \\ &\leq C\Big(R^{N_1+N_2(\theta+1)-\frac{\alpha r}{r-p}} + R^{N_1+N_2(\theta+1)+\frac{[2\theta-\beta(\theta+1)]r}{r-q}}\Big). \end{aligned} \tag{2.10}$$

Now, we impose the conditions

$$\begin{gathered} N_1 + N_2(\theta+1) - \frac{\alpha r}{r-p} < 0, \\ N_1 + N_2(\theta+1) + \frac{[2\theta-\beta(\theta+1)]r}{r-q} < 0, \end{gathered}$$

which are equivalent to

$$r < Q \min\Big\{\frac{p}{Q-\alpha}, \frac{q}{\theta(2-\beta)+Q-\beta}\Big\}.$$

Therefore, under the condition (2.9), passing to the limit as $R \to \infty$ in (2.10), we obtain

$$\int_{\mathbb{R}^N} u^r\,dx\,dy = 0,$$

which proves that u is trivial. $\square$

For the limit cases $\alpha \to 2^-$ and $\beta \to 2^-$, we obtain the following Liouville-type theorem.

Corollary 2.4. *Let u be a non-negative weak solution of the elliptic inequality*

$$(-\Delta_x)u^p + |x|^{2\theta}(-\Delta_y)u^q \geq u^r, \quad (x,y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},$$

where $\theta \geq 0$ and $N = N_1 + N_2 \geq 2$. Suppose that

$$1 \leq \max\{p,q\} < r < \frac{Q\min\{p,q\}}{Q-2}.$$

Then u is trivial.

The above corollary follows by taking $\alpha = \beta = 2$ in Theorem 2.3, The following Liouville-type result which was established by Dolcetta and Cutri [7] is an immediate consequence of Corollary 2.4.

Corollary 2.5. *Let u be a non-negative weak solution of the elliptic inequality*

$$(-\Delta_x)u + |x|^{2\theta}(-\Delta_y)u \geq u^r, \quad (x,y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},$$

where $\theta \geq 0$ and $N = N_1 + N_2 \geq 2$. Suppose that

$$1 < r < \frac{Q}{Q-2}.$$

Then u is trivial.

The above corollary follows by taking $p = q = 1$ in Corollary 2.4.

Remark 2.6. The obtained results in this paper can be extended to various classes of systems of elliptic inequalities including the system

$$\begin{gathered}
(-\Delta_x)^{\alpha/2} u^{p(x,y)} + |x|^{2\theta}(-\Delta_y)^{\beta/2} u^{q(x,y)} \geq v^{r(x,y)},\\
(-\Delta_x)^{\gamma/2} v^{\mu(x,y)} + |x|^{2\lambda}(-\Delta_y)^{\tau/2} v^{\sigma(x,y)} \geq u^{\xi(x,y)},
\end{gathered}$$

with appropriate functional parameters.

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