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CRITICAL QUASILINEAR SCHRÖDINGER EQUATION WITH SIGN-CHANGING POTENTIAL

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Abstract. We study the existence of nontrivial solutions for a class of quasilinear Schrödinger equations in $\mathbb{R}^N$ with critical nonlinearity, where the potential is allowed to change signs. The quasilinear equations are reduced to semilinear equations by using a change of variable. The geometric hypotheses of a mountain pass theorem without compactness conditions are satisfied so that the equation possesses a nontrivial solution.

1. Introduction

In this article we discuss the existence of nontrivial solutions for quasilinear Schrödinger equation

$$-\Delta u + V(x)u - \Delta(u^2)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

which has atracted a great deal of attention during recent years (see [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 17]), because not only it provides an important model for developing mathematical methods but it represents a special case of modeling for many physical phenomena, see [2, 12] for an explanation. Some existence results for (1.1) have been concluded when the potential $V(x)$ is bounded from below or coercive, we refer to [7, 9, 12] where they have focused on the existence of solutions for (1.1) in the subcritical case when $f(x,u) = |u|^{p-1}u$, $4 \le p+1 < 22^*$, $N \ge 3$, and have suggested the results by using direct variational methods, such as constrained minimization arguments. To overcome the undefiniteness of natural functional associated to (1.1), we rewrite the functional with a new variable which reduces the problem to looking for solutions of an auxiliary semilinear equation by employing the ideas in [4, 14, 7]. We establish a new potential function $V(x)$ which can be sign-changing and may be unbounded from below without any periodic hypotheses. A new nonlinearity $f(x,u) = K(x)|u|^{22^*-2}u + g(x,u) + h(x)$ is established which is more general than in other papers, for example [3, 6, 7, 9, 12, 16, 19].

First we consider the following quasilinear Schrödinger equation with critical growth

$$-\Delta u + V(x)u - \Delta(u^2)u = K(x)|u|^{22^*-2}u + g(x,u) + h(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

where the functions V, K, $h : \mathbb{R}^N \to \mathbb{R}$ and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are continuous and satisfy the following assumptions:

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(A1) $\int |\nabla u|^2 + V(x)u^2 > 0$ for all $u \in E \setminus \{0\}$.
(A2) $V(x)$ is sign-changing, $V^+(x) \in L^\infty(\mathbb{R}^N)$, $\lim_{|x|\to+\infty} V^+(x) = a_0 > 0$ and $\|V^-\|_{N/2} < \frac{S(\theta-4)}{\theta-2}$, where $V^\pm(x) := \max\{\pm V(x), 0\}$, S denotes the Sobolev optimal constant and θ is the constant in (A6).
(A3) $0 < C \le K(x) \in L^\infty(\mathbb{R}^N)$.
(A4) $g(x,u) = o(u)$ uniformly in $x \in \mathbb{R}^N$ as $u \to 0^+$.
(A5) There are constants $a_1, a_2 > 0$ and $4 \le p < 22^*$ such that

$$|g(x,u)| \le a_1 + a_2|u|^{p-1}, \quad \forall (x,u) \in \mathbb{R}^N \times [0,+\infty).$$

(A6) There exists a constant $\theta \in (4, 22^*)$ satisfying

$$0 < G(x,u) \le \frac{1}{\theta} g(x,u)u, \quad \forall (x,u) \in \mathbb{R}^N \times (0,+\infty),$$

where $G(x,u) := \int_0^u g(x,s)ds$.
(A7) $h \not\equiv 0$ and $\|h\|_{2N/(N+2)} < \frac{\alpha}{4} S^{1/2}\rho$, where α and ρ are given in Lemma 3.2.

We remark that the potential may be unbounded from below and the associated functional does not satisfy any compactness conditions. Note that $22^* = \frac{4N}{N-2}$, here and in the sequel, $N \ge 3$. Let

$$E := \{u \in H^1(\mathbb{R}^N) : \int V^+(x)u^2 < \infty\},$$

we observe that E is a Hilbert space equipped with the inner product

$$(u,v) := \int \nabla u \nabla v + V^+(x)uv$$

and the norm $\|u\| = (u,u)^{1/2}$. Obviously, it follows from (A2) that $\|\cdot\|$ is an equivalent norm with the standard one in $H^1(\mathbb{R}^N)$ and hence E is continuously embedded into $L^p(\mathbb{R}^N)$, $2 \le p \le 2^*$, i.e., there is a constant $\tau_p > 0$ such that

$$\|u\|_p \le \tau_p \|u\|, \quad \forall u \in E, \tag{1.3}$$

where $\|\cdot\|_p$ is used for the usual norm in $L^p(\mathbb{R}^N)$. Now we state our main result.

Theorem 1.1. *If the conditions* (A1)–(A7) *hold. Then problem* (1.2) *possesses a nontrivial nonnegative solution in E.*

Also, we consider a more general problem

$$-\Delta u + V(x)u - \Delta(u^2)u = |u|^{22^*-2}u + g(u), \quad x \in \mathbb{R}^N, \tag{1.4}$$

under hypotheses (A1) and

(A8) $V(x)$ is sign-changing, $\lim_{|x|\to+\infty} V^+(x) = V^+(\infty) > 0$, $V^+(x) \le V^+(\infty)$ in $\mathbb{R}^N$ and $\|V^-\|_{N/2} < \frac{S(\theta-4)}{\theta-2}$.
(A4') $g(u) = o(u)$ as $u \to 0^+$.
(A5') There are constants $a_1, a_2 > 0$ and $4 \le p < 22^*$ such that

$$|g(u)| \le a_1 + a_2|u|^{p-1}, \ \forall u \in [0,+\infty).$$

(A6') There exists a constant $\theta \in (4, 22^*)$ with

$$0 < G(u) \le \frac{1}{\theta} g(u)u, \ \forall u \in (0,+\infty),$$

where $G(u) = \int_0^u g(s)ds$.

(A9) $(i) G(u)/(u^{22^*-1}) \to +\infty$ as $u \to +\infty$, if $3 \leq N < 10$;
$(ii) G(u)/u^4 \to +\infty$ as $u \to +\infty$, if $N \geq 10$.

(A10) The function $\frac{g(u)}{u^3}$ is nondecreasing for all $u > 0$.

Now we state the second main result.

Theorem 1.2. *Assume that* (A1), (A8), (A4')–(A6'), (A9), (A10) *are satisfied. Then problem* (1.4) *admits a nontrivial nonnegative solution in* E.

Remark 1.3. Regarding the the results suggested in [6], Theorems 1.1 and 1.2 give an extension from their results to quasilinear Schrödinger equation including critical terms case.

Remark 1.4. A problem of type (1.4) for $N = 2$ was studied in [11] where V and g are two continuous 1-periodic functions, V is nonnegative and bounded from below and g is critical growth. Moreover in [15] a similar result to Theorem 1.2 is provided under a more restricted hypotheses on the periodic potential V. While our results in both Theorems 1.1 and 1.2 do not need any periodic conditions and the potential $V(x)$ may be unbounded from below. Also, the method of our proof is different from that in [15].

The article is organized as follows: in Section 2, we reduce the quasilinear problem into a semilinear one by the dual method and show some preliminary results. Section 3 is devoted to prove that the mountain pass level of I is well defined, show the boundedness for the $(PS)_c$ sequence of the associated functional, and finish Theorem 1.1. Finally we bring results that complete the proof of Theorem 1.2 in Section 4.

Throughout this article, C will denote various positive constants whose exact value is not essential. The domain of an integral is $\mathbb{R}^N$ unless otherwise indicated. $\int f(x)dx$ is abbreviated to $\int f(x)$.

2. Preliminary results

We show that the energy functional corresponding to (1.2) given by

$$\begin{aligned} J(u) :=& \frac{1}{2}\int (1+2u^2)|\nabla u|^2 + \frac{1}{2}\int V(x)u^2 - \frac{1}{22^*}\int K(x)|u|^{22^*} \\ & - \int G(x,u) - \int h(x)u, \end{aligned}$$

which is not well defined in general, such as in $H^1(\mathbb{R}^N)$. To avoid this trouble, we use of the change of variable $v := f^{-1}(u)$ introduced by [7], where f is defined by

$$f'(t) = \frac{1}{\sqrt{1+2f^2(t)}} \text{ on } [0,+\infty) \text{ and } f(t) = -f(-t) \text{ on } (-\infty, 0].$$

We list some properties of f, and the proofs of which may be found in [4, 14].

Lemma 2.1. *The function f satisfies the following properties:*

(1) *f is uniquely defined, C^∞ and invertible;*
(2) *$|f'(t)| \leq 1$ for all $t \in \mathbb{R}$;*
(3) *$|f(t)| \leq |t|$ for all $t \in \mathbb{R}$;*
(4) *$f(t)/t \to 1$ as $t \to 0$;*
(5) *$f(t)/\sqrt{t} \to 2^{1/4}$ as $t \to +\infty$;*
(6) *$f(t)/2 \leq tf'(t) \leq f(t)$ for all $t \geq 0$;*

(7) $|f(t)| \le 2^{1/4}|t|^{1/2}$ *for all* $t \in \mathbb{R}$*;*
(8) *there exists a positive constant* C *such that* $|f(t)| \ge C|t|$ *for* $|t| \le 1$ *and* $|f(t)| \ge C|t|^{1/2}$ *for* $|t| \ge 1$*;*
(9) $|f(t)f'(t)| < 1/\sqrt{2}$ *for all* $t \in \mathbb{R}$*;*
(10) *the function* $f(t)t^{-1}$ *is nonincreasing for all* $t \in \mathbb{R}\backslash\{0\}$*;*
(11) *the function* $f(t)f'(t)t^{-1}$ *is decreasing for all* $t > 0$*;*
(12) *the function* $f^3(t)f'(t)t^{-1}$ *is increasing for all* $t > 0$*;*
(13) *the function* $f^{22^*-1}(t)f'(t)t^{-1}$ *is increasing for all* $t > 0$*.*

After the change of variable we obtain the functional

$$\begin{aligned} I(v) :=&\frac{1}{2}\int |\nabla v|^2 + \frac{1}{2}\int V(x)f^2(v) - \frac{1}{22^*}\int K(x)|f(v)|^{22^*} \\ &- \int G(x, f(v)) - \int h(x)f(v). \end{aligned}$$

Then I is well-defined on E and belongs to C^1 in view of the hypotheses (A2)–(A5) and (A7). Furthermore, it is easy to check that

$$\begin{aligned} \langle I'(v), w\rangle =& \int \nabla v\nabla w + \int V(x)f(v)f'(v)w - \int K(x)|f(v)|^{22^*-2}f(v)f'(v)w \\ &- \int g(x, f(v))f'(v)w - \int h(x)f'(v)w, \quad \forall v, w \in E, \end{aligned}$$

and the critical point of I are weak solutions of the problem

$$-\Delta v + V(x)f(v)f'(v) = K(x)|f(v)|^{22^*-2}f(v)f'(v) + g(x, f(v))f'(v) + h(x)f'(v),$$

for $x \in \mathbb{R}^N$. We observe that if $v \in E$ is a critical point of the functional I, then the function $u = f(v) \in E$ is a solution of (1.2) (cf:[4]). To obtain a nonnegative solution for (1.2), we set $g(x, u) = 0$ for all $x \in \mathbb{R}^N$ and $u \le 0$. By (A4) and (A5) we also see that, given $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$|g(x, u)| \le \varepsilon|u| + C_\varepsilon|u|^{p-1}, \quad \forall (x, u) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.1}$$

3. Proof of Theorem 1.1

In this section we assume that (A1)–(A7) are satisfied. The following lemmas are crucial for the proof of Theorem 1.1.

Lemma 3.1. *There exist constants* $\rho, \alpha > 0$ *such that* $\int |\nabla v|^2+V(x)f^2(v) \ge \alpha\|v\|^2$, *whenever* $\|v\| = \rho$.

The proof of the above lemma is similar to that of [6, Lemma 3.1]. So we omit it.

Lemma 3.2. *For the above* ρ*, there exists a constant* $\beta > 0$ *such that* $\inf_{\|v\|=\rho} I(v) \ge \beta$.

Proof. By (A3), Lemma 2.1(7) and the Sobolev imbedding inequality, it is easy to obtain

$$\begin{aligned} \int K(x)|f(v)|^{22^*} &\le 2^{2^*/2}\|K\|_\infty \int |v|^{2^*} \\ &\le 2^{2^*/2}\|K\|_\infty S^{-2^*/2}\Big(\int |\nabla v|^2\Big)^{2^*/2} \\ &\le 2^{2^*/2}\|K\|_\infty S^{-2^*/2}\|v\|^{2^*}. \end{aligned}$$

By (2.1), Lemma 2.1(3,7) and (1.3), we have

$$\begin{aligned}\int G(x,f(v)) &\le \frac{\varepsilon}{2}\int |f(v)|^2 + \frac{C_\varepsilon}{p}\int |f(v)|^p\\ &\le \frac{\varepsilon}{2}\int |v|^2 + C_\varepsilon\int |v|^{p/2}\\ &\le \frac{\varepsilon}{2}\tau_2^2\|v\|^2 + C_\varepsilon\|v\|^{p/2}.\end{aligned}$$

It follows from (A7), Lemma 2.1(3), the Hölder inequality and the Sobolev imbedding inequality that

$$\int h(x)f(v) \le \|h\|_{\frac{2N}{N+2}}\|v\|_{2^*} \le \|h\|_{\frac{2N}{N+2}}S^{-1/2}\Big(\int |\nabla v|^2\Big)^{1/2} \le \|h\|_{\frac{2N}{N+2}}S^{-1/2}\|v\|.$$

Therefore, combining the above inequalities with Lemma 3.1, we obtain

$$I(u) \ge \frac{\alpha}{2}\|v\|^2 - \frac{2^{2^*/2}}{22^*}\|K\|_\infty S^{-2^*/2}\|v\|^{2^*} - \frac{\varepsilon}{2}\tau_2^2\|v\|^2 - C_\varepsilon\|v\|^{p/2} - \|h\|_{\frac{2N}{N+2}}S^{-1/2}\|v\|.$$

Choosing $\varepsilon \le \alpha/(2\tau_2^2)$ and for every $\|v\| = \rho$ we obtain

$$I(u) \ge \rho\big[\frac{\alpha}{4}\rho - \|h\|_{\frac{2N}{N+2}}S^{-1/2}\big] - \frac{2^{2^*/2}}{22^*}\|K\|_\infty S^{-2^*/2}\rho^{2^*} - C\rho^{p/2}.$$

For ρ sufficiently small, we derive that there exists a constant $\beta > 0$ such that $\inf_{\|v\|=\rho} I(v) \ge \beta$ by (A7). $\square$

Lemma 3.3. *There exists $v_0 \in E$ such that $\|v_0\| > \rho$ and $I(v_0) < 0$.*

Proof. Given $\varphi \in C_0^\infty(\mathbb{R}^N,[0,1])$ with $B := supp\varphi$, we derive that $I(t\varphi) \to -\infty$ as $t \to +\infty$, which completes the proof if we take $v_0 = t\varphi$ with t large enough. Note that $0 < t\varphi \le t$ in B and then

$$f(t\varphi) \ge f(t)\varphi \tag{3.1}$$

by Lemma 2.1(10). It follows from (A2), (A3), (A6), (A7), Lemma 2.1(3) and (3.1) that

$$\begin{aligned}I(t\varphi) &\le \frac{t^2}{2}\int_B |\nabla\varphi|^2 + \frac{1}{2}\int_B V^+(x)f^2(t\varphi) - \frac{1}{2}\int_B V^-(x)f^2(t\varphi)\\ &\quad - \frac{C}{22^*}\cdot f^{22^*}(t)\int_B |\varphi|^{22^*} + t\|h\|_{\frac{2N}{N+2}}\|\varphi\|_{2^*}\\ &\le \frac{t^2}{2}\|\varphi\|^2 - \frac{C}{22^*}\cdot f^{22^*}(t)\int_B |\varphi|^{22^*} + t\|h\|_{\frac{2N}{N+2}}\|\varphi\|_{2^*}\\ &\to -\infty \quad \text{as } t \to +\infty,\end{aligned}$$

since $f^{22^*}(t)/t^2 \to +\infty$ as $t \to +\infty$. $\square$

Lemma 3.4. *The $(PS)_c$ sequence $(v_n) \subset E$ is bounded.*

Proof. Set $(v_n) \subset E$ be a $(PS)_c$ sequence: $I(v_n) \to c$ and $I'(v_n) \to 0$ as $n \to \infty$. Using Lemma 2.1(3,6), (A3), (A6) and the Sobolev imbedding inequality we easily deduce that

$$\begin{aligned}&c + o_n(1) + o_n(1)\|v_n\|\\ &= I(v_n) - \frac{2}{\theta}I'(v_n)v_n\end{aligned}$$

$$
\begin{aligned}
&\geq (\frac{1}{2}-\frac{2}{\theta})\int |\nabla v_n|^2+V^+(x)f^2(v_n)-(\frac{1}{2}-\frac{1}{\theta})\int V^-(x)f^2(v_n)\\
&\quad -(\frac{1}{22^*}-\frac{1}{\theta})\int K(x)|f(v_n)|^{22^*}+\frac{1}{\theta}\int g(x,f(v_n))f(v_n)\\
&\quad -\int G(x,f(v_n))-(1+\frac{2}{\theta})\int |h(x)f(v_n)|\\
&\geq (\frac{1}{2}-\frac{2}{\theta})\int |\nabla v_n|^2+V^+(x)f^2(v_n)-(\frac{1}{2}-\frac{1}{\theta})\|V^-\|_{N/2}\|v_n\|_{2^*}^2\\
&\quad +(\frac{1}{\theta}-\frac{1}{22^*})\int K(x)|f(v_n)|^{22^*}-(1+\frac{2}{\theta})\|h\|_{\frac{2N}{N+2}}\|v_n\|_{2^*}\\
&\geq \Big[(\frac{1}{2}-\frac{2}{\theta})-(\frac{1}{2}-\frac{1}{\theta})\|V^-\|_{N/2}S^{-1}\Big]\int |\nabla v_n|^2+V^+(x)f^2(v_n)\\
&\quad +(\frac{1}{\theta}-\frac{1}{22^*})\int K(x)|f(v_n)|^{22^*}-(1+\frac{2}{\theta})\|h\|_{\frac{2N}{N+2}}S^{-1/2}\Big(\int |\nabla v_n|^2\Big)^{1/2}.
\end{aligned}
$$

It follows from (A2) that $(\frac{1}{2}-\frac{2}{\theta})-(\frac{1}{2}-\frac{1}{\theta})\|V^-\|_{N/2}S^{-1}>0$ and hence

$$
\begin{gathered}
\int |\nabla v_n|^2+V^+(x)f^2(v_n)\leq C+C\|v_n\|,\\
\int K(x)|f(v_n)|^{22^*}\leq C+C\|v_n\|.
\end{gathered}
\tag{3.2}
$$

From (3.2), we only prove that $\int V^+(x)v_n^2\leq C+C\|v_n\|$. In fact, from (A2), (A3), Lemma 2.1(8) and (3.2) it follows that

$$
\begin{aligned}
\int_{|v_n|\geq 1} V^+(x)v_n^2 &\leq \|V^+\|_\infty\int_{|v_n|\geq 1} v_n^2\leq C\|V^+\|_\infty\int |f(v_n)|^{22^*}\\
&\leq C\|V^+\|_\infty\int K(x)|f(v_n)|^{22^*}\leq C+C\|v_n\|
\end{aligned}
$$

and

$$
\int_{|v_n|\leq 1} V^+(x)v_n^2\leq C\int_{|v_n|\leq 1} V^+(x)f^2(v_n)\leq C+C\|v_n\|.
$$

Thus we have $\|v_n\|^2\leq C+C\|v_n\|$ and then $(v_n)\subset E$ is bounded. □

Lemma 3.5. *Suppose that $(v_n)\subset E$ is a bounded $(PS)_c$ sequence for the functional I, then up to a subsequence, $v_n\rightharpoonup v$ in E and v is a nontrivial critical point of the functional I.*

Proof. The argument is similar as in [15]. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, we only need to show that $\langle I'(v),\varphi\rangle=0$ for all $\varphi\in C_0^\infty(\mathbb{R}^N)$. Notice that $\langle I'(v_n),\varphi\rangle\to 0$, for all $\varphi\in C_0^\infty(\mathbb{R}^N)$, it suffices to derive that $\langle I'(v_n),\varphi\rangle\to\langle I'(v),\varphi\rangle$. In fact,

$$
\begin{aligned}
&\langle I'(v_n),\varphi\rangle-\langle I'(v),\varphi\rangle-\int(\nabla v_n-\nabla v)\nabla\varphi\\
&=\int[f(v_n)f'(v_n)-f(v)f'(v)]V^+(x)\varphi+\int[f(v)f'(v)-f(v_n)f'(v_n)]V^-(x)\varphi\\
&\quad +\int\big[|f(v)|^{22^*-2}f(v)f'(v)-|f(v_n)|^{22^*-2}f(v_n)f'(v_n)\big]K(x)\varphi\\
&\quad +\int\big[g(x,f(v))f'(v)-g(x,f(v_n))f'(v_n)\big]\varphi+\int[f'(v)-f'(v_n)]h(x)\varphi.
\end{aligned}
$$

Since E is continuously embedded into $H^1(\mathbb{R}^N)$, we know that

$$\int \nabla v_n \nabla \varphi \to \int \nabla v \nabla \varphi.$$

Besides, it follows from $v_n \rightharpoonup v$ in E that $v_n \to v$ in $L^p_{\rm loc}(\mathbb{R}^N)$, $p \in [1, 2^*)$. Then, up to subsequence, $v_n \to v$ a.e. on $B := supp\varphi$ as $n \to \infty$ and $|v_n(x)| \le |w_p(x)|$ a.e. on B with $w_p \in L^p(B)$ for every $n \in \mathbb{N}$. Therefore, we have

$$\begin{gathered}
f'(v_n) \to f'(v) \quad \text{a.e. on } B \text{ as } n \to \infty, \\
f(v_n)f'(v_n) \to f(v)f'(v) \quad \text{a.e. on } B \text{ as } n \to \infty, \\
|f(v_n)|^{22^*-2} f(v_n)f'(v_n) \to |f(v)|^{22^*-2} f(v)f'(v) \quad \text{a.e. on } B \text{ as } n \to \infty, \\
g(x, f(v_n))f'(v_n) \to g(x, f(v))f'(v) \quad \text{a.e. on } B \text{ as } n \to \infty.
\end{gathered}$$

Furthermore, by (A2), (A3), (A7), Lemma 2.1(2,7,9) and the Hölder inequality we have

$$\begin{gathered}
|V^+(x)f(v_n)f'(v_n)\varphi| \le C\|V^+\|_\infty |\varphi| \in L^1(B), \\
|V^-(x)f(v_n)f'(v_n)\varphi| \le |V^-(x)||\varphi| \in L^1(B), \\
|K(x)|f(v_n)|^{22^*-2} f(v_n)f'(v_n)\varphi| \le \|K\|_\infty 2^{\frac{2^*-1}{2}} |w_{2^*-1}|^{2^*-1}|\varphi| \in L^1(B), \\
|h(x)f'(v_n)\varphi| \le |h(x)||\varphi| \in L^1(B).
\end{gathered}$$

Hence, the Lebesgue Dominated Convergence Theorem implies

$$\begin{gathered}
\int V^+(x)f(v_n)f'(v_n)\varphi \to \int V^+(x)f(v)f'(v)\varphi, \\
\int V^-(x)f(v_n)f'(v_n)\varphi \to \int V^-(x)f(v)f'(v)\varphi, \\
\int K(x)|f(v_n)|^{22^*-2} f(v_n)f'(v_n)\varphi \to \int K(x)|f(v)|^{22^*-2} f(v)f'(v)\varphi, \\
\int h(x)f'(v_n)\varphi \to \int h(x)f'(v)\varphi.
\end{gathered}$$

For $|v_n| \le 1$, by (2.1) and Lemma 2.1(2,3), we have

$$|g(x, f(v_n))f'(v_n)\varphi| \le \varepsilon |f(v_n)||\varphi| + C_\varepsilon |f(v_n)|^{p-1}|\varphi| \le (\varepsilon + C_\varepsilon)|\varphi|.$$

For $|v_n| > 1$, by (2.1) and Lemma 2.1(2,3,7,9) we conclude that

$$\begin{aligned}
|g(x, f(v_n))f'(v_n)\varphi| &\le \varepsilon |v_n||\varphi| + C_\varepsilon |f(v_n)|^{p-1}|f'(v_n)||\varphi| \\
&\le \varepsilon |w_2||\varphi| + C_\varepsilon |f(v_n)|^{p-2}|\varphi| \\
&\le \varepsilon |w_2||\varphi| + C_\varepsilon |v_n|^{\frac{p}{2}-1}|\varphi| \\
&\le \varepsilon |w_2||\varphi| + C_\varepsilon |w_{2^*-1}|^{2^*-1}|\varphi|.
\end{aligned}$$

Combining the above facts and using the Lebesgue Dominated Convergence Theorem implies

$$\int g(x, f(v_n))f'(v_n)\varphi \to \int g(x, f(v))f'(v)\varphi.$$

Hence, v is a critical point of I. From the condition (A7), v is nontrivial. $\square$

Proof of Theorem 1.1. Lemmas 3.2 and 3.3 imply that the functional I satisfies the mountain pass geometry, thus the $(PS)_c$ sequence exists, where

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)), \quad \Gamma := \{\gamma \in C([0,1],E) : \gamma(0)=0, \gamma(1)=v_0\}.$$

Assume that $(v_n) \subset E$ is a $(PS)_c$ sequence, (v_n) is bounded by Lemma 3.3. Going if necessary to a subsequence, $v_n \rightharpoonup v$ in E. We obviously get that v is a nontrivial critical point of the functional I by Lemma 3.5. □

4. Proof of Theorem 1.2

In this section we assume that (A1), (A8), (A4')–(A6'), (A9), (A10) are satisfied. We study the existence of nontrivial critical points for the functional $I_0 \in C^1(E,\mathbb{R})$ given by

$$I_0(v) := \frac{1}{2}\int |\nabla v|^2 + \frac{1}{2}\int V(x) f^2(v) - \frac{1}{22^*}\int |f(v)|^{22^*} - \int G(f(v)).$$

We also denote the corresponding limiting functional by

$$I_1(v) := \frac{1}{2}\int |\nabla v|^2 + \frac{1}{2}\int V^+(\infty) f^2(v) - \frac{1}{22^*}\int |f(v)|^{22^*} - \int G(f(v)).$$

We set $g(u) = 0$ if $u \le 0$. Some propositions and lemmas are needed and their proofs are similar as in [15], we just state them in brief and omit their proofs as follows.

Proposition 4.1. *Assume* (A8), (A4'), (A5') *hold. Let* $(v_n) \subset E$ *be a* $(PS)_c$ *sequence with* $0 < c < \frac{1}{2N}S^{\frac{N}{2}}$*, and* $v_n \rightharpoonup 0$ *in* E*. Then there exist a sequence* $(y_n) \subset \mathbb{R}^N$ *and* $r, \eta > 0$ *such that* $|y_n| \to +\infty$ *and*

$$\limsup_{n\to\infty} \int_{B_r(y_n)} v_n^2 \ge \eta > 0.$$

Given $\varepsilon > 0$, we study the function $w_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ defined by

$$w_\varepsilon(x) = C(N)\frac{\varepsilon^{\frac{N-2}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{N-2}{2}}},$$

where $C(N) = [N(N-2)]^{\frac{N-2}{4}}$. Recall that by ([18, 1, 13]), $\{w_\varepsilon\}_{\varepsilon>0}$ is a family of functions on which the infimum, that defines the best constant S, for the Sobolev imbedding $D^{1,2}(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N)$, is attained. Moreover, one has

$$w_\varepsilon \in L^{2^*}(\mathbb{R}^N), \quad \nabla w_\varepsilon \in L^2(\mathbb{R}^N), \quad \int |\nabla w_\varepsilon|^2 = \int |w_\varepsilon|^{2^*} = S^{\frac{N}{2}}.$$

We also consider $\phi \in C_0^\infty(\mathbb{R}^N,[0,1])$, $\phi \equiv 1$ in $B_1(0)$, $\phi \equiv 0$ in $\mathbb{R}^N\backslash B_2(0)$ and define

$$u_\varepsilon = \phi w_\varepsilon, \quad v_\varepsilon = \frac{u_\varepsilon}{(\int u_\varepsilon^{2^*})^{1/2^*}}.$$

Lemma 4.2. *There exist positive constants* k_1*,* k_2 *and* ε_0 *such that*

$$\int_{\mathbb{R}^N\backslash B_1(0)} |\nabla u_\varepsilon|^2 = O(\varepsilon^{N-2}) \quad \text{as } \varepsilon \to 0^+,$$

$$k_1 < \int u_\varepsilon^{2^*} < k_2, \quad \forall 0 < \varepsilon < \varepsilon_0,$$

$$\int_{|x|\le 1}|x|^{N-2}w_\varepsilon^{2^*}=O(\varepsilon^{N-2})\quad \text{as } \varepsilon\to 0^+,$$
$$\int|\nabla v_\varepsilon|^2\le S+O(\varepsilon^{N-2})\quad \text{as } \varepsilon\to 0^+.$$

Lemma 4.3. *As $\varepsilon\to 0$, we have*

$$\|v_\varepsilon\|_2^2=\begin{cases}O(\varepsilon), & \text{if } N=3,\\ O(\varepsilon^2|\log\varepsilon|), & \text{if } N=4,\\ O(\varepsilon^2), & \text{if } N\ge 5,\end{cases}$$
$$\|v_\varepsilon\|_{2^*-\frac12}^{2^*-\frac12}=O(\varepsilon^{\frac{N-2}{4}}).$$

Proposition 4.4. *If conditions* (A4'), (A5'), (A8), (A9) *hold. Then there exists $v\in E\backslash\{0\}$ such that*

$$\max_{t\ge 0}I_0(tv)<\frac{1}{2N}S^{\frac N2}.$$

Lemma 4.5. *If $\{v_n\}\subset E$ is a bounded $(PS)_c$ sequence for the functional I_0, then up to a subsequence, $v_n\rightharpoonup v\not\equiv 0$ with $I_0'(v)=0$.*

Proof. Since $\{v_n\}$ is bounded, going if necessary to a subsequence, $v_n\rightharpoonup v$ in E. It is obvious that $I_0'(v)=0$. If $v\not\equiv 0$, the proof is complete.

If $v=0$, we claim that $\{v_n\}$ is also a $(PS)_c$ sequence for I_1. Indeed, we have

$$I_1(v_n)-I_0(v_n)=\frac12\int[V^+(\infty)-V^+(x)]f^2(v_n)+\frac12\int V^-(x)f^2(v_n)\to 0,$$

using (A8), Lemma 2.1(3) and $v_n^2\rightharpoonup 0$ in $L^{N/(N-2)}$. Similarly we derive

$$\begin{aligned}\sup_{\|u\|\le 1}|\langle I_1'(v_n)-I_0'(v_n),u\rangle|&=\sup_{\|u\|\le 1}\Big|\int(V^+(\infty)-V^+(x))f(v_n)f'(v_n)u\Big|\\&\quad+\sup_{\|u\|\le 1}\Big|\int V^-(x)f(v_n)f'(v_n)u\Big|\to 0.\end{aligned}$$

In view of Proposition 4.4, we observe that $0<\beta_0\le c<\frac{1}{2N}S^{\frac N2}$, where the constant β_0 will be stated in the proof of Theorem 1.2. Furthermore, by Proposition 4.1, there exists a sequence $(y_n)\subset\mathbb{R}^N$ and $r,\eta>0$ such that $|y_n|\to+\infty$ and

$$\limsup_{n\to\infty}\int_{B_r(y_n)}v_n^2\ge\eta>0,\ \forall n\in\mathbb{N}.$$

Defining $u_n(x)=v_n(x+y_n)$, we know $\{u_n(x)\}$ is also a $(PS)_c$ sequence for I_1. Thus, going to a subsequence if necessary, there exists $u\in E$ such that $u_n\rightharpoonup u$ in E and $I_1'(u)=0$ with $u\not\equiv 0$. We obtain that by Fatou's Lemma

$$c=\limsup_{n\to\infty}[I_1(u_n)-\frac12 I_1'(u_n)u_n]\ge I_1(u)-\frac12 I_1'(u)u=I_1(u).$$

Our next task is to verify that $\max_{t\ge0}I_1(tu)=I_1(u)\le c$. For that, we define the function $\eta(t):=I_1(tu)$ for $t\ge 0$. Since u is a critical point of I_1, it follows that $u>0$ (see the proof in [15]). Then we obtain

$$\eta'(t)=t\int|\nabla u|^2+\int V^+(\infty)f(tu)f'(tu)u$$

$$
\begin{aligned}
&- \int |f(tu)|^{22^*-2} f(tu) f'(tu) u - \int g(f(tu)) f'(tu) u \\
&= t\Big\{ \int |\nabla u|^2 - \int \Big[\frac{|f(t|u|)|^{22^*-2} f(t|u|) f'(t|u|)}{t|u|} \\
&\quad + \frac{g(f(t|u|)) f'(t|u|)}{t|u|} - \frac{V^+(\infty) f(t|u|) f'(t|u|)}{t|u|} \Big] u^2 \Big\}.
\end{aligned}
$$

Note that, fixed $x \in \mathbb{R}^N$, the function $\vartheta : (0, +\infty) \to \mathbb{R}$ defined by

$$
\begin{aligned}
\vartheta(s) &= \frac{f^{22^*-1}(s) f'(s)}{s} + \frac{g(f(s)) f'(s)}{s} - \frac{V^+(\infty) f(s) f'(s)}{s} \\
&= \frac{f^{22^*-1}(s) f'(s)}{s} + \frac{g(f(s))}{f^3(s)} \cdot \frac{f^3(s) f'(s)}{s} + V^+(\infty)(-\frac{f(s) f'(s)}{s})
\end{aligned}
$$

is increasing by Lemma 2.1(11,12,13) and (A10). Now we observe that $\eta'(1) = 0$, since u is a critical point of I_1. Moreover, we have that $\eta'(t) > 0$ for $0 < t < 1$ and $\eta'(t) < 0$ for $t > 1$. Therefore, $I_1(u) = \eta(1) = \max_{t \geq 0} \eta(t) = \max_{t \geq 0} I_1(tu)$ and then

$$
c \leq \max_{t \geq 0} I_0(tu) \leq \max_{t \geq 0} I_1(tu) = I_1(u) \leq c.
$$

This implies that there exists a way $r_0 \in \Gamma$ such that $c = \max_{t \in [0,1]} I_0(r_0(t)) > 0$, and hence, I_0 possesses a critical point v on level c. It follows from $c \geq \beta_0 > 0 = I_0(0)$ that v is a nonzero critical point of I_0. □

Proof of Theorem 1.2. The proof is similar as the one of Theorem 1.1. Only we modify the proof of Lemma 3.2 that

$$
I_0(v) \geq \frac{\alpha}{2} \rho^2 - \frac{\varepsilon}{2} \tau_2^2 \rho^2 - \frac{2^{2^*/2}}{22^*} S^{-2^*/2} \rho^{2^*} - C_\varepsilon \rho^{p/2},
$$

for every $\|v\| = \rho$. Choosing for all $\varepsilon \in (0, \frac{\alpha}{\tau_2^2})$ and ρ sufficiently small, we derive that there exists a constant β_0 such that $\inf_{\|v\| = \rho} I_0(v) \geq \beta_0 > 0$. Combining this fact with Lemma 3.3, the functional I_0 has a mountain pass geometry. So the $(PS)_c$ sequence (v_n) exists, where

$$
c := \inf_{r \in \Gamma} \max_{t \in [0,1]} I_0(r(t)), \quad \Gamma := \{ r \in C([0,1], E) : r(0) = 0, I_0(r(1)) < 0 \}.
$$

It follows from Lemma 3.4 that (v_n) is a bounded $(PS)_c$ sequence for the functional I_0. Lemma 4.5 ensures that $I_0'(v) = 0$ and $v \not\equiv 0$. □

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