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EXISTENCE AND NONEXISTENCE OF SOLUTIONS OF ASYMPTOTICALLY LINEAR KLEIN-GORDON EQUATION

PAULO C. CARRIÃO, RAQUEL LEHRER, ANDRÉ VICENTE

Abstract. In this article we study a nonlinear Klein-Gordon equation when the nonlinear term asymptotically linear at infinity. We used the Pohozaev manifold to separate a subspace of $H^1(\mathbb{R}^N)$ on a global existence region and on a blow up region.

1. Introduction

We consider the Cauchy problem for the nonlinear Klein-Gordon equation

$$\begin{gathered} u_{tt} - \Delta u + \lambda u = a(x) f(u) \quad \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } \mathbb{R}^N, \end{gathered} \tag{1.1}$$

where $N \geq 3$; $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator; λ is a positive constant; and $a : \mathbb{R} \to \mathbb{R}^+$, $f : \mathbb{R} \to \mathbb{R}^+$, $u_0, u_1 : \mathbb{R}^N \to \mathbb{R}$ are given functions.

The motivation for studying problem (1.1) was the paper of Kaitai and Quande [11], where they studied the case $a(x) \equiv 1$ and $f(u) = u^2 + u^3$. In that case the equation $(1.1)_1$ is associated with the study of crystals dislocation. The authors proved a result of global existence and finite time blow up to the problem when $N \leq 3$.

It is know that the evolution problem (1.1) has a strong connection with the elliptic problem:

$$- \Delta u + \lambda u = a(x) f(u) \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

In fact, defining the functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{\lambda}{2} \int_{\mathbb{R}^N} u^2 \, dx - \int_{\mathbb{R}^N} a(x) F(u) dx,$$

where $F(\xi) = \int_0^\xi f(s) ds$, the critical values of I are the weak solutions of (1.2). When the known Ambrosetti-Rabinowitz condition (see [3])

$$0 < \theta F(\xi) < \xi f(\xi), \quad \text{for all } \xi \in \mathbb{R} \backslash \{0\},$$

for some $\theta > 2$ holds, it is possible to control the projection of $u \in H^1(\mathbb{R}^N)$ over the Nehari manifold

$$\mathcal{N} = \{u \in H^1(\mathbb{R}^N) \backslash \{0\}; I'(u) u = 0\},$$

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where $I'(u)u$ is the Gateaux derivative of u applied on u, and to prove that the level

$$d = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)), \tag{1.3}$$

where

$$\Gamma = \{\gamma \in C([0,1]; H^1(\mathbb{R}^N));\ \gamma(0) = 0 \text{ and } \gamma(1) = e\},$$

here $e \in H^1(\mathbb{R}^N)$; $\|e\| > r > 0$ and $\inf_{\|u\|=r} I(u) > I(0) > I(e))$, is reached over $\mathcal{N}$, i e,

$$d = \inf_{u\in\mathcal{N}} I(u).$$

The number d is a critical value of I and is called of Mountain pass level. Details can be found in Willem [20].

Therefore, it is possible defining the energy of (1.1) by

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^N} |u_t|^2 dx + I(u), \tag{1.4}$$

with appropriate assumptions on f, to prove an existence/nonexistence result when the energy at $t = 0$, i e, $E(0)$ is bellow of the mountain pass level. In many papers the Nehari manifold has an important role because it allows to "separate" one $H^1(\mathbb{R}^N)$ subset on an existence region and on a nonexistence region, see [1, 2, 6, 13, 18] and references therein. See also a more recent work of Wang [19].

On the other hand, when we do not have the Ambrosetti-Rabinowitz condition the work to get blow up results involving (1.1) can be hard. It holds when f is, for example, defined by $f(u) = \frac{u^3}{1+u^2}$. In the elliptic context, this function is a prototype of a class of nonlinearity so called *asymptotic linear at infinity* which was recently solved, in a more general context, by Lehrer and Maia [12]. The presence of $a(x)$ let the problem *nonautonomous* and it gives some technical difficulties. Quickly speaking, the authors showed that the mountain pass level is not attained. To solve the problem they worked with an alternative to the use of the Nehari manifold, namely, it was used the Pohozaev manifold:

$$\mathcal{P} = \{u \in H^1(\mathbb{R}^N)\backslash\{0\}; J(u) = 0\},$$

where $J : H^1(\mathbb{R}^N) \to \mathbb{R}$ is given by

$$\begin{aligned} J(u) = \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \frac{N\lambda}{2}\int_{\mathbb{R}^N} u^2\, dx - N\int_{\mathbb{R}^N} a(x)F(u)\, dx \\ - \int_{\mathbb{R}^N} \nabla a(x)\cdot xF(u)dx. \end{aligned}$$

The key of their paper was to show that the functional I has a critical value, above the mountain pass level, in a subset of $\mathcal{P}$.

Let $a : \mathbb{R}^N \to \mathbb{R}$ be a radial function satisfying the following assumptions

- (A1) $a \in C^2(\mathbb{R}^N, \mathbb{R}^+)$, with $\inf_{x\in\mathbb{R}^N} a(x) > 0$;
- (A2) $\lim_{|x|\to\infty} a(x) = a_\infty > \lambda$;
- (A3) $\nabla a(x)\cdot x \geq 0$, for all $x \in \mathbb{R}^N$, with the strict inequality holding on a subset of positive Lebesgue measure of $\mathbb{R}^N$;
- (A4) $a(x) + \frac{\nabla a(x)\cdot x}{N} < a_\infty$, for all $x \in \mathbb{R}^N$;
- (A5) $\nabla a(x)\cdot x + \frac{x\cdot H(x)\cdot x}{N} \geq 0$, for all $x \in \mathbb{R}^N$, where H represents the Hessian matrix of the function a.

On the nonlinearity f we assume:

(A6) $f \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\lim_{s\to 0} \frac{f(s)}{s} = 0$;
(A7) $\lim_{s\to\infty} \frac{f(s)}{s} = 1$;
(A8) if $F(s) = \int_0^s f(t)dt$ and $Q(s) = \frac{1}{2}f(s)s - F(s)$, then there exists a constant $D \geq 1$ such that

$$0 < Q(s) \leq D\, Q(t), \quad \text{for all } 0 < s \leq t,$$
$$\lim_{s\to\infty} Q(s) = +\infty.$$

We observe that the assumptions (A1)–(A8) are the same of [12] and the condition (A8) was introduced by Jeanjean and Tanaka [9]. Note that (A8) is more general than the usual assumption that $f(s)/s$ begin an increasing function of $s > 0$. In particular, if f is differentiable, then $f(s)/s$ is increasing if and only if (A8) holds with $D = 1$. We extend f to $\mathbb{R}^-$ by $f(s) = 0$ if $s < 0$.

Conditions (A6) and (A7) imply that, given $\varepsilon > 0$ and $2 \leq p \leq p^* := \frac{2N}{N-2}$, there exists a positive constant $C = C(\varepsilon, p)$ such that for all s in $\mathbb{R}$

$$|F(s)| \leq \frac{\varepsilon}{2}|s|^2 + C|s|^p \,. \tag{1.5}$$

Therefore (1.1) is asymptotic linear at infinity and nonautonomous. As we do not have the Ambrosetti-Rabinowitz condition then the Nehari manifold is not appropriate, therefore we used the Pohozaev manifold to find an existence and a nonexistence region. Precisely, defining

$$W_1 = \{u \in H^1_{\rm rad}(\mathbb{R}^N)\backslash\{0\}; I(u) < c \text{ and } J(u) > 0\} \cup \{0\},$$
$$W_2 = \{u \in H^1_{\rm rad}(\mathbb{R}^N)\backslash\{0\}; I(u) < c \text{ and } J(u) < 0\},$$

where c is the mountain pass level and will be defined posteriorly, we proved that when the initial data is taken in W_1, the problem (1.1) has a global solution which there exist for all $t \geq 0$. Moreover, if the initial data is in W_2, then the solution blow up (in finite or infinite time).

We also would like to cite the classical paper of Shatah [16], where $a \equiv 1$, f satisfies the Berestycki-Lions assumptions and $N \geq 3$. In this case the author also used the Pohozaev manifold. The work of Shatah was extended to $N = 2$ by Jeanjean and Le Coz [8].

The goal of our paper is to prove an existence/nonexistence result to (1.1) when a and f satisfy (A1)-(A5) and (A6)-(A8), respectively. This work extends, in a sence, the results of [11, 16] to an other class of nonlinearities. Our paper is organized as follows: in Section 2 we give the notations, the preliminaries and we stablish a linear existence theorem, which is analogous to Serrin, Todorova and Vitillaro [15, Theorem 3]. In Section 3 we prove the main result.

2. Preliminaries

The norms in $L^2(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$ are denoted, respectively, by

$$\|u\|_2 = \Big(\int_{\mathbb{R}^N} |u(x)|^2\,dx\Big)^{1/2}, \quad \|u\|_\lambda = \Big(\lambda\int_{\mathbb{R}^N} |u(x)|^2\,dx + \int_{\mathbb{R}^N} \nabla u\cdot\nabla u\,dx\Big)^{1/2},$$

here $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ is the gradient operator in spatial variable and $\cdot$ is the usual scalar product in $\mathbb{R}^N$.

To prove our blow up result we need two technical lemmas which are connected with the autonomous elliptic problem:

$$-\Delta u + \lambda u = a_\infty f(u) \quad \text{in } \mathbb{R}^N, \tag{2.1}$$

where the constant a_∞ was given in (A2). Associated to (2.1) we have the Pohozaev manifold

$$\mathcal{P}_\infty = \{u \in H^1(\mathbb{R}^N)\backslash\{0\}; J_\infty(u) = 0\},$$

where

$$J_\infty(u) = \frac{(N-2)}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\lambda N}{2}\int_{\mathbb{R}^N} u^2\, dx - N\int_{\mathbb{R}^N} a_\infty F(u) dx.$$

Define the functional I_∞ associated with (2.1) by

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \lambda u^2 - \int_{\mathbb{R}^N} a_\infty F(u) dx,$$

and the mountain pass level

$$c_\infty = \min_{\gamma\in\Gamma_\infty} \max_{0\le t\le 1} I_\infty(\gamma(t)),$$

where the set of paths is given by

$$\Gamma_\infty = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) | \gamma(0) = 0, I_\infty(\gamma(1)) < 0\}.$$

We have the following result which show that, in the autonomous case, the mountain pass level is the minimum of I_∞ over $\mathcal{P}_\infty$.

Lemma 2.1. *Let φ_∞ be the ground state solution of the autonomous elliptic problem. Then*

$$I_\infty(\varphi_\infty) = c_\infty = \min_{v\in\mathcal{P}_\infty} I_\infty(v) > 0.$$

For a proof of the above lemma, see Jeanjean and Tanaka [10, Lemma 3.1].

Let I, E, J, W_1 and W_2 as in the preview section and c the mountain pass level, associated to (1.1), defined in (1.3), namely,

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)).$$

The following lemma establishes that the mountain pass level of the nonautonomous problem is attained in the Pohozaev manifold and it is the same level of the autonomous problem.

Lemma 2.2. *It holds that*

$$c = c_\infty = \inf_{u\in\mathcal{P}} I(u).$$

For a proof of the above lemma, see [12, lemmas 3.13 and 4.2]. We will need one more characterization to the level c which is given by the next lemma.

Lemma 2.3. *We have*

$$c = \min\{T_\infty(v);\ J_\infty(v) \le 0\},$$

where $T_\infty(v) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v|^2\, dx$.

Proof. We observe that, for all $u \in H^1(\mathbb{R}^N)$,

$$I_\infty(u) = T_\infty(u) + \frac{1}{N} J_\infty(u).$$

Let $v \in H^1(\mathbb{R}^N)$ be such that $J_\infty(v) \leq 0$. If $J_\infty(v) = 0$, then $v \in \mathcal{P}_\infty$ and $I_\infty(u) = T_\infty(u)$. Therefore

$$\inf_{J_\infty(v)=0} T_\infty(v) = \inf_{J_\infty(v)=0} I_\infty(v) = \inf_{v\in\mathcal{P}_\infty} I_\infty(v) = c, \tag{2.2}$$

where in the last equality we used Lemmas 2.1 and 2.2. If $J_\infty(v) < 0$, then for $\beta > 0$ define $v_\beta(x) = v(\frac{x}{\beta})$. Thus

$$\begin{aligned} J_\infty(v_\beta) = \beta^{N-2}\Big[\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v(x)|^2 dx \\ - N\beta^2\Big(\int_{\mathbb{R}^N} a_\infty F(v(x))dx + \frac{\lambda N}{2}\int_{\mathbb{R}^N} |v(x)|^2 dx\Big)\Big]. \end{aligned}$$

Note that for $\beta = 1$ we have $J_\infty(v_1) = J_\infty(v) < 0$ and for $\beta > 0$ sufficiently small $J_\infty(v_\beta) > 0$. Therefore, there exists $\beta_0 \in (0,1)$ such that $J_\infty(v_{\beta_0}) = 0$. Thus, we infer that

$$\begin{aligned} I_\infty(v_{\beta_0}) = T_\infty(v_{\beta_0}) = \frac{\beta_0^{N-2}}{N}\int_{\mathbb{R}^N} |\nabla v(x)|^2 dx \\ < \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v(x)|^2 dx = T_\infty(v). \end{aligned} \tag{2.3}$$

Taking into account (2.2) and (2.3) we complete the proof. □

To prove the existence of solution we will need a theorem that gives us the existence of solution to a linear problem. The next lemma establishes this result and its proof is analogous to Serrin, Todorova and Vitillaro [15, Theorem 3].

Lemma 2.4. *Let $u_0 \in H^1(\mathbb{R}^N)$, $u_1 \in L^2(\mathbb{R}^N)$ and $g \in L^2(\mathbb{R}^N \times (0,T))$ be given functions. Then, for all $T > 0$ there exists a unique weak solution, $u : \mathbb{R}^N \times (0,T) \to \mathbb{R}$, of the linear problem*

$$\begin{gathered} u_{tt} - \Delta u + \lambda u = g(x,t) \quad \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x) \quad \text{in } \mathbb{R}^N, \end{gathered} \tag{2.4}$$

in the class

$$u \in C([0,T]; H^1(\mathbb{R}^N)), \quad u_t \in C([0,T]; L^2(\mathbb{R}^N)).$$

Moreover, the energy identity satisfies

$$E_l(t) - E_l(s) = \int_s^t \int_{\mathbb{R}^N} g(x,t) u_t(x,t)\, dxdt, \tag{2.5}$$

for all $0 \leq s \leq t$, where

$$E_l(t) = \frac{1}{2}\left[\int_{\mathbb{R}^N} |u_t(x,t)|^2 dx + \int_{\mathbb{R}^N} |\nabla u(x,t)|^2 dx + \lambda\int_{\mathbb{R}^N} |u(x,t)|^2 dx\right].$$

3. Main results

Theorem 3.1 (Local Existence). *Suppose that assumptions* (A1)-(A8) *hold. Then for each set of initial conditions* $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ *there exists a* $T > 0$ *such that the problem* (1.1) *has a unique weak solution,* $u : \mathbb{R}^N \times (0,T) \to \mathbb{R}$*, in the class*

$$u \in C([0,T]; H^1(\mathbb{R}^N)) \cap C^1([0,T]; L^2(\mathbb{R}^N)).$$

Proof. Define de set

$$X_T = \{u \in C([0,T]; H^1(\mathbb{R}^N)) \cap C^1([0,T]; L^2(\mathbb{R}^N))\}$$

endowed with the norm

$$\|u\|^2 = \|u\|^2_{L^\infty(0,T;H^1(\mathbb{R}^N))} + \|u_t\|^2_{L^\infty(0,T;L^2(\mathbb{R}^N))}$$

and, for each $\alpha > 0$, we consider the set

$$X_{T,\alpha} = \{u \in X_T; \|u\| \le \alpha,\ u(0) = u_0 \text{ and } u_t(0) = u_1\}.$$

Define $g : X_{T,\alpha} \to L^2(\mathbb{R}^N \times (0,T))$ by $g(v) = f(v)$.

For each $v \in X_{T,\alpha}$ let u be the solution of

$$\begin{gathered} u_{tt} - \Delta u + \lambda u = g(v) \quad \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \quad \text{in } \mathbb{R}^N \end{gathered} \tag{3.1}$$

given by Lemma 2.4. Now, we show that for α large enough and T small enough $u \in X_{T,\alpha}$. The energy identity (2.5) gives us

$$E_l(t) = E_l(0) + \int_0^t \int_{\mathbb{R}^N} f(v) u_t \, dx d\xi. \tag{3.2}$$

As $|f(s)| \le Cs$, for all s, then

$$\Big| \int_0^t \int_{\mathbb{R}^N} f(v) u_t \, dx\, dt \Big| \le C \int_0^t \int_{\mathbb{R}^N} |v||u_t| \, dx d\xi \le C \int_0^t \|v(\xi)\|_2 \|u_t(\xi)\|_2 \, d\xi,$$

but $v \in X_{T,\alpha}$, therefore

$$\Big| \int_0^t \int_{\mathbb{R}^N} f(v) u_t \, dx dt \Big| \le \sqrt{2} C\alpha \int_0^t E_l^{1/2}(\xi) \, d\xi. \tag{3.3}$$

Putting (3.3) into (3.2) we obtain

$$\frac{E_l(t)}{2} \le \frac{E_l(0)}{2} + \frac{C\alpha}{\sqrt{2}} \int_0^t E_l^{1/2}(\xi) \, d\xi. \tag{3.4}$$

Applying Lemma A.5 in Brezis [5, page 157], we conclude that

$$E_l^{1/2}(t) \le E_l^{1/2}(0) + \frac{C\alpha}{\sqrt{2}} T, \quad \text{for all } t \in [0,T]. \tag{3.5}$$

Choosing $\alpha > \sqrt{2} E_l(0)^{1/2}$ and, posteriorly, $T < \frac{\alpha - \sqrt{2} E_l^{1/2}(0)}{C\alpha}$ from (3.5) and using the definition of E_l we obtain

$$\Big(\|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \lambda \|u(t)\|_2^2 \Big)^{1/2} \le \alpha,$$

i e, $u \in X_{T,\alpha}$. This allows us to define the application $\Phi : X_{T,\alpha} \to X_{T,\alpha}$ by $\Phi(v) = u$.

We will prove that for $T > 0$ sufficiently small Φ is a contraction. In fact, consider $v_1, v_2 \in X_{T,\alpha}$ and denote $u_1 = \Phi(v_1)$ and $u_2 = \Phi(v_2)$. Define also $z = u_1 - u_2$ the unique solution of

$$\begin{gathered} z_{tt} - \Delta z + \lambda z = g(u_1) - g(u_2) \quad \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = 0, \quad u_t(x,0) = 0 \quad \text{in } \mathbb{R}^N. \end{gathered} \tag{3.6}$$

The energy identity (2.5) gives

$$\frac{1}{2}\Big(\|z_t(t)\|_2^2 + \|z(t)\|_\lambda^2\Big) = \int_0^t \int_{\mathbb{R}^N} (f(v_1) - f(v_2)) z_t \, dx \, d\xi. \tag{3.7}$$

From the assumptions over f and Hölder inequality we obtain

$$\int_{\mathbb{R}^N} (f(v_1) - f(v_2)) z_t \, dx \le C \int_{\mathbb{R}^N} |v_1 - v_2||z_t| dx \le C\|v_1(t) - v_2(t)\|_2 \|z_t(t)\|_2.$$

Combining this with (3.7) we obtain

$$\frac{1}{2}\Big(\|z_t(t)\|_2^2 + \|z(t)\|_\lambda^2\Big) \le C\|v_1 - v_2\| \Big(\|z_t(t)\|_2^2 + \|z(t)\|_\lambda^2\Big)^{1/2}. \tag{3.8}$$

The inequality (3.8) and Brezis [5, Lemma A.5] gives

$$\Big(\|z_t(t)\|_2^2 + \|z(t)\|_\lambda^2\Big)^{1/2} \le C\alpha T\|v_1 - v_2\|,$$

and since $z = u_1 - u_2 = \Phi(v_1) - \Phi(v_2)$ it follows that

$$\|\Phi(v_1) - \Phi(v_2)\| \le C\alpha T\|v_1 - v_2\|.$$

Taking $T > 0$ sufficiently small we conclude that Φ is a contraction. Therefore, Φ has a unique fixed point which is the solution of (1.1). $\square$

Let u be the local solution of (1.1) given by Theorem 3.1. From the linear energy identity (2.5) we have the identity

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + I(u(t)) = E(0), \tag{3.9}$$

for all t in the interval of existence of solution u, where E was defined in (1.4).

We say that a subset V of $H^1(\mathbb{R}^N)$ is an *invariant region* for the solution of (1.1) when if the initial data $u_0 \in V$, then the solution of (1.1) is in V.

Define a subset of $H^1(\mathbb{R}^N)$:

$$W = \{u \in H^1_{\rm rad}(\mathbb{R}^N); I(u) < c\}.$$

Lemma 3.2. *Under the assumption $E(0) < c$, we have that $W = W_1 \cup W_2$ and W_1, W_2 are invariant regions for the solutions of* (1.1).

Proof. We note that if $J(u) = 0$, then $u \in \mathcal{P}$. Thus $I(u) \ge \inf_{u\in\mathcal{P}} I(u) = c$. From this, for all $u \in W$ we can conclude that $J(u) \ne 0$, i.e., $u \in W_1$ or $u \in W_2$, thus $W = W_1 \cup W_2$.

Now, let $u_0 \in W_1$ and u be the solution of (1.1) associated to u_0. From (3.9) and the assumption $E(0) < c$ we obtain

$$I(u(t)) \le \frac{1}{2}\|u_t(t)\|_2^2 + I(u(t)) = E(0) < c,$$

therefore $u(t) \in W$ for all t in the interval of existence of solution. We affirm that $u(t) \in W_1$ for all t in the interval of existence of the solution. If it is not hold, then there exists $t_0 > 0$ such that $u(t_0) \notin W_1$, therefore, as $W = W_1 \cup W_2$ we have

$u(t_0) \in W_2$. From the definition of W_1 and W_2 there exists $t^* \in (0, t_0)$ such that $J(u(t^*)) = 0$. If $u(t^*) \neq 0$, then $u(t^*) \in \mathcal{P}$ and this implies

$$I(u(t^*)) \geq \inf_{u \in \mathcal{P}} I(u) = c,$$

therefore $u(t^*) \notin W$, which is a contradiction. If $u(t^*) = 0$ then

$$\lim_{t \to t^{*+}} (\|\nabla u(t)\|_2^2 + \lambda \|u(t)\|_2^2) = 0. \tag{3.10}$$

and

$$J(u(t)) < 0, \text{ for all } t^* < t \leq t_0 \tag{3.11}$$

From the definition of J and (3.11) we have

$$\begin{aligned} &\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{N\lambda}{2} \int_{\mathbb{R}^N} |u|^2 dx \\ &< N \int_{\mathbb{R}^N} a(x) F(u) dx + \int_{\mathbb{R}^N} \nabla a(x) \cdot x F(u) dx, \end{aligned}$$

for all $t^* < t \leq t_0$. This inequality, (A4) and (1.5) imply that given $\varepsilon > 0$ and $2 \leq p \leq 2^*$ such that

$$\|u(t)\|_\lambda^2 \leq \frac{\varepsilon}{2} \int_{\mathbb{R}^N} |u|^2 dx + c(\varepsilon, p) \int_{\mathbb{R}^N} |u|^p dx.$$

Thus

$$\|u(t)\|_\lambda^2 \leq C \|u(t)\|_{L^p(\mathbb{R}^N)}^p \leq C \|u(t)\|_\lambda^p,$$

consequently

$$\frac{1}{C} \leq \|u(t)\|_\lambda^{p-2},$$

but this and (3.10) give us a contradiction. Therefore $u(t) \in W_1$, for all t in the interval of existence of solution. The proof for W_2 is analogous. $\square$

Theorem 3.3 (Global solution)**.** *Suppose that* (A1)–(A8) *are satisfied and that* $(u_0, u_1) \in W_1 \times L^2(\mathbb{R}^N)$ *and* $E(0) < c$. *Then the local solution given by Theorem 3.1 can be extended for all* $t > 0$.

Proof. It is sufficient to estimate the $H^1(\mathbb{R}^N)$ norm. We observe that

$$\|\nabla u(t)\|_2^2 + J(u) + \int_{\mathbb{R}^N} \nabla a(x) \cdot x F(u) dx = N I(u(t)). \tag{3.12}$$

As $u_0 \in W_1$, then for all t, $u(t) \in W_1$ and $J(u(t)) > 0$. Therefore, from (3.12) we have

$$\|\nabla u(t)\|_2^2 < N I(u(t)) - \int_{\mathbb{R}^N} \nabla a(x) \cdot x F(u) dx < Nc, \tag{3.13}$$

for all $t \geq 0$. By Sobolev, Gagliardo, Nirenberg inequality there exist $C > 0$ such that

$$\|u(t)\|_{2^*} < C, \tag{3.14}$$

for all $t \geq 0$. Using (1.5) for all $\varepsilon > 0$ we have

$$\int_{\mathbb{R}^N} a(x) F(u) dx \leq \frac{\|a\|_\infty \varepsilon}{2} \|u(t)\|_2^2 + C(\varepsilon) \|u(t)\|_{2^*}^{2^*}.$$

This allows us to conclude that

$$\begin{aligned} I(u(t)) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\lambda}{2}\int_{\mathbb{R}^N} |u|^2 dx - \frac{\|a\|_\infty \varepsilon}{2}\|u(t)\|_2^2 - C(\varepsilon)\|u(t)\|_{2^*}^{2^*} \\ &\le \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\lambda}{2}\int_{\mathbb{R}^N} |u|^2 dx - \int_{\mathbb{R}^N} a(x)F(u)dx. \end{aligned} \tag{3.15}$$

Using the assumption we obtain

$$I(u(t)) < E(t) = E(0) < c. \tag{3.16}$$

From (3.12) and (3.16) we have

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\lambda}{2}\int_{\mathbb{R}^N} |u|^2 dx - \frac{\|a\|_\infty \varepsilon}{2}\|u(t)\|_2^2 - C(\varepsilon)\|u(t)\|_{2^*}^{2^*} < c$$

This inequality, (3.13), (3.14) allow us to infer that

$$\Big(\frac{\lambda}{2} - \frac{\|a\|_\infty \varepsilon}{2}\Big)\|u(t)\|_2^2 \le C.$$

Choosing $\varepsilon > 0$ such that $\varepsilon < \frac{\lambda}{\|a\|_\infty}$, we conclude that $\|u(t)\|_2^2$ is bounded, this and (3.13) give the result. $\square$

The next step is to prove the blow up result, for which we need some auxiliary results.

Lemma 3.4. *Let $(u_0, u_1) \in W_2 \times L^2(\mathbb{R}^N)$ be such that $E(0) < c$ and $u(t)$ the associated solution of (1.1) defined in $[0, T)$. Then there exists $\delta > 0$ such that $J(u(t)) < -\delta$, for all $t \in [0, T)$.*

Proof. Since $u_0 \in W_2$ then $u(t) \in W_2$, for all $t \in (0, T)$, this implies

$$J_\infty(u(t)) \le J(u(t)) < 0, \text{ for all } t \in (0, T).$$

On the other hand, it is easy to see that

$$\frac{1}{N}\int_{\mathbb{R}^N} |\nabla u(t)|^2 dx = I(u(t)) - \frac{J(u(t))}{N} - \frac{1}{N}\int_{\mathbb{R}^N} \nabla a(x)\cdot xF(u(t))\,dx.$$

As $I(u(t)) \le E(t) = E(0)$ and $\nabla a(x)\cdot xF(u(t)) \ge 0$ we have

$$\frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^2 dx \le E(0) - \frac{J(u(t))}{N} = c - \nu - \frac{J(u(t))}{N}, \tag{3.17}$$

where $\nu := c - E(0)$ is a positive constant, since $E(0) < c$. Suppose that does not exist δ satisfying the lemma. Then there exists a sequence $(t_k)_{k\in\mathbb{N}} \subset (0, T)$ such that $J(u(t_k)) \to 0$, when $k \to \infty$. Therefore, for k large enough, we have

$$-\frac{\nu N}{2} < J(u(t_k)) \le 0. \tag{3.18}$$

From (3.17) and (3.18) we obtain

$$T_\infty(u(t_k)) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u(t_k)|^2 dx \le c - \frac{\nu}{2},$$

for k large enough, but this contradicts the Lemma 2.3. $\square$

For each $\varepsilon > 0$, define $\Phi_\varepsilon : \mathbb{R} \to \mathbb{R}$ by

$$\Phi_\varepsilon(r) = \begin{cases} N & \text{if } 0 \le r \le \exp(\frac{1}{\varepsilon}), \\ 2N - N\varepsilon ln(r) & \text{if } \exp(\frac{1}{\varepsilon}) < r \le \exp(\frac{2}{\varepsilon}), \\ 0 & \text{if } r > \exp(\frac{2}{\varepsilon}) \end{cases}$$

and $\Psi_\varepsilon : \mathbb{R} \to \mathbb{R}$ by

$$\Psi_\varepsilon(r) = \frac{1}{r^{N-1}} \int_0^r s^{N-1} \Phi_\varepsilon(s) ds.$$

The next lemma summarizes some properties of Φ_ε and Ψ_ε.

Lemma 3.5. *For each $\varepsilon > 0$, we have*

$$\Phi_\varepsilon(r) = N, \quad \Psi_\varepsilon(r) = r, \quad 0 \le r \le \exp(\frac{1}{\varepsilon}); \tag{3.19}$$

$$\Psi'_\varepsilon(r) + \frac{N-1}{r} \Psi_\varepsilon(r) = \Phi_\varepsilon(r), \quad \text{for all } r \ge 0; \tag{3.20}$$

$$\|\Psi'_\varepsilon - \frac{1}{r} \Psi_\varepsilon\|_{L^\infty} < \varepsilon; \tag{3.21}$$

$$|\Phi_\varepsilon(r)| \le k, \quad \Psi'_\varepsilon(r) \le 1, \quad \text{for all } r \ge 0; \tag{3.22}$$

$$\Big(\frac{r^{N-1}}{N-1} \Psi_\varepsilon(r) \Big)' = \frac{r^{N-1}}{N-1} \Phi_\varepsilon(r), \quad \text{for all } r \ge 0. \tag{3.23}$$

For a proof of the above lemma see Ohta and Todorova [14] and Jeanjean and Le Coz [8].

Lemma 3.6. *Let $(u_0, u_1) \in W_2 \times L^2(\mathbb{R}^N)$ be such that $E(0) < c$ and $u(t)$ the associated solution of (1.1) defined in $[0, \infty)$. If there exists a constant $K > 0$ such that $\|u(t)\|_\lambda \le K$, then*

$$\nu t \le C(1 + \|u_t(t)\|_2 \|\nabla u(t)\|_2),$$

for all $t \in [0, \infty)$, where ν and C are positive constants.

Proof. As $u_0 \in W_2$, by the uniqueness of solution, $u(t)$ is a radial function for all $t \ge 0$, namely, $u(x,t) = u(r,t)$, where $r = |x|$. Therefore, equation $(1.1)_1$ becomes

$$u_{tt} - \frac{N-1}{r} u' - u'' + \lambda u = a f(u). \tag{3.24}$$

Multiplying (3.24) by $\frac{\Psi_\varepsilon(r) u' r^{N-1}}{N-1}$ and integrating over $[0, \infty)$ we obtain

$$\begin{aligned} &\int_0^\infty u_{tt} u' \Psi_\varepsilon(r) \frac{r^{N-1}}{N-1} dr - \int_0^\infty |u'|^2 \Psi_\varepsilon(r) r^{N-2} dr \\ &- \int_0^\infty u'' u' \Psi_\varepsilon(r) \frac{r^{N-1}}{N-1} dr + \lambda \int_0^\infty u u' \Psi_\varepsilon(r) \frac{r^{N-1}}{N-1} dr \\ &= \int_0^\infty a(r) f(u) u' \Psi_\varepsilon(r) \frac{r^{N-1}}{N-1} dr. \end{aligned} \tag{3.25}$$

Now estimate the terms of the above equation.

$$\begin{aligned}
&\int_0^\infty u_{tt}u'\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr\\
&=\frac{\partial}{\partial t}\int_0^\infty u_t u'\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr-\int_0^\infty u_t u_t'\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr\\
&=\frac{\partial}{\partial t}\int_0^\infty u_t u'\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr-\frac{1}{2}\int_0^\infty (u_t^2)'\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr\\
&=\frac{\partial}{\partial t}\int_0^\infty u_t u'\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr+\frac{1}{2}\int_0^\infty u_t^2\Phi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr;
\end{aligned}\tag{3.26}$$

$$\begin{aligned}
&-\int_0^\infty u''u'\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr\\
&=-\frac{1}{2}\int_0^\infty (|u'|^2)'\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr\\
&=\frac{1}{2}\int_0^\infty |u'|^2\Big(\Psi_\varepsilon'(r)\frac{r^{N-1}}{N-1}+\Psi_\varepsilon(r)r^{N-2}\Big)dr;
\end{aligned}\tag{3.27}$$

$$\begin{aligned}
\lambda\int_0^\infty uu'\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr&=\frac{\lambda}{2}\int_0^\infty (u^2)'\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr\\
&=-\frac{\lambda}{2}\int_0^\infty u^2\Phi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr;
\end{aligned}\tag{3.28}$$

$$\begin{aligned}
&\int_0^\infty a(r)F(u)\Phi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr\\
&=\int_0^\infty a(r)F(u)\Big(\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}\Big)'dr\\
&=-\int_0^\infty (a'(r)F(u)+a(r)f(u)u')\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr.
\end{aligned}$$

From here,

$$\begin{aligned}
&\int_0^\infty a(r)f(u)u'\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr\\
&=-\int_0^\infty a(r)F(u)\Phi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr-\int_0^\infty a'(r)F(u)\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr.
\end{aligned}\tag{3.29}$$

Substituting (3.26)–(3.29) in (3.25) we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_0^\infty u_t u'\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr\\
&+\frac{1}{2}\int_0^\infty u_t^2\Phi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr-\int_0^\infty |u'|^2\Psi_\varepsilon(r)r^{N-2}dr\\
&+\frac{1}{2}\int_0^\infty |u'|^2\Big(\Psi_\varepsilon'(r)\frac{r^{N-1}}{N-1}+\Psi_\varepsilon(r)r^{N-2}\Big)dr-\frac{\lambda}{2}\int_0^\infty u^2\Phi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr\\
&=-\int_0^\infty a(r)F(u)\Phi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr-\int_0^\infty a'(r)F(u)\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr.
\end{aligned}\tag{3.30}$$

Define

$$
\begin{aligned}
J_T(u) = \frac{N-2}{2N}\int_0^\infty |u'|^2\Phi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr - \int_0^\infty a(r)F(u)\Phi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr \\
+ \frac{\lambda}{2}\int_0^\infty u^2\Phi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr - \frac{1}{N}\int_0^\infty a'(r)rF(u)\Phi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr.
\end{aligned} \tag{3.31}
$$

From (3.30) and (3.31) we have

$$
\begin{aligned}
&-\frac{\partial}{\partial t}\int_0^\infty u_t u'\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr \\
&= \underbrace{\frac{1}{2}\int_0^\infty u_t^2\Phi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr}_{\geq 0} - \int_0^\infty |u'|^2\Psi_\varepsilon(r)r^{N-2}dr \\
&\quad + \frac{1}{2}\int_0^\infty |u'|^2\Big(\Psi_\varepsilon'(r)\frac{r^{N-1}}{N-1} + \Psi_\varepsilon(r)r^{N-2}\Big)dr \\
&\quad + \frac{N-2}{2N}\int_0^\infty |u'|^2\Phi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr \\
&\quad - \frac{1}{N}\int_0^\infty a'(r)rF(u)\Phi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr \\
&\quad + \int_0^\infty a'(r)F(u)\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr - J_T(u).
\end{aligned} \tag{3.32}
$$

As

$$
\begin{aligned}
&\int_0^\infty |u'|^2\Phi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr \\
&= \int_0^\infty |u'|^2\Big(\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}\Big)'dr \\
&= \int_0^\infty |u'|^2\Big(\Psi_\varepsilon'(r)\frac{r^{N-1}}{N-1} + \Psi_\varepsilon(r)r^{N-2}\Big)dr,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&-\int_0^\infty |u'|^2\Psi_\varepsilon(r)r^{N-2}dr + \frac{1}{2}\int_0^\infty |u'|^2\Big(\Psi_\varepsilon'(r)\frac{r^{N-1}}{N-1} + \Psi_\varepsilon(r)r^{N-2}\Big)dr \\
&+ \frac{N-2}{2N}\int_0^\infty |u'|^2\Phi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr \\
&= \frac{1}{N}\int_0^\infty |u'|^2\Big(\Psi_\varepsilon'(r) - \frac{1}{r}\Psi_\varepsilon(r)\Big)r^{N-1}dr.
\end{aligned}
$$

From this identity, the assumption $\|u(t)\|_\lambda \leq K$ and (3.21) we conclude that

$$
\begin{aligned}
&\Big| -\int_0^\infty |u'|^2\Psi_\varepsilon(r)r^{N-2}dr + \frac{1}{2}\int_0^\infty |u'|^2\Big(\Psi_\varepsilon'(r)\frac{r^{N-1}}{N-1} + \Psi_\varepsilon(r)r^{N-2}\Big)dr \\
&+ \frac{N-2}{2N}\int_0^\infty |u'|^2\Phi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr\Big| \leq \varepsilon,
\end{aligned}
$$

$$\begin{aligned} &-\int_0^\infty |u'|^2\Psi_\varepsilon(r)r^{N-2}dr+\frac{1}{2}\int_0^\infty |u'|^2\Big(\Psi_\varepsilon'(r)\frac{r^{N-1}}{N-1}+\Psi_\varepsilon(r)r^{N-2}\Big)dr \\ &+\frac{N-2}{2N}\int_0^\infty |u'|^2\Phi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr\geq -\varepsilon. \end{aligned}\tag{3.33}$$

On the other hand,

$$\begin{aligned} &-\frac{1}{N}\int_0^\infty a'(r)rF(u)\Phi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr+\int_0^\infty a'(r)F(u)\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr \\ &-\frac{1}{N}\int_0^\infty a'(r)rF(u)\Big(\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}\Big)'dr+\int_0^\infty a'(r)F(u)\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr \\ &=\frac{1}{N-1}\int_0^\infty a'(r)rF(u)\Big(\frac{1}{r}\Psi_\varepsilon(r)-\frac{1}{N}\Psi_\varepsilon'(r)-\frac{N-1}{rN}\Psi_\varepsilon(r)\Big)r^{N-1}dr \\ &=-\frac{1}{N(N-1)}\int_0^\infty a'(r)rF(u)\Big(\Psi_\varepsilon'(r)-\frac{1}{r}\Psi_\varepsilon(r)\Big)r^{N-1}dr. \end{aligned}$$

Using the assumptions that $a'(r)r\to 0$, when $r\to\infty$, $|F(u)|\leq Cu^2$, $\|u(t)\|_\lambda\leq K$ and (3.21), we obtain

$$\Big|-\frac{1}{N}\int_0^\infty a'(r)rF(u)\Phi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr+\int_0^\infty a'(r)F(u)\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr\Big|<\varepsilon,$$

or

$$-\frac{1}{N}\int_0^\infty a'(r)rF(u)\Phi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr+\int_0^\infty a'(r)F(u)\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr>-\varepsilon.\tag{3.34}$$

Substituting (3.33) and (3.34) in (3.32) we have

$$-\frac{\partial}{\partial t}\int_0^\infty u_tu'\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr>-J_T(u)-2\varepsilon.\tag{3.35}$$

We know that $J_T(u)\to J(u)$, when $\varepsilon\to 0$, therefore for all $\mu>0$, there exists $\gamma>0$ such that $|J_T(u)-J(u)|<\mu$, when $\varepsilon<\gamma$. From here,

$$-J(u)-\mu<-J_T(u),\quad \text{when } \varepsilon<\gamma.\tag{3.36}$$

Inequalities (3.35) and (3.36) give us

$$-\frac{\partial}{\partial t}\int_0^\infty u_tu'\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr>-J(u)-\mu-2\varepsilon.$$

Taking $\mu=\delta/4$, where δ was given by Lemma 3.4, $\varepsilon=\min\{\frac{\delta}{8},\frac{\gamma}{2}\}$ and using Lemma 3.4 we obtain

$$-\frac{\partial}{\partial t}\int_0^\infty u_tu'\Psi_\varepsilon(r)\frac{r^{N-1}}{N-1}dr>\frac{\delta}{2},$$

integrating over $(0,\infty)$ we conclude that

$$\delta t<C(1+\|u_t(t)\|_2\|\nabla u(t)\|_2),$$

for all $t\geq 0$, this inequality allows us to complete the proof. □

Theorem 3.7 (Blow up)**.** *Suppose that* (A1)–(A8) *hold. Then for all* $(u_0,u_1)\in W_2\times L^2(\mathbb{R}^N)$ *such that* $E(0)<c$ *there exists* $0<T\leq\infty$ *and a unique function* $u:\mathbb{R}^N\times[0,T)\to\mathbb{R}$ *solution of* (1.1) *in the class*

$$u\in C([0,T];H^1(\mathbb{R}^N))\cap C^1([0,T];L^2(\mathbb{R}^N)).$$

such that $u(t)\in W_2$ *for all* $t\in(0,T)$ *and either*

(a) *the solution exists locally, i.e.* $T < \infty$, *and there exists a sequence* $(t_k)_{k\in\mathbb{N}} \subset (0, T)$ *with* $t_k \to T^-$ *such that*
$$\|u(t_k)\|_\lambda \to \infty, \quad \text{when } t_k \to T^-;$$
or

(b) *the solution exists globally on* $[0, \infty)$ *and there exists a sequence* $(t_k)_{k\in\mathbb{N}} \subset (0, \infty)$ *with* $t_k \to \infty$ *such that*
$$\|u(t_k)\|_\lambda \to \infty, \quad \text{when } t_k \to \infty.$$

Proof. The blow up result of the item (a) is a consequence of $T < \infty$, see Georgiev and Todorova [7] and Segal [17]. Suppose that $T = \infty$. We will prove by contradiction. Suppose that there exists a constant $k_1 > 0$ such that
$$\lambda\|u(t)\|_2^2 + \|\nabla u(t)\|_2^2 \le \|u(t)\|_\lambda^2 \le k_1 \tag{3.37}$$
for all $t \ge 0$. By the identity (3.9) we have
$$\frac{1}{2}\|u_t(t)\|_2^2 = E(0) - \frac{1}{2}\|u(t)\|_\lambda^2 + \int_{\mathbb{R}^N} a(x)F(u)dx. \tag{3.38}$$
From (3.37) and (3.38) we conclude that there exists a constant $k_2 > 0$ such that
$$\frac{1}{2}\|u_t(t)\|_2^2 \le k_2, \tag{3.39}$$
for all $t \ge 0$. But (3.37) and (3.39) give a contradiction with Lemma 3.6. □

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Paulo C. Carrião
Universidade Federal de Minas Gerais - ICEX - Matemática, CEP: 31270-901, Belo Horizonte, MG, Brazil
E-mail address: pauloceca@gmail.com

Raquel Lehrer
Universidade Estadual do Oeste do Paraná - CCET, Rua Universitária, 2069, Jd. Universitário, CEP: 85819-110, Cascavel, PR, Brazil
E-mail address: rlehrer@gmail.com

André Vicente
Universidade Estadual do Oeste do Paraná - CCET, Rua Universitária, 2069, Jd. Universitário, CEP: 85819-110, Cascavel, PR, Brazil
E-mail address: andre.vicente@unioeste.br