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EXISTENCE OF INFINITELY MANY SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS

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Abstract. In this article, we study the existence and infinitely many solutions for the elliptic boundary-value problem

$$\begin{gathered}-\Delta u + a(x)u = f(x,u) \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega.\end{gathered}$$

Our main tools are the local linking and symmetric mountain pass theorem in critical point theory.

1. Introduction and statement of main results

In this article, we investigate the elliptic boundary-value problem

$$\begin{gathered}-\Delta u + a(x)u = f(x,u) \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is an open bounded domain with smooth boundary $\partial\Omega$, $a \in L^{N/2}(\Omega)$, and the nonlinearity $f \in \mathcal{C}(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfies some of the following hypotheses:

(H1) There exist constants $\alpha \geq 1$, $C_0 \geq 0$ such that
$$\alpha G(x,t) + C_0 \geq G(x,st) \quad \forall t \in \mathbb{R},\ x \in \bar{\Omega},\ s \in [0,1],$$
where
$$G(x,t) := tf(x,t) - 2F(x,t), F(x,t) = \int_0^t f(x,s)ds.$$

(H1') There exists $t^* > 0$ such that for all $x \in \Omega$, $f(x,t)/t$ is increasing for $t \geq t^*$ and decreasing for $t \leq -t^*$.

(H2) $\lim_{|t|\to\infty} f(x,t)/(t|t|^{2^*-2}) = 0$ uniformly for almost every (a.e.) $x \in \Omega$, $2^* = 2N/(N-2)$.

(H3) $\lim_{|t|\to\infty} F(x,t)/t^2 = +\infty$ uniformly for a.e. $x \in \Omega$.

(H4) $\lim_{t\to 0} f(x,t)/t = 0$ uniformly in $x \in \Omega$.

(H5) $f \in \mathcal{C}(\Omega \times \mathbb{R}, \mathbb{R})$, and there exists constants $C_1 > 0$ and $p \in (2, 2^*)$ such that
$$|f(x,t)| \leq C_1(1 + |t|^{p-1}), \quad \forall (x,t) \in \Omega \times \mathbb{R};$$

(H6) $\frac{\lambda_n}{2} t^2 \leq F(x,t)$, for all $(x,t) \in \bar{\Omega} \times \mathbb{R}$ in which λ_n is an eigenvalue of $-\Delta + a$.

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(H7) There exists a constant $C > 0$ such that
$$G(x,t) \le G(x,s) + C$$
for each $x \in \Omega$, $0 < t < s$ or $s < t < 0$ where $G(x,t)$ is the same as in (H1).

(H8) For some $\delta > 0$, either
$$F(x,t) \ge 0 \quad \text{for } |t| \le \delta, x \in \Omega,$$
or
$$F(x,t) \le 0 \quad \text{for } |t| \le \delta, x \in \Omega.$$

There has been a great deal of interest in semilinear elliptic equations in previous years. With the aid of variational methods, the existence and multiplicity of solutions for (1.1) have been extensively investigated in the literature [1]-[12] and references therein. According to the growth of the primitive $F(x,t) := \int_0^t f(x,s)ds$ of the nonlinearity f near infinity in t, the existing literature usually distinguishes between the situations of the sub-quadratic and super-quadratic. For the later situation, most of the results were obtained under the (AR) condition (see [1]): there exits $\mu > 2, l_0 > 0$ such that
$$0 < \mu F(x,t) \le t f(x,t), \quad \forall |t| \ge l_0, x \in \Omega.$$
In [1], the authors developed the dual variational methods and obtained infinitely many solutions of (1.1) under the (AR) condition. There are many other results obtained under the (AR) condition. See [20]-[23] and the references therein. However, this condition eliminates many nonlinearities, among them the function given in [11],
$$f(x,t) = 2t\ln(1 + |t|)\,.$$

Some new super-quadratic conditions are established instead of (AR) in [8]-[10] and [16]. Among them, a few are weaker than (AR), but most complement it, such as the monotonicity condition on $f(x,t)/t$. In [8], the authors obtained the infinitely many solutions of problem (1.1) under some weak super-quadratic conditions, but the conditions there actually imply that $F(x,t)$ is of μ-order ($\mu > 2$) growth near infinity with respect to t. After that, many efforts have been made to extend the results. In [14], the authors obtained problem (1.1) possesses at least one nontrivial solutions with $a \in L^\infty(\Omega)$, $f(x,u)$ satisfies the (AR) condition and (H4), (H5), (H8).

Based on linking theorem, Li and Wang obtained the following theorem:

Theorem 1.1 ([13, Theorem 1.1])**.** *Suppose that Ω is a bounded domain in $\mathbb{R}^N$ with $N \ge 3$ and $a \in L^{N/2}(\Omega)$. Under the hypotheses* (H3)–(H7)*, problem* (1.1) *has at least one nontrivial solution.*

In [11], the authors obtained the following theorem:

Theorem 1.2 ([11, Theorem 1.2])**.** *Suppose that* (H1)–(H4) *hold and $a(x) = 0$. Then* (1.1) *has a weak nontrivial solution.*

Motivated by [11, 13, 14], we show that (1.1) possesses at least one, or infinitely many nontrivial solutions by using critical point theorem. Then we have the following theorems

Theorem 1.3. *Suppose that f satisfies* (H1)–(H4), (H8)*, and 0 is an eigenvalue of $-\Delta + a$. Then* (1.1) *has at least one nontrivial solution.*

Theorem 1.4. *Suppose that f is odd and* (H1)–(H3) *hold. Then* (1.1) *has infinitely many nontrivial solutions.*

Corollary 1.5. *Suppose that f is odd and the assumptions* (H1'), (H2), (H3) *hold. Then problem* (1.1) *has infinitely many nontrivial solutions.*

Remark 1.6. Comparing our results with [14, 13, 11], we obtain at least one, or infinitely many solutions of (1.1) under fewer and weaker conditions.

Theorem 1.3 has weaker conditions than Theorem 1.1. It is obvious that (H5) implies (H2). We can easily prove that (H1) is equivalent to (H7) if $\alpha = 1$, and (H1) gives some general sense of monotony when $\alpha > 1$. There are functions satisfying (H1) but not (H7). For example (see in [11]), if

$$F(x,t) = t^2 \ln(1+t^2) + t \sin t,$$

then

$$f(x,t) = 2t \ln(1+t^2) + t^2 \cdot \frac{2t}{1+t^2} + \sin t + t \cos t,$$

and $G(x,t) = tf(x,t) - 2F(x,t)$ satisfies (H1) but not (H7) when α large enough. This means (H1) is weaker than (H7).

Comparing Theorem 1.4 with Theorem 1.2, the condition on $a(x)$ is weaker, (H4) is eliminated, we obtain infinitely many solutions rather than one nontrivial solution.

In (H2), we have functionals satisfying the so-called nonstandard growth conditions. Because the lack of compactness of the embedding in $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, we cannot use the standard variational directly. We overcome this difficulty by using the Vitaly convergence theorem and some analysis technics.

This article is organized as follows. In section 2 we present some definitions and preliminary results. In section 3 we give the proof of our results.

2. Preliminaries

In this section we give some definitions and preliminary results, which are used in Section 3. Let $E := H_0^1(\Omega)$ be the Sobolev space equipped with the inner product and the norm:

$$\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v dx, \quad \|u\| = \langle u, u\rangle^{1/2}.$$

Recall that a function $u \in E$ is called a weak solution of (1.1) if

$$\int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega a(x) uv dx = \int_\Omega f(x,u) v dx, \quad \forall v \in E,$$

which is equivalent to a critical point of the $\mathcal{C}^1$ functional

$$I(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 + a(x) u^2 dx - \int_\Omega F(x,u) dx, \quad u \in E.$$

We denote a subsequence of a sequence $\{u_n\}$ as $\{u_n\}$ to simplify the notation unless specified. We need the following concept which is a weak version of the (PS) condition (see [3]):

Definition 2.1. We say that $I \in \mathcal{C}^1(E, \mathbb{R})$ satisfies the Cerami condition at level $c \in \mathbb{R}$ ($(Ce)_c$ for short) if any sequence $\{u_n\} \subseteq E$ with

$$I(u_n) \to c, \quad (1 + \|u_n\|)\|I'(u_n)\| \to 0$$

possesses a convergent subsequence in E; I satisfies the (Ce) condition if I satisfies condition $(Ce)_c$ for all $c \in \mathbb{R}$.

The definition below is a weak version of the $(PS)^*$ condition.

Definition 2.2 ([17, Definition 2.1]). *A functional $I \in \mathcal{C}^1(E, \mathbb{R})$ satisfies the $(Ce)^*$ condition if every sequence $\{u_{\alpha_n}\}$ such that $\{\alpha_n\}$ is admissible and*

$$u_{\alpha_n} \in X_{\alpha_n}, \quad \sup I(u_{\alpha_n}) < +\infty, \quad (1 + \|u_{\alpha_n}\|) I'(u_{\alpha_n}) \to 0$$

contains a subsequence which converges to a critical point of I.

The following propositions are our main tools, which can be found in [14] and [19] respectively.

Proposition 2.3 ([17, Theorem 2.2]). *For a real Banach space B with a direct decomposition $B = B^1 \oplus B^2$, the following two sequence of subspace satisfies that*

$$B_0^1 \subset B_1^1 \subset \dots \subset B^1, B_0^2 \subset B_1^2 \subset \dots \subset B^2, \quad B^j = \overline{\cup_{n\in\mathbb{N}} B_n^j},\ j = 1, 2.$$

and $\dim B_n^j < \infty$, $j = 1, 2, n \in \mathbb{N}$. Then $I \in \mathcal{C}(B, \mathbb{R})$ satisfies the following:

- (i) *I has a local linking at 0 and $B^1 \neq 0$,*
- (ii) *I satisfies the $(Ce)^*$ condition,*
- (iii) *I maps bounded sets into bounded sets,*
- (iv) *for every $m \in \mathbb{N}$, $I(u) \to -\infty$, $\|u\| \to \infty$, $u \in B_m^1 \oplus B^2$.*

Then I has at least two critical points.

Proposition 2.4 ([19, Theorem 9.12]). *Let E be an infinite dimensional Banach space and let $I \in \mathcal{C}^1(E, \mathbb{R})$ be even, satisfy (PS), and $I(0) = 0$. If $E = V \oplus X$, where V is finite dimensional, and I satisfies*

- (I1') *there are constants $\rho, \alpha > 0$ such that $I\mid_{\partial B_\rho \cap X} \geq \alpha$, and*
- (I2') *for each finite dimensional subspace $\widetilde{E} \subset E$, there is an $R = R(\widetilde{E})$ such that $I \leq 0$ on $\widetilde{E} \backslash B_R(\widetilde{E})$.*

Then I possesses an unbounded sequence of critical values.

Next we recall something about the eigenvalues of elliptic operators (see [20]). According to the theory of spectrum of compact operators, we let

$$-\infty < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n < 0 \leq \lambda_{n+1} \leq \lambda_{n+2} \leq \dots$$

be the sequence for the eigenvalue problem

$$\begin{gathered} -\Delta u + a(x)u = \lambda u, \\ u \in E \end{gathered} \tag{2.1}$$

where each eigenvalue is replaced according to its multiplicity. $\lim_{j\to\infty} \lambda_j = +\infty$ and $\lambda_1 = \inf_{u \in E, |u|_2 = 1} \int_\Omega [|\nabla u|^2 + a(x)u^2] dx$. Let $e_1, e_2, \dots, e_n, e_{n+1} \dots$ be the corresponding orthonormal eigenfunctions in $L^2(\Omega)$. Then a direct decomposition of E can be defined as follows:

$$V := \operatorname{span}\{e_1, e_2, \dots, e_n\},$$
$$X := \big\{u \in E : \int_\Omega uv dx = 0, v \in V\big\}.$$

Then $\dim V < +\infty$, $\dim X = +\infty$, $E = V \oplus X$.

3. Proof of Theorems

In this section, we prove our results.

Proof of Theorem 1.3. We shall apply Proposition 2.3 to the functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega a(x)u^2 dx - \int_\Omega F(x,u)dx$$

defined on E. We consider only the case when 0 is an eigenvalue of $-\Delta + a$ and

$$F(x,u) \le 0 \quad \text{for } |u| \le \delta. \tag{3.1}$$

Then other cases are similar and simpler.

Suppose that $E = V \oplus X$ and V be the (finite dimensional) space spanned by the eigenfunctions corresponding to negative eigenvalues of $-\Delta + a$ and X be its orthogonal complement in E. Choose an Hilbertain basis $(e_n)_{n\ge 0}$ for X and define

$$X_m = span(e_0, e_1, \dots, e_m), m \in \mathbb{R}$$

(i) We claim that I has a local linking at 0 with respect to (V, X). Decompose X into $X^1 + X^2$ where $X^1 = ker(-\Delta + a), X^2 = (V + X^1)^\perp$. For $u \in X$, we have $u = u_1 + u_2$, $u_1 \in X^1, u_2 \in X^2$. Since $\dim X^1 < \infty$, there exists $C > 0$ such that

$$\|u_1\|_\infty \le C_2 \|u_1\|, \quad \text{for all } u_1 \in X^1. \tag{3.2}$$

It follows from (H2) and (H4) that, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(x,t)| \le \varepsilon t^2 + C_\varepsilon |t|^{2^*}. \tag{3.3}$$

Then, on V, for some $C > 0$,

$$I(u) \le \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega a(x)u^2 dx + \varepsilon \int_\Omega u^2 dx + C\|u\|^{2^*},$$

and hence, for $r > 0$ small enough,

$$I(u) \le 0, \quad u \in V, \quad \|u\| \le r.$$

Let $u = u_1 + u_2 \in X$ such that $\|u\| \le \frac{\delta}{2C_2}$ and set

$$\Omega_1 = \{x \in \Omega : |u_2(x)| \in \delta/2\}, \quad \Omega_2 = \Omega \setminus \Omega_1.$$

On Ω_1, we have, by (3.2),

$$|u(x)| \le |u_1(x)| + |u_2(x)| \le \|u_1\|_\infty + \frac{\delta}{2} \le \delta,$$

hence, by (3.1),

$$\int_{\Omega_1} F(x,u)dx \le 0.$$

On Ω_2, we have, also by (3.2),

$$|u(x)| \le |u_1(x)| + |u_2(x)| \le 2|u_2(x)|.$$

Hence, by (3.3),

$$|F(x,u)| \le \varepsilon |u|^2 + C_\varepsilon |u|^{2^*} \le 4\varepsilon |u_2|^2 + 2^{2^*} C_\varepsilon |u_2|^{2^*}$$

and for some $c > 0$,

$$\int_{\Omega_2} F(x,u)dx \le 4\varepsilon \int_\Omega u_2^2 dx + c\|u_2\|^{2^*}.$$

Therefore,

$$I(u) \geq \frac{1}{2}\int_{\Omega} |\nabla u_2|^2 dx + \frac{1}{2}\int_{\Omega} a(x)u_2^2 dx - 4\varepsilon \int_{\Omega} u_2^2 dx - c\|u_2\|^{2^*} - \int_{\Omega_1} F(x,u)dx$$

and for $0 < r < \delta/(2C)$ small enough,

$$I(u) \geq 0, \quad u \in X, \quad \|u\| \leq r.$$

(ii) We claim that I satisfies $(Ce)^*$ condition. Consider a sequence $\{u_{\alpha_n}\}$ such that $\{\alpha_n\}$ is admissible and

$$u_{\alpha_n} \in E_{\alpha_n}, \quad c = \sup I(u_{\alpha_n}) < +\infty, \quad (1 + \|u_{\alpha_n}\|)I'(u_{\alpha_n}) \to 0. \tag{3.4}$$

Here, $c \in \mathbb{R}$, $E_{\alpha_n} = V_{\alpha_n} \oplus X_{\alpha_n}$, $\alpha_n \in \mathbb{N}$, and $V_{\alpha_1} \subset V_{\alpha_2} \subset \cdots \subset V = \overline{\cup_{\alpha_n \in \mathbb{N}} V_{\alpha_n}}$, $X_{\alpha_1} \subset X_{\alpha_2} \subset \cdots \subset X = \overline{\cup_{\alpha_n \in \mathbb{N}} X_{\alpha_n}}$, V_{α_i} and X_{α_i} are subspaces, $i = \alpha_1, \dots, \alpha_n$. We note u_{α_n} with u_n for short.

We first prove that $\{u_n\}$ is bounded in E. If not, then $\|u_n\| \to \infty$ as $n \to \infty$. Let $\omega_n = \frac{u_n}{\|u_n\|}$, then $\omega_n \in E$ and $\|\omega_n\| = 1$. Then there is an $\omega \in E$ such that

$$\begin{gathered} \omega_n \rightharpoonup \omega \quad \text{in } E; \\ \omega_n \to \omega \quad \text{in } L^p(\Omega), \text{ where } 2 \leq p < 2^*; \\ \omega_n \to \omega \quad \text{a.e. in } \Omega. \end{gathered}$$

By the Sobolev Embedding theorem one gets

$$|\omega_n|_{2^*} \leq C_3\|\omega_n\| = C_3,$$

where C_3 is a positive constant. Denote $\Omega_{\neq} = \{x \in \Omega : \omega(x) \neq 0\}$. Then $|\Omega_{\neq}| = 0$. In fact,

$$\lim_{n\to\infty} \frac{u_n(x)}{\|u_n\|} = \lim_{n\to\infty} \omega_n(x) = \omega(x) \neq 0 \text{in} \Omega_{\neq}.$$

Which implies $|u_n(x)| \to +\infty$ a.e. in $\Omega_{\neq}$. Then we obtain

$$\lim_{n\to+\infty} \frac{F(x,u(x))}{|u_n(x)|^2} = +\infty \quad \text{a.e. in } \Omega_{\neq}. \tag{3.5}$$

By (H3), there exists a constant $C_4 > 0$ such that

$$\frac{F(x,t)}{|t|^2} > 1$$

for all $x \in \Omega$ and $t \geq C_4$. Since $F(x,t)$ is continuous on $\bar{\Omega} \times [-C_4, C_4]$, there exists $C > 0$ such that

$$|F(x,t)| \leq C \quad \text{for all } (x,t) \in \bar{\Omega} \times [-C_4, C_4].$$

Then we see that there exists a constant $\widetilde{C}$ such that

$$F(x,t) \geq \widetilde{C} \quad \text{for all } (x,t) \in \bar{\Omega} \times \mathbb{R}. \tag{3.6}$$

This implies

$$\frac{F(x,u_n(x))}{|u_n(x)|^2}|\omega_n(x)|^2 - \frac{\widetilde{C}}{\|u_n\|^2} \geq 0. \tag{3.7}$$

By the definition of $(Ce)^*$ condition, we have

$$c \geq I(u_n) = \frac{1}{2}\|u_n\|^2 + \frac{1}{2}\int_{\Omega} a(x)u_n^2 dx - \int_{\Omega} F(x,u_n)dx. \tag{3.8}$$

We have

$$\frac{1}{2}+\frac{1}{2}\int_\Omega a(x)\omega_n^2dx=\int_\Omega \frac{F(x,u_n)}{|u_n|^2}\omega_n^2dx+o(1). \tag{3.9}$$

If $|\Omega_{\neq}|>0$, then by (H3), (3.5) and (3.7), combining with Fatou's Lemma, one has

$$\begin{aligned}
+\infty &= \int_{\Omega_{\neq}} \liminf_{n\to\infty}\frac{F(x,u_n(x))}{|u_n(x)|^2}|\omega_n(x)|^2dx-\int_{\Omega_{\neq}}\limsup_{n\to\infty}\frac{\widetilde{C}}{\|u_n\|^2}dx\\
&\le \int_{\Omega_{\neq}} \liminf_{n\to\infty}\Big(\frac{F(x,u_n(x))}{|u_n(x)|^2}|\omega_n(x)|^2-\frac{\widetilde{C}}{\|u_n\|^2}\Big)dx\\
&\le \liminf_{n\to\infty}\int_{\Omega_{\neq}} \Big(\frac{F(x,u_n(x))}{|u_n(x)|^2}|\omega_n(x)|^2-\frac{\widetilde{C}}{\|u_n\|^2}\Big)dx\\
&\le \liminf_{n\to\infty}\int_{\Omega} \Big(\frac{F(x,u_n(x))}{|u_n(x)|^2}|\omega_n(x)|^2-\frac{\widetilde{C}}{\|u_n\|^2}\Big)dx\\
&= \liminf_{n\to\infty}\int_{\Omega}\frac{F(x,u_n(x))}{\|u_n(x)\|^2}dx\\
&\le \frac{1}{2}+\frac{1}{2}\int_\Omega a(x)\omega_n^2dx+o(1)\\
&\le \frac{1}{2}+C_3^2|a(x)|_{\frac{N}{2}}+o(1).
\end{aligned}$$

It is a contradiction. Then we obtain $|\Omega_{\neq}|=0$. Hence $\omega(x)=0$ a.e. in Ω.

Since $I(tu_n)$ is continuous in $t\in[0,1]$, there exists $t_n\in[0,1]$ such that

$$I(t_nu_n)=\max_{t\in[0,1]}I(tu_n).$$

As $\langle I'(u_n),u_n\rangle=o(1)$, we see that

$$\langle I'(t_nu_n),t_nu_n\rangle=o(1).$$

From (H1), for $t\in[0,1]$, we obtain

$$\begin{aligned}
2I(tu_n)&\le 2I(t_nu_n)\\
&=2I(t_nu_n)-\langle I'(t_nu_n),t_nu_n\rangle+o(1)\\
&=\int_\Omega[t_nu_nf(x,t_nu_n)-2F(x,t_nu_n)]dx+o(1)\\
&\le \int_\Omega[\alpha(u_nf(x,u_n)-2F(x,u_n))+C_0]dx+o(1)\\
&=\alpha[2I(u_n)-\langle I'(u_n),u_n\rangle]+C_0|\Omega|+o(1)\\
&\le 2\alpha c+C_0|\Omega|+o(1)
\end{aligned} \tag{3.10}$$

Furthermore, by (H2), for any $\varepsilon\ge 0$, there exists $C_\varepsilon>0$ such that

$$|F(x,t)|\le \frac{1}{2C_3^{2^*}}\varepsilon|t|^{2^*}+C_\varepsilon,\quad \text{for } t\in\mathbb{R},\ \text{a.e. } x\in\Omega.$$

Let $\delta=\varepsilon/(2C_\varepsilon)>0$, $A\subseteq\Omega$, $\operatorname{meas}A<\delta$. Then

$$\begin{aligned}
\Big|\int_A F(x,\omega_n)dx\Big| &\le \int_A|F(x,\omega_n)|dx\\
&\le \int_A C_\varepsilon dx+\frac{1}{2C_3^{2^*}}\varepsilon\int_A|\omega_n|^{2^*}dx
\end{aligned}$$

$$\begin{aligned}&\le \int_A a(\varepsilon)dx + \frac{1}{2C_3^{2^*}}\varepsilon\int_\Omega |\omega_n|^{2^*}dx\\&\le \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon\end{aligned}$$

So we obtain $\{\int_\Omega F(x,\omega_n)dx, n \in N\}$ is equi-absolutely continuous. Then

$$\int_\Omega F(x,\omega_n)dx \to \int_\Omega F(x,0)dx = 0$$

from the Vitali's convergence theorem.

On the other hand, the functional

$$\chi : u \mapsto \int_\Omega a(x)u^2 dx$$

is weakly continuous when $a \in L^{\frac{N}{2}}(\Omega)$. Then

$$\int_\Omega a(x)\omega_n^2 dx \to 0 \quad \text{when } n \to \infty.$$

This implies for any $s > 0$,

$$\begin{aligned}2I(s\omega_n) &= \|s\omega_n\|^2 + s^2\int_\Omega a(x)\omega_n^2 dx - 2\int_\Omega F(x,s\omega_n)dx\\&= s^2 + o(1).\end{aligned}$$

Combining with (3.10) we obtain

$$s^2 + o(1) = 2I(s\omega_n) \le 2\alpha c + C_0|\Omega| + o(1).$$

For the arbitrariness of s, we obtain a contradiction. Hence $\|u_n\|$ is bounded in E.

Then, going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in X. Then we have

$$\begin{aligned}\|u_n - u\|^2 =& \langle I'(u_n) - I'(u), u_n - u\rangle - \int_\Omega [a(u_n-u)^2\\&- (f(x,u_n) - f(x,u))(u_n-u)]dx.\end{aligned}$$

this means that $u_n \to u$ in E and $I'(u) = 0$.

(iii) It is obvious that I maps bounded sets into bounded sets.

(iv) Finally, we claim that, for every $m \in \mathbb{N}$,

$$I(u) \to -\infty \quad \text{for } \|u\| \to \infty,\ u \in V \oplus X_m.$$

In fact, from (H3), we know that for all $M > 0$, there exists C_M such that

$$F(x,u) \ge Mu^2 - C_M. \tag{3.11}$$

Then

$$\begin{aligned}I(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_\Omega au^2dx - \int_\Omega F(x,u)dx\\&\le \frac{1}{2}\|u\|^2 + |a|_{\frac{N}{2}}|u|_{2^*}^2 - \int_\Omega F(x,u)dx\\&\le \frac{1}{2}\|u\|^2 + C\|u\|^2 - M\overline{C}\|u\|^2 - C_M|\Omega|\\&= \Big(\frac{1}{2} + C - M\overline{C}\Big)\|u\|^2 - C_M|\Omega|.\end{aligned} \tag{3.12}$$

In the above inequality, one can always find $M > 0$ large enough such that $\frac{1}{2} + C - M\overline{C} < 0$. This implies $I(u) \to -\infty$ for $\|u\| \to \infty, u \in V \oplus X_m$. The proof is complete. $\square$

Proof of Theorem 1.4. We use Proposition 2.4. It is clear that $I(0) = 0$. Similar to the proof of (ii) in Theorem 1.3, we know I satisfies the (Ce) condition. According to [21] we know, Proposition 2.4 holds under the (Ce) condition. Then in this section, we need only to prove I satisfies (I1') and (I2'). Similar to the analysis in [22], we obtain E possesses the orthogonal decomposition $E = E^- \oplus E^0 \oplus E^+$ with $E^- = \mathcal{L}^- = \operatorname{span}\{e_1, e_2, \dots, e_{n-1}\}$, $E^0 = \mathcal{L}^0 = \ker(-\Delta + a)$, $E^+ = \mathcal{L}^+ = \overline{\operatorname{span}\{e_n, e_{n+1}, \dots\}}$. Then for all $u \in E$, we have $u = u^- + u^0 + u^+ \in E^- \oplus E^0 \oplus E^+$ and the corresponding functional of (1.1) as follows:

$$
\begin{aligned}
I(u) &= \frac{1}{2}\int_\Omega (|\nabla u|^2 + a(x)u^2)dx - \int_\Omega F(x,u)dx \\
&= \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_\Omega F(x,u)dx.
\end{aligned}
$$

For $f \in \mathcal{C}(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ and (H2), then for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that $f(x,u) \le 2^*\varepsilon|u|^{2^*-1} + C_\varepsilon$. Then

$$
F(x,u) \le \varepsilon u^{2^*} + C_\varepsilon u. \tag{3.13}
$$

Then for all $u \in E^+$, we have $u = u^+$ and $|u|_2^2 \le \frac{1}{\lambda_n}\|u\|^2, \lambda_n$ is an eigenvalue of $-\Delta + a$. Let $\varepsilon = 1$ in (3.13), combining with the Hölder inequality, for all $u \in E^+$, we obtain

$$
\begin{aligned}
I(u) &= \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_\Omega F(x,u)dx \\
&= \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u)dx \\
&\ge \frac{1}{2}\|u\|^2 - |u|_{2^*}^{2^*} - C|u|_1 \\
&\ge \frac{1}{2}\|u\|^2 - C\|u\|^{2^*} - C|u|_1 \\
&\ge \frac{1}{2}\|u\|^2 - C\|u\|^{2^*} - C\frac{1}{\sqrt{\lambda_n}}\|u\| \\
&= \Big(\frac{1}{4}\|u\|^2 - C\|u\|^{2^*}\Big) + \Big(\frac{1}{4}\|u\|^2 - C\frac{1}{\sqrt{\lambda_n}}\|u\|\Big).
\end{aligned}
$$

In the above inequality, one can find a $u_0 \in E^+$ such that $\frac{1}{4}\|u_0\|^2 - C\|u_0\|^{2^*} > 0$. When $\lambda_n \ge (\frac{4C_\varepsilon}{\|u_0\|})^2$, we have $\frac{1}{4}\|u_0\|^2 - C_\varepsilon\frac{1}{\sqrt{\lambda_n}}\|u_0\| \ge 0$.

For $k \in N$ such that $\lambda_k \ge (\frac{4C}{\|u_0\|})^2$, and let

$$
Z = \overline{\operatorname{span}\{e_k, e_{k+1}, \dots\}}, \quad Y = \{u \in E : \int_\Omega uv\,dx = 0, v \in Z\},
$$

then $E = Y \oplus Z$. Let $\alpha = \frac{1}{4}\|u_0\|^2 - \varepsilon C\|u_0\|^{2^*} > 0$, then we obtain for all $\|u\| = \|u_0\|$ in Z, $I(u) \ge \alpha > 0$. This implies $I(u)$ satisfies (I1').

Now we prove $I(u)$ satisfies (I2'). Take $\widetilde{E}$ as a finite dimensional subspace of E. Then for any $u \in \widetilde{E}$, combining with (3.11), we have

$$\begin{aligned} I(u) &= \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_\Omega F(x,u)dx \\ &\le \frac{1}{2}\|u^+\|^2 - \int_\Omega (Mu^2 - C_M)dx \\ &= \frac{1}{2}\|u^+\|^2 - M|u|_2^2 + C \\ &= \frac{1}{2}\|u^+\|^2 - M|u^+|_2^2 - M|u^-|_2^2 + C \\ &\le \Big(\frac{1}{2} - M\widetilde{C}\Big)\|u^+\|^2 - M_2\|u^-\|^2 + C. \end{aligned}$$

From the above inequality, one can always find a $u_0 \in \widetilde{E}$ and M large enough such that $\frac{1}{2} - M\widetilde{C} < 0$ and $I(u_0) < 0$. Then there exists $R = R(\widetilde{E})$ such that $I(u) \le 0$ for all $\|u\| \ge \|u_0\| > R(\widetilde{E})$. This means $I(u)$ satisfies (I2'). Then the proof is complete. □

Proof of Corollary 1.5. According to [15, Lemma 2.3], one can show that (H1') implies (H7). Combining Remark 1.6 and the proof of Theorem 1.4, Corollary 1.5 is obtained. □

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