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ON RECENT DEVELOPMENTS TREATING THE EXACT CONTROLLABILITY OF ABSTRACT CONTROL PROBLEMS

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Abstract. In this note we comment on an error present in the recent and extensive literature on exact controllability of abstract control differential problems. Our observations are valid for first order, second order, integro-differential and fractional differential control problems.

1. Introduction

In this short note we point out an error in the recent literature on controllability of abstract control differential problems. To this end, we establish a simple result on lack of exact controllability for abstract integral control problems of the form

$$x(t) = \mathcal{R}(t,0)x_0 + \int_0^t \mathcal{R}(t,s)[Bu(s) + f(s,x(s))]ds, \quad \forall t \in [0,a], \tag{1.1}$$

where $(\mathcal{R}(t,s))_{a\geq t\geq s\geq 0}$ is a family of bounded linear operators defined on a Banach space $(X,\|\cdot\|)$, $\mathcal{R}(0)x = x$ for all $x \in X$, $\mathcal{R}(t,\cdot)x \in C((0,t];X)$ for all $t \in (0,a]$ and every $x \in X$, $u \in L^p([0,a],U)$ where $p \geq 1$ and $(U,\|\cdot\|_U)$ is a Banach space, $B : U \to X$ is a bounded linear operator and $f \in C([0,a] \times X;X)$.

In the first section we establish a general result on the lack of controllability of problem (1.1). In the second section, we apply this result to study the lack of controllability of different models of abstract differential control problems, including first and second order differential equations, abstract integro-differential and fractional differential control problems. In each case, we include some bibliographic comments.

In the remainder of this note, $\mathcal{L}(Z,W)$ represents the space of bounded linear operator from a Banach space $(Z,\|\cdot\|_Z)$ into a Banach space $(W,\|\cdot\|_W)$ endowed with the operator norm denoted by $\|\cdot\|_{\mathcal{L}(Z,W)}$. We write $\mathcal{L}(Z)$ and $\|\cdot\|_{\mathcal{L}(Z)}$ when $Z = W$.

2. Abstract integral control problems

To study the lack of controllability of the control problem (1.1) and prove our affirmations on the associate literature, we introduce the following general condition,

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(H1) There exits a Banach space $(Y, \|\cdot\|_Y)$ continuously embedded in X and $p > 1$ such that $X \neq Y$, $R(t,s)X \subset Y$ for all $t > s$ and $R(a,\cdot) \in L^{p'}([0,a], \mathcal{L}(X,Y))$ where $\frac{1}{p} + \frac{1}{p'} = 1$.

From control theory, we adopt the following concept of p-controllability.

Definition 2.1. We say that control problem (1.1) is p-controllable on $[0,a]$ if for given $x_0 \in X$, $x_1 \in X$ there exists $u \in L^p([0,a];U)$ and a solution $x \in C([0,a];X)$ of (1.1) such that $x(a) = x_1$.

The next Lemma is the key for proving our next results.

Lemma 2.2. *If condition* (H1) *is satisfied, then the integral problem* (1.1) *is not p-controllable on* $[0,a]$.

Proof. Let $x_1 \in X \setminus Y$. If the problem is p-controllable then there exists $u \in L^p([0,a];U)$ and a solution $x(\cdot)$ of (1.1) such that $x(a) = x_1$. On the other hand, from the estimate

$$\begin{aligned}\|x(a)\|_Y &\leq \|R(a,0)x_0\|_Y + \int_0^a \|R(a,s)[Bu(s) + f(s,x(s))]\|_Y ds \\ &\leq \|R(a,0)x_0\|_Y + \int_0^a \|R(a,s)\|_{\mathcal{L}(X,Y)} \|Bu(s) + f(s,x(s))\| ds \\ &\leq \|R(a,0)x_0\|_Y + \|R(a,\cdot)\|_{L^{p'}([0,a],\mathcal{L}(X,Y))} \|B\|_{\mathcal{L}(U,X)} \|u\|_{L^p([0,a],U)} \\ &\quad + \|R(a,\cdot)\|_{L^{p'}([0,a],\mathcal{L}(X,Y))} \|f(\cdot,u(\cdot))\|_{L^p([0,a],X)},\end{aligned}$$

we have that $\|x(a)\|_Y < \infty$, which implies that $x(a) = x_1 \in Y$. This proves that problem (1.1) is not p-controllable on $[0,a]$ if the abstract condition (H1) is satisfied. □

3. Abstract differential control problems

In this section, we use Lemma 2.2 to establish the lack of p-controllability for different models of abstract control differential systems and we prove our affirmations on the associated literature. For convenience, we include some notations and a useful Lemma.

Considering the related literature, next we assume that $A : D(A) \subset X \to X$ is an unbounded sectorial operator, $0 \in \rho(A)$ and A is the infinitesimal generator of an analytic semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on X. In addition, $[D(A)]$ denotes the domain of A endowed with the norm $\|x\|_{\mathcal{D}} = \|Ax\| + \|x\|$ and X_β represents the domain of the β-fractional power $(-A)^\beta$ of $-A$ endowed with the norm $\|x\|_\beta = \|x\| + \|(-A)^\beta x\|$, see [30, 25] for details. From semigroup theory we known that X_β is a Banach space continuously embedded in X, $T(t)X \subset X_\beta$ for all $t > 0$ and there exists $C_\beta > 0$ such that $\|T(t)\|_{\mathcal{L}(X,X_\beta)} \leq C_\beta t^{-\beta}$ for all $t > 0$.

Lemma 3.1. $X_\alpha \neq X$ *for all* $\alpha \in (0,1)$.

Proof. Assume that $X_\alpha = X$ for some $\alpha \in (0,1)$. Since $\|x\| \leq \|x\|_\alpha$ for all $x \in X$ and $(X_\beta, \|\cdot\|_\alpha) \hookrightarrow (X, \|\cdot\|)$, from a consequence of the Open Mapping Theorem we have $(X, \|\cdot\|)$ and $(X_\alpha, \|\cdot\|_\alpha)$ are isomorphic. Thus, there exists $C > 0$ such that $\|x\|_\alpha \leq C\|x\|$ for all $x \in X$. Using this fact, $x \in X$ we obtain

$$\|T(t)x - x\| = \|A \int_0^t T(s)x ds\|$$

$$\begin{aligned}
&\le \int_0^t \|(-A)^{1-\alpha}T(s)(-A)^{\alpha}x\|ds \\
&\le \int_0^t \frac{C_{1-\alpha}}{s^{1-\alpha}}\|x\|_{\alpha}ds \\
&\le \frac{C_{1-\alpha}C}{\alpha}t^{\alpha}\|x\|,
\end{aligned}$$

which implies that $\|T(t)-I\|_{\mathcal{L}(X)} \to 0$ as $t \to 0$. This proves that A is bounded on X (see [30, Theorem 1.1.2]), which is a contradiction. $\square$

We divide the remainder of this paper in several subsections.

3.1. **First order abstract control problems.** Consider the control problem

$$x'(t) = Ax(t) + Bu(t) + f(t, x(t)), \quad \forall t \in [0, a], \tag{3.1}$$

$$x(0) = x_0. \tag{3.2}$$

Definition 3.2. A function $x \in C([0, a]; X)$ is said to be a mild solution of (3.1)-(3.2) if

$$x(t) = T(t)x_0 + \int_0^t T(t-s)[Bu(s) + f(s, x(s))]ds, \quad \forall t \in [0, a].$$

Definition 3.3. We say the system (3.1)-(3.2) is p-controllable on $[0, a]$ if for all $x_0, x_1 \in X$ there exists $u \in L^p([0, a]; U)$ and a mild solution $x(\cdot)$ of (3.1)-(3.2) such that $x(a) = x_1$.

From Lemmas 2.2 and 3.1 we have the following result.

Proposition 3.4. *System* (3.1)-(3.2) *is never* p*-controllable on* $[0, a]$.

Proof. Let $\alpha \in (0, \frac{p-1}{p})$, $Y = X_\alpha$ and $(\mathcal{R}(t,s))_{a \ge t \ge s \ge 0}$ be the operator family defined by $R(t,s) = T(t-s)$. From semigroup theory and Lemma 3.1 we have that $Y \hookrightarrow X$, $R(t,s)X \subset Y$ for all $t > s$ and $Y \ne X$. Moreover, since $\|R(t-s)\|^{p'}_{\mathcal{L}(X,X_\alpha)} \le C_\alpha^{p'}(t-s)^{-p'\alpha}$ and $1 - p'\alpha > 0$, we have that $R(a, \cdot) \in L^{p'}([0, a]; \mathcal{L}(X, Y))$. From the above remarks and Lemma 2.2 we obtain that problem (3.1)-(3.2) is never p-controllable on $[0, a]$. $\square$

Remark 3.5. From Proposition 3.4, we note that the results in [11, 16, 28, 33] are not valid if A is unbounded and that the examples in [6, 8, 10, 16, 23, 27, 28, 33, 4, 12, 20, 22] are not controllable as claimed by the authors.

3.2. **Second order abstract Cauchy control problems.** Consider the problem

$$x''(t) = Ax(t) + Bu(t) + f(t, x(t)), \quad \forall\, t \in [0, a], \tag{3.3}$$

$$x(0) = x_0, \quad x'(0) = x_1, \tag{3.4}$$

where $u \in L^p([0, a], U)$ and $A : D(A) \subset X \to X$ is the generator of a strongly continuous cosine function of bounded linear operators $(\mathcal{C}(t))_{t \ge 0}$ on X, with associated sine function $(\mathcal{S}(t))_{t \in \mathbb{R}}$ given by $\mathcal{S}(t) = \int_0^t \mathcal{C}(s)ds$ for $t \in \mathbb{R}$. From the cosine function theory (see Fattorini [15]) we adopt the following concept.

Definition 3.6. A function $x \in C([0,a];X)$ is said to be a mild solution of problem (3.3)-(3.4) if

$$x(t) = \mathcal{C}(t)x_0 + \mathcal{S}(t)x_1 + \int_0^t \mathcal{S}(t-s)[Bu(s) + f(s,x(s))]ds, \quad \forall t \in [0,a].$$

Definition 3.7. We say the control system (3.3)-(3.4) is p-controllable on $[0,a]$ if for all $x_0, x_1, x_2 \in X$ there exists $u \in L^p([0,a];U)$ and a mild solution $x(\cdot)$ of (3.3)-(3.4) such that $x(a) = x_2$.

Let $E = \{x \in X : \mathcal{C}(\cdot)x \in C^1(\mathbb{R},X)\}$. From [19], the space E endowed with the norm $\|x\|_E = \|x\| + \sup_{0 \le t \le 1} \|AS(t)x\|$, is a Banach space and the operator-valued function

$$\mathcal{H}(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$$

is a strongly continuous group of bounded linear operators on $E \times X$, with generator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. From the above, $S(\cdot)x \in C([0,a];E)$ for all $x \in X$ and there exists $N_E > 0$ such that $\|S(t)\|_{\mathcal{L}(X,E)} \le N_E$ for all $t \in [0,a]$. In addition, from Rankin [32], Lemma 3.1 and Lemma 2.2, we obtain the following result.

Proposition 3.8. *Assume that the general conditions in* [32] *are satisfied. Then* $E \subset X_\alpha$ *for all* $\alpha \in (0, \frac{1}{2})$, $E \neq X$ *and the abstract control system* (3.3)-(3.4) *is never* p*-controllable on* $[0,a]$.

Proof. Let $R(t,s) = S(t-s)$ and $Y = E$. From our remarks on cosine function theory, $R(t,s)X \subset E$ for $t > s$, $E \neq X$ and $R(a,\cdot) \in L^{p'}([0,a];\mathcal{L}(X,E))$ which implies that condition (H1) is satisfied. Using $x_0 = 0$, from the above it is easy to see that the problem (3.3)-(3.4) is not p-controllable on $[0,a]$. □

Remark 3.9. The general assumptions in [32] are usually verified by sectorial operators. In particular, these conditions are satisfied in [2, 3, 6, 7, 9, 13, 29]. Thus, the results in [2, 3, 6, 7, 9, 13, 29] are only valid for the case in which A is bounded and the examples in [2, 3, 29] are not controllable as affirmed in these works.

3.3. **Integro-differential control systems.** Consider the abstract integro-differential control problem

$$x'(t) = Ax(t) + \int_0^t C(t-s)x(s)ds + Bu(t) + f(t,x(t)), \tag{3.5}$$

$$x(0) = x_0, \tag{3.6}$$

where $(C(\tau))_{\tau \in [0,a]}$ is a family of closed linear operator defined with domain $D(A)$ and $u \in L^p([0,a],U)$. Next we assume that the assumptions in [18] are fulfilled. Under these conditions, the problem

$$x'(t) = Ax(t) + \int_0^t C(t-s)x(s)ds, \quad t \ge 0,$$
$$x(0) = x_0,$$

has an associated analytic resolvent operator $(Q(s))_{s \ge 0}$ on X.

Definition 3.10 ([18, Definition 2.1])**.** A family of bounded linear operators $(Q(s))_{s\geq 0}$ on X is said to be a resolvent operator for (3.5)-(3.6) if $Q(0)x = x$ for all $x \in X$, $Q(\cdot)x \in C([0,\infty),X) \cap C^1((0,\infty);X)$ for all $x \in X$, there are positive constants M,γ such that $\|Q(t)\| \leq Me^{\gamma t}$ for all $t > 0$, $Q(t) \in \mathcal{L}([D(A)])$ for all $t \geq 0$, $Q(\cdot)x \in C^1([0,\infty);[D(A)])$ for all $x \in D(A)$ and

$$Q'(t)y = AQ(t)y + \int_0^t C(t-s)Q(s)y ds, \quad y \in D(A),$$
$$Q'(t)y = Q(t)Ay + \int_0^t Q(t-s)C(s)y ds, \quad y \in D(A),$$

From the theory of resolvent operators, we adopt the following concepts.

Definition 3.11. A function $x \in C([0,a];X)$ is called a mild solution of (3.5)-(3.6) if

$$x(t) = Q(t)x_0 + \int_0^t Q(t-s)[Bu(s) + f(s,x(s))]ds, \quad \forall t \in [0,a].$$

Definition 3.12. System (3.5)-(3.6) is called p-controllable on $[0,a]$ if for all $x_1 \in X$ there exists $u \in L^p([0,a];U)$ and a mild solution $x(\cdot)$ of (3.5)-(3.6) such that $x(a) = x_1$.

Proposition 3.13. *Assume the conditions in* [18, Theorem 3.4] *are fulfilled. Then the abstract control system* (3.5)-(3.6) *is not p-controllable on* $[0,a]$.

Proof. Let $\alpha \in (0, \frac{p-1}{p})$, $Y = X_\alpha$ and $R(t,s) = Q(t-s)$. From [18, Theorem 3.4] we know that $Q \in C((0,a];\mathcal{L}(X,X_\alpha))$ and there exists $K > 0$ such that $\|Q(t)\|_{\mathcal{L}(X,X_\alpha)} \leq Kt^{-\alpha}$ for all $t \in (0,a]$, which implies that

$$R(a,\cdot) \in L^{p'}([0,a],\mathcal{L}(X,X_\alpha)).$$

Finally, from Lemma 3.1 and Lemma 2.2 it follows that the problem is not p-controllable. □

Remark 3.14. From the above, we note that the abstract problem in [34] is not controllable if the involved operator is unbounded and that the example in [34] is not controllable.

3.4. **Abstract evolution control systems.** Assume that $A(t) : D(A) \subset X \to X$, $t \in [0,a]$, is a family of sectorial operator and consider the abstract control system

$$x'(t) = A(t)x(t) + Bu(t) + f(t,x(t)), \quad \forall t \in [0,a], \tag{3.7}$$
$$x(0) = x_0. \tag{3.8}$$

Let $A = A(0)$ and suppose that $(A(t))_{t\in[0,a]}$ verifies the conditions in [25, Chapter VI]. In this case, there exists an evolution operator $(U(t,s))_{a\geq t\geq s\geq 0}$ associated with the problem

$$x'(t) = A(t)x(t), \quad \forall t \in [s,a], \tag{3.9}$$
$$x(s) = x_0, \quad a \geq s \geq 0. \tag{3.10}$$

Definition 3.15. A function $x \in C([0,a];X)$ is called a mild solution of (3.7)-(3.8) if

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)[Bu(s) + f(s,x(s))]ds, \quad \forall t \in [0,a].$$

Definition 3.16. Control system (3.7)-(3.8) is said to be p-controllable on $[0,a]$ if for all $x_1 \in X$ there exists $u \in L^2([0,a];U)$ and a mild solution $x(\cdot)$ of (3.7)-(3.8) such that $x(a) = x_1$.

Proposition 3.17. *Control system* (3.7)-(3.8) *is not p-controllable on* $[0,a]$.

Proof. Let $\alpha \in (0, \frac{p-1}{p})$ and $(X,\mathcal{D})_{\alpha,\infty}$ be the space in [25, Chapter I] with $\mathcal{D} = [D(A)]$. From [25, Chapter I, Chapter VI], we know that $(X,\mathcal{D})_{\alpha,\infty}$ is an interpolation space between $[D(A)]$ and X, $U(t,\cdot) \in C([0,t); \mathcal{L}(X; (X,\mathcal{D})_{\alpha,\infty})$ for all $t > 0$ and there exists a constant $E_{\alpha,\infty} > 0$ such that $\|U(t,s)\|_{\mathcal{L}(X,(X,\mathcal{D})_{\alpha,\infty})} \leq E_{\alpha,\infty}(t-s)^{-\alpha}$ for all $a \geq t > s \geq 0$.

Let the space

$$D_A(\alpha,\infty) = \{x \in X : [x]_{\alpha,\infty} = \sup_{t\in(0,a)} \|t^{1-\alpha} AT(t)x\| < \infty\},$$

be endowed with the norm $\|x\|_{D_A(\alpha,\infty)} = [x]_{\alpha,\infty} + \|x\|$. From [25, Section 2.2.1] we know that $(X,\mathcal{D})_{\alpha,\infty} = D_A(\alpha,\infty)$ and that the norms in $D_A(\alpha,\infty)$ and $(X,\mathcal{D})_{\alpha,\infty}$ are equivalent. Moreover, from the proof of Lemma 3.1 it is easy to note that $(X,\mathcal{D})_{\alpha,\infty} \neq X$.

From the above remarks, we have that condition (H1) is satisfied with $R(t,s) = U(t,s)$ and $Y = (X,\mathcal{D})_{\alpha,\infty}$. Thus, the problem (3.7)-(3.8) is not p-controllable on $[0,a]$. □

Remark 3.18. The main results in [1, 5, 17, 24] are not applicable to partial differential control systems and the examples in these works are not controllable.

We complete this note by considering a problem involving a general class of abstract control fractional differential problems.

3.5. **Fractional differential control problems.** Consider the fractional differential control problem

$$dJ_t^{1-\alpha}x(t) = Ax(t) + Bu(t) + f(t,x(t)), \quad \forall t \in I = [0,b], \tag{3.11}$$

$$x(0) = x_0, \tag{3.12}$$

where $\frac{1}{2} < \alpha < 1$, $J_t^{1-\alpha}$ is the $(1-\alpha)$-order Riemman fractional integral operator and f is a continuous function.

In the next definitions, $(T_\alpha(t))_{t\geq 0}$ and $(S_\alpha(t))_{t\geq 0}$ are the operators families in [21].

Definition 3.19. A function $x \in C([0,a];X)$ is say to be a mild solution of (3.11)-(3.12) if $x(0) = x_0$ and

$$x(t) = T_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)Bu(s)ds + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)f(s,x(s))ds, \tag{3.13}$$

for all $t \in [0,b]$.

Definition 3.20. Control problem (3.11)-(3.12) is p-controllable if for every $x_0, x_1 \in X$ there exists a control $u \in L^p(I,U)$ and a mild solution $x(\cdot)$ of (3.11)-(3.12) such that $x(a) = x_1$.

From [21], for $\gamma > 0$ there exist constants $D_\alpha, C_{\gamma,\alpha}$ such that $\|S_\alpha(t)\|_{\mathcal{L}(X,X_\gamma)} \leq C_{\gamma,\alpha} t^{-\alpha\gamma}$ and $\|S_\alpha(x)\|_{\mathcal{L}(X_1,X_1)} \leq D_\alpha$ for all $t \in [0,b]$.

Lemma 3.21. *Problem* (3.11)-(3.12) *is never p-controllable.*

Proof. To simplify, we assume that $x_0 = 0$. From the properties of the operator family $(S_\alpha(t))_{t\geq 0}$, for $\gamma < \frac{1}{\alpha}$ such that $p > \frac{1}{1-\alpha\gamma}$, we have that the condition (H1) is satisfied with $Y = X_\gamma$. This prove the assertion. □

Remark 3.22. The result in [21] is not applicable to partial differential control systems and the considered example is not controllable as claimed. Our affirmation follows from Lemma 3.21 with $f \equiv 0$ in [21]. For additional details, see [31].

Remark 3.23. The control problems considered above corresponds to semi-linear ordinary differential control problems with a linear part dominated by a sectorial operator, which include the case where the operator A is a realization on a Banach space of a strongly elliptic operator. Thus, our remarks are valid for control problems involving parabolic and hyperbolic differential equations. For additional details on the lack of controllability in partial control differential equations and differential control problems in abstract spaces we cite [26, 35, 36] and the references therein.

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