LOW REGULARITY SOLUTIONS FOR CHERN-SIMONS-DIRAC SYSTEMS IN THE TEMPORAL AND COULOMB GAUGE

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ABSTRACT. We prove low regularity local well-posedness results in Bourgain-Klainerman-Machedon spaces for the Chern-Simons-Dirac system in the temporal gauge and the Coulomb gauge. Under slightly stronger assumptions on the data we also obtain "unconditional" uniqueness in the natural solution spaces.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

Consider the Chern-Simons-Dirac system in two space dimensions

$$i\partial_t\psi + i\alpha^j\partial_j\psi = m\beta\psi - \alpha^\mu A_\mu\psi \tag{1.1}$$

$$\partial_\mu A_\nu - \partial_\nu A_\mu = -2\epsilon_{\mu\nu\lambda}\langle\psi, \alpha^\lambda\psi\rangle \tag{1.2}$$

with initial data

$$\psi(0) = \psi_0, \quad A_\mu(0) = a_\mu, \tag{1.3}$$

where we use the convention that repeated upper and lower indices are summed, Latin indices run over 1,2 and Greek indices over 0,1,2 with Minkowski metric of signature $(+,-,-)$. Here $\psi : \mathbb{R}^{1+2} \to \mathbb{C}^2$, $A_\nu : \mathbb{R}^{1+2} \to \mathbb{R}$, $m \in \mathbb{R}$. $\alpha^1, \alpha^2, \beta$ are hermitian (2×2)-matrices satisfying $\beta^2 = (\alpha^1)^2 = (\alpha^2)^2 = I$, $\alpha^j\beta + \beta\alpha^j = 0$, $\alpha^j\alpha^k + \alpha^k\alpha^j = 2\delta^{jk}I$, $\alpha^0 = I$. $\langle\cdot,\cdot\rangle$ denotes the $\mathbb{C}^2$-scalar product. A particular representation is given by

$$\alpha^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$\epsilon_{\mu\nu\lambda}$ is the totally skew-symmetric tensor with $\epsilon_{012} = 1$.

This model was proposed by Cho, Kim and Park [5], and by Li and Bhaduri [11]. The equations are invariant under the gauge transformations

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\chi, \quad \psi \to \psi' = e^{i\chi}\phi.$$

The most common gauges are the Coulomb gauge $\partial^j A_j = 0$, the Lorenz gauge $\partial^\mu A_\mu = 0$ and the temporal gauge $A_0 = 0$.

Local well-posedness for data with minimal regularity assumptions was shown by Huh [7] in the Lorenz gauge for data $\psi_0 \in H^{\frac{5}{8}}$, $a_\mu \in H^{1/2}$ using a null structure, in the Coulomb gauge for $\psi_0 \in H^{\frac{1}{2}+\epsilon}$, $a_i \in L^2$, and in temporal gauge for $\psi_0 \in H^{\frac{3}{4}+\epsilon}$,

$a_j \in H^{\frac{3}{4}+\epsilon} + L^2$, both without using a null structure. The result in Lorenz gauge was improved by Huh-Oh [8] where the regularity of the data was lowered down to $\psi_0 \in H^s$, $a_\mu \in H^s$ with $s > 1/4$. Their proof relies also on a null structure in the nonlinear terms of the Dirac equation as well as the wave equation. They apply a Picard iteration in Bourgain-Klainerman-Machedon spaces $X^{s,b}$, which implies uniqueness in these spaces. Independently Okamoto [12] proved a similar result in Lorenz as well as Coulomb gauge also using a null structure of the system. The methods of Okamoto and Huh-Oh are different. Okamoto reduces the problem to a single Dirac equation with cubic nonlinearity for ψ, which does not contain A_μ any longer. From a solution ψ of this equation the potentials A_μ can be constructed by solving a wave equation in Lorenz gauge and an elliptic equation in Coulomb gauge. Huh-Oh on the other hand directly solve a coupled system of a Dirac equation for ψ and a wave equation for A_μ. Recently Bournaveas-Candy-Machihara [4] proved local well-posedness in Coulomb gauge under similar regularity assumptions without use of a null structure. Their proof relies on a bilinear Strichartz estimate given by Klainerman-Tataru [10].

A low regularity local well-posedness result in temporal gauge was given by Tao [14] for the Yang-Mills equations.

In the present paper we consider the temporal gauge as well as the Coulomb gauge. In temporal gauge we improve the result of Huh [7] to data $\psi_0 \in H^s$, $a_j \in H^{s+\frac{1}{8}}$ with $s > 3/8$. We use Bourgain-Klainerman-Machedon spaces $X^{s,b}$ adapted to the phase functions $\tau \pm |\xi|$ on one hand and τ on the other hand. We decompose A_j into its divergence-free part A_j^{df} and its curl-free part A_j^{cf}. The main problem here is that there seems to be no null structure in the nonlinearity $A_j^{cf} \alpha^j \psi$ in the Dirac equation whereas in Lorenz gauge $A_\mu^{cf} \alpha^\mu \psi$ has such a null structure. In fact all the other terms possess such a null structure. However we are not able to use it for an improvement of our result. We apply the bilinear estimates in wave-Sobolev spaces established in d'Ancona-Foschi-Selberg [2] which rely on Strichartz estimates. Morover we use a variant of an estimate for the $L_x^6 L_t^2$ - norm for the solution of the wave equation which goes back to Tataru and Tao. When applying this estimate we partly follow Tao's arguments in the case of the Yang-Mills equations [14]. We prove existence and uniqueness in $X^{s,b}$ - spaces first (Theorem 1.1). Then we prove unconditional uniqueness under the stronger assumption $s > \frac{19}{40}$ (Theorem 1.2) by using an idea of Zhou [16].

In Coulomb gauge we make the same regularity assumptions as Okamoto [12] and Bournaveas-Candy-Machihara [4], namely $\psi_0 \in H^{\frac{1}{4}+\epsilon}$, and also reduce the problem to a single Dirac equation with cubic nonlinearity. We give a short (alternative) proof of local well-posedness in $X^{s,b}$ - spaces without use of a null structure (Theorem 1.3) using d'Ancona-Foschi-Selberg [2] (cf. Proposition 1.5). We also prove unconditional uniqueness in the space $\psi \in C^0([0,T], H^s)$ under the assumption $s > 1/3$ (Theorem 1.4).

We first give some notation. We denote the Fourier transform with respect to space and time by $\widehat{\ }$. The operator $|\nabla|^\alpha$ is defined by $\mathcal{F}(|\nabla|^\alpha f)(\xi) = |\xi|^\alpha (\mathcal{F}f)(\xi)$, where $\mathcal{F}$ is the Fourier transform, and similarly $\langle\nabla\rangle^\alpha$, where $\langle\cdot\rangle := (1+|\cdot|^2)^{1/2}$. The inhomogeneous and homogeneous Sobolev spaces are denoted by $H^{s,p}$ and $\dot{H}^{s,p}$, respectively. For $p = 2$ we simply denote them by H^s and $\dot{H}^s$. We repeatedly use the Sobolev embeddings $\dot{H}^{s,p} \hookrightarrow L^q$ for $1 < p \leq q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{s}{2}$, and also

$\dot{H}^{1+} \cap \dot{H}^{1-} \hookrightarrow L^\infty$ in two space dimensions. $a+ := a + \epsilon$ for a sufficiently small $\epsilon > 0$, so that $a < a+ < a++$, and similarly $a-- < a- < a$.

We define the standard spaces $X_\pm^{s,b}$ of Bourgain-Klainerman-Machedon type belonging to the half waves as the completion of the Schwarz space $\mathcal{S}(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{X_\pm^{s,b}} = \|\langle\xi\rangle^s \langle\tau \pm |\xi|\rangle^b \widehat{u}(\tau,\xi)\|_{L_{\tau\xi}^2}.$$

Similarly we define the wave-Sobolev spaces $X_{|\tau|=|\xi|}^{s,b}$ with norm

$$\|u\|_{X_{|\tau|=|\xi|}^{s,b}} = \|\langle\xi\rangle^s \langle|\tau| - |\xi|\rangle^b \widehat{u}(\tau,\xi)\|_{L_{\tau\xi}^2}$$

and also $X_{\tau=0}^{s,b}$ with norm

$$\|u\|_{X_{\tau=0}^{s,b}} = \|\langle\xi\rangle^s \langle\tau\rangle^b \widehat{u}(\tau,\xi)\|_{L_{\tau\xi}^2}.$$

We also define $X_\pm^{s,b}[0,T]$ as the space of the restrictions of functions in $X_\pm^{s,b}$ to $[0,T] \times \mathbb{R}^2$ and similarly $X_{|\tau|=|\xi|}^{s,b}[0,T]$ and $X_{\tau=0}^{s,b}[0,T]$. We frequently use the obvious embeddings $X_{|\tau|=|\xi|}^{s,b} \hookrightarrow X_\pm^{s,b}$ for $b \le 0$ and $X_\pm^{s,b} \hookrightarrow X_{|\tau|=|\xi|}^{s,b}$ for $b \ge 0$.

We now formulate our main results in the case of the temporal gauge.

Theorem 1.1. *Let $\epsilon > 0$ and $s > 3/8$. The Chern-Simons-Dirac system (1.1), (1.2), (1.3) in temporal gauge $A_0 = 0$ with data $\psi_0 \in H^s(\mathbb{R}^2)$, $a_j \in H^{s+\frac{1}{8}}(\mathbb{R}^2)$, satisfying the compatibility condition $\partial_1 a_2 - \partial_2 a_1 = -2\langle\psi_0, \psi_0\rangle$, has a local solution*

$$\psi \in C^0([0,T], H^s(\mathbb{R}^2)), \quad |\nabla|^\epsilon A_j \in C^0([0,T], H^{s+\frac{1}{8}-\epsilon}(\mathbb{R}^2)).$$

More precisely $\psi = \psi_+ + \psi_-$ with $\psi_\pm \in X_\pm^{s,\frac{1}{2}+}[0,T]$. If $A = A^{df} + A^{cf}$ is the decomposition into its divergence-free part and its "curl-free" part, where

$$A^{df} = (-\Delta)^{-1}(\partial_2(\partial_1 A_2 - \partial_2 A_1), \partial_1(\partial_2 A_1 - \partial_1 A_2)),$$

$$A^{cf} = -(-\Delta)^{-1}(\partial_1(\partial_1 A_1 + \partial_2 A_2), \partial_2(\partial_1 A_1 + \partial_2 A_2)) = -(-\Delta)^{-1}\nabla \operatorname{div} A,$$

one has

$$A^{cf} \in X_{\tau=0}^{s+\frac{1}{8},\frac{1}{2}+}[0,T], \quad |\nabla|^\epsilon A^{df} \in X_{|\tau|=|\xi|}^{s+\frac{3}{8}-\epsilon,\frac{1}{2}+}[0,T]$$

and in these spaces uniqueness holds. Moreover we have $\psi_\pm \in X_\pm^{s,1}[0,T]$.

Remark. The Chern-Simons-Dirac system is invariant under the scaling

$$\psi^{(\lambda)}(t,x) = \lambda\psi(\lambda t, \lambda x), \ A^{(\lambda)}(t,x) = \lambda A_\mu(\lambda t, \lambda x).$$

Thus in 2+1 dimensions the scaling critical Sobolev exponent is $s = 0$, i.e. ψ_0, $a_\mu \in H^s = L^2$. In Lorenz gauge Huh-Oh [8] remarked that their result $s > 1/4$ is probably optimal in view of Zhou [15], who proved that is the case for a system of nonlinear wave equations with nonlinearities, which fulfill a null condition. In our case of the temporal gauge however the system is reduced to a coupled system of a wave equation for ψ and a transport equation for A^{cf} where null conditions seem to be not useful because they are only adapted for wave equations. Nevertheless it would be desirable to improve our result to $s > 1/4$ for ψ_0 and a_j.

Theorem 1.2. *Let the assumptions of Theorem 1.1 be fulfilled. If $s > 19/40$, the solution of (1.1), (1.2), (1.3) is unique in the space $\psi \in C^0([0,T], H^s(\mathbb{R}^2))$, $A^{cf} \in C^0([0,T], H^{s+\frac{1}{8}}(\mathbb{R}^2))$, $|\nabla|^\epsilon A^{df} \in C^0([0,T], H^{s+\frac{3}{8}-\epsilon}(\mathbb{R}^2))$.*

Consider now the Coulomb gauge condition $\partial_j A^j = 0$. In this case one easily checks using (1.2) that the potentials A_μ satisfy the elliptic equations

$$
\begin{aligned}
A_0 &= \Delta^{-1}(\partial_2 \langle \psi, \alpha^1 \psi \rangle - \partial_1 \langle \psi, \alpha^2 \psi \rangle), \\
A_1 &= \Delta^{-1}\partial_2 \langle \psi, \psi \rangle, \quad A_2 = -\Delta^{-1}\partial_1 \langle \psi, \psi \rangle.
\end{aligned} \tag{1.4}
$$

Inserting this into (1.1) we obtain

$$
i\partial_t \psi + i\alpha^j \partial_j \psi = m\beta\psi + N(\psi, \psi, \psi), \tag{1.5}
$$

where

$$
\begin{aligned}
&N(\psi_1, \psi_2, \psi_3) \\
&= \Delta^{-1}\left(\partial_1 \langle \psi_1, \alpha_2 \psi_2 \rangle - \partial_2 \langle \psi_1, \alpha_1 \psi_2 \rangle + \partial_2 \langle \psi_1, \psi_2 \rangle \alpha_1 - \partial_1 \langle \psi_1, \psi_2 \rangle \alpha_2\right)\psi_3.
\end{aligned}
$$

In the sequel we consider this nonlinear Dirac equation with initial condition

$$
\psi(0) = \psi_0. \tag{1.6}
$$

Using an idea of d'Ancona - Foschi -Selberg [1] we simplify (1.5) by considering the projections onto the one-dimensional eigenspaces of the operator $-i\alpha \cdot \nabla = -i\alpha^j \partial_j$ belonging to the eigenvalues $\pm|\xi|$. These projections are given by $\Pi_\pm = \Pi_\pm(D)$, where $D = \frac{\nabla}{i}$ and $\Pi_\pm(\xi) = \frac{1}{2}(I \pm \frac{\xi}{|\xi|} \cdot \alpha)$. Then $-i\alpha \cdot \nabla = |D|\Pi_+(D) - |D|\Pi_-(D)$ and $\Pi_\pm(\xi)\beta = \beta\Pi_\mp(\xi)$. Defining $\psi_\pm := \Pi_\pm(D)\psi$, the Dirac equation can be rewritten as

$$
(-i\partial_t \pm |D|)\psi_\pm = m\beta\psi_\mp + \Pi_\pm N(\psi_+ + \psi_-, \psi_+ + \psi_-, \psi_+ + \psi_-). \tag{1.7}
$$

The initial condition is transformed into

$$
\psi_\pm(0) = \Pi_\pm \psi_0. \tag{1.8}
$$

We now formulate our results in the case of the Coulomb gauge.

Theorem 1.3. *Assume $\psi_0 \in H^s(\mathbb{R}^2)$ with $s > 1/4$. Then (1.5),(1.6) is locally well-posed in $H^s(\mathbb{R}^2)$. More precisely there exists $T > 0$, such that there exists a unique solution $\psi = \psi_+ + \psi_-$ with $\psi_\pm \in X_\pm^{s, \frac{1}{2}+}[0, T]$. This solution belongs to $C^0([0, T], H^s(\mathbb{R}^2))$.*

The unconditional uniqueness result is the following.

Theorem 1.4. *Assume $\psi_0 \in H^s(\mathbb{R}^2)$ with $s > 1/3$. The solution of (1.5), (1.6) is unique in $C^0([0, T], H^s(\mathbb{R}^2))$.*

Fundamental for the proof of our theorems are the following bilinear estimates in wave-Sobolev spaces which were proven by d'Ancona, Foschi and Selberg in the two dimensional case $n = 2$ in [2] in a more general form which include many limit cases which we do not need.

Proposition 1.5. *Let $n = 2$. The estimate*

$$
\|uv\|_{X_{|\tau|=|\xi|}^{-s_0,-b_0}} \lesssim \|u\|_{X_{|\tau|=|\xi|}^{s_1,b_1}} \|v\|_{X_{|\tau|=|\xi|}^{s_2,b_2}}
$$

holds, provided the following conditions hold:

$$
\begin{aligned}
b_0 + b_1 + b_2 &> \frac{1}{2}, \quad b_0 + b_1 \geq 0, \\
b_0 + b_2 &\geq 0, \quad b_1 + b_2 \geq 0,
\end{aligned}
$$

$$s_0 + s_1 + s_2 > \frac{3}{2} - (b_0 + b_1 + b_2),$$

$$s_0 + s_1 + s_2 > 1 - \min(b_0 + b_1, b_0 + b_2, b_1 + b_2),$$

$$s_0 + s_1 + s_2 > \frac{1}{2} - \min(b_0, b_1, b_2), \quad s_0 + s_1 + s_2 > \frac{3}{4},$$

$$(s_0 + b_0) + 2s_1 + 2s_2 > 1, \quad 2s_0 + (s_1 + b_1) + 2s_2 > 1,$$

$$2s_0 + 2s_1 + (s_2 + b_2) > 1, \quad s_1 + s_2 \geq \max(0, -b_0),$$

$$s_0 + s_2 \geq \max(0, -b_1), \quad s_0 + s_1 \geq \max(0, -b_2).$$

Another decisive tool are the estimates for the wave equation in the following proposition.

Proposition 1.6. *The following estimates hold*

$$\|u\|_{L^6_{xt}} \lesssim \|u\|_{X^{\frac{1}{2},\frac{1}{2}+}_{|\tau|=|\xi|}}, \tag{1.9}$$

$$\|u\|_{L^p_x L^2_t} \lesssim \|u\|_{X^{\frac{1}{2}-\frac{2}{p},\frac{1}{2}+}_{|\tau|=|\xi|}} \quad for\ 6 \leq p < \infty, \tag{1.10}$$

$$especially \quad \|u\|_{L^6_x L^2_t} \lesssim \|u\|_{X^{\frac{1}{6},\frac{1}{2}+}_{|\tau|=|\xi|}}, \tag{1.11}$$

$$\|u\|_{L^\infty_x L^2_t} \lesssim \|u\|_{X^{\frac{1}{2}+,\frac{1}{2}+}_{|\tau|=|\xi|}}, \tag{1.12}$$

$$\|u\|_{L^\infty_x L^{2+}_t} \lesssim \|u\|_{X^{\frac{1}{2}+,\frac{1}{2}+}_{|\tau|=|\xi|}}, \tag{1.13}$$

$$\|u\|_{L^6_x L^{2+}_t} \lesssim \|u\|_{X^{\frac{1}{6}+,\frac{1}{2}+}_{|\tau|=|\xi|}}, \tag{1.14}$$

$$\|u\|_{L^4_x L^{2+}_t} \lesssim \|u\|_{X^{\frac{1}{8}+,\frac{3}{8}+}_{|\tau|=|\xi|}}, \tag{1.15}$$

$$\|u\|_{L^p_x L^{2+}_t} \lesssim \|u\|_{X^{\frac{1}{2}-\frac{2}{p}+,\frac{1}{2}+}_{|\tau|=|\xi|}} \quad for\ 6 \leq p < \infty. \tag{1.16}$$

Proof. (1.9) is the standard Strichartz estimate combined with the transfer principle. Concerning (1.10) we use [9] (appendix by D. Tataru) Thm. B2:

$$\|\mathcal{F}_t u\|_{L^2_\tau L^6_x} \lesssim \|u_0\|_{\dot{H}^{\frac{1}{6}}_x},$$

if $u = e^{it|\nabla|}u_0$, and $\mathcal{F}$ denotes the Fourier transform with respect to time. This implies by Plancherel, Minkowski's inequality and Sobolev's embedding theorem

$$\|u\|_{L^p_x L^2_t} = \|\mathcal{F}_t u\|_{L^p_x L^2_\tau} \lesssim \|\mathcal{F}_t u\|_{L^2_\tau L^p_x} \lesssim \|\mathcal{F}_t u\|_{L^2_\tau H^{\frac{1}{3}-\frac{2}{p},6}_x} \lesssim \|u_0\|_{H^{\frac{1}{2}-\frac{2}{p},2}_x}.$$

The transfer principle gives (1.10). (1.12) follows similarly using $H^{\frac{1}{3}+,6}_x \hookrightarrow L^\infty_x$. (1.14) is obtained by interpolation between (1.11) and (1.9), and (1.15) by interpolation between (1.14) and the trivial identity $\|u\|_{L^2_{xt}} = \|u\|_{X^{0,0}_{|\tau|=|\xi|}}$. Moreover we obtain (1.13) and (1.16) by interpolation between (1.12) and (1.10), resp., and the estimate $\|u\|_{L^\infty_{xt}} \lesssim \|u\|_{X^{1+,\frac{1}{2}+}_{|\tau|=|\xi|}}$. □

2. Reformulation of the problem in temporal gauge

Imposing the temporal gauge condition $A_0 = 0$ the system (1.1), (1.2) is equivalent to

$$i\partial_t \psi + i\alpha^j \partial_j \psi = m\beta\psi - \alpha^j A_j \psi \tag{2.1}$$

$$\partial_t A_1 = -2\langle \psi, \alpha^2 \psi \rangle, \quad \partial_t A_2 = 2\langle \psi, \alpha^1 \psi \rangle \tag{2.2}$$

$$\partial_1 A_2 - \partial_2 A_1 = -2\langle \psi, \psi \rangle. \tag{2.3}$$

We first show that (2.3) is fulfilled for any solution of (2.1), (2.2), if it holds initially, i.e., if the following compatability condition holds:

$$\partial_1 A_2(0) - \partial_2 A_1(0) = -2\langle \psi(0), \psi(0) \rangle, \tag{2.4}$$

which we assume from now on. Indeed one easily calculates using (2.1):

$$\partial_t \langle \psi, \psi \rangle = -\partial_j \langle \psi, \alpha^j \psi \rangle, \tag{2.5}$$

which implies by (2.2)

$$\partial_t(\partial_1 A_2 - \partial_2 A_1) = 2\partial_j \langle \psi, \alpha^j \psi \rangle = -2\partial_t \langle \psi, \psi \rangle,$$

so that (2.3) holds, if it holds initially. Thus we only have to solve (2.1) and (2.2).

We decompose $A = (A_1, A_2)$ into its divergence-free part A^{df} and its "curl-free" part A^{cf}, namely $A = A^{df} + A^{cf}$, where

$$A^{df} = (-\Delta)^{-1}(\partial_2(\partial_1 A_2 - \partial_2 A_1), \partial_1(\partial_2 A_1 - \partial_1 A_2)),$$

$$A^{cf} = -(-\Delta)^{-1}(\partial_1(\partial_1 A_1 + \partial_2 A_2), \partial_2(\partial_1 A_1 + \partial_2 A_2)) = -(-\Delta)^{-1}\nabla \operatorname{div} A.$$

Then (2.3) and (2.2) imply

$$A^{df} = -2(-\Delta)^{-1}(\partial_2\langle \psi, \psi \rangle, -\partial_1\langle \psi, \psi \rangle), \tag{2.6}$$

$$\partial_t A_j^{cf} = -2(-\Delta)^{-1}\partial_j(\partial_2\langle \psi, \alpha^1 \psi \rangle - \partial_1\langle \psi, \alpha^2 \psi \rangle). \tag{2.7}$$

Reversely, defining $A = A^{df} + A^{cf}$, we show that our new system (2.1), (2.6), (2.7) implies (2.1), (2.2), (2.3), so that both systems are equivalent. It only remains to show that (2.2) holds. By (2.6), (2.7), (2.5) we obtain

$$\begin{aligned}
\partial_t A_1 &= \partial_t A_1^{df} + \partial_t A_1^{cf} \\
&= -2(-\Delta)^{-1}\big(\partial_2\partial_t\langle \psi, \psi \rangle + \partial_1(\partial_2\langle \psi, \alpha^1 \psi \rangle - \partial_1\langle \psi, \alpha^2 \psi \rangle)\big) \\
&= 2(-\Delta)^{-1}\big(\partial_2\partial_j\langle \psi, \alpha^j \psi \rangle - \partial_1(\partial_2\langle \psi, \alpha^1 \psi \rangle - \partial_1\langle \psi, \alpha^2 \psi \rangle)\big) \\
&= 2(-\Delta)^{-1}(\partial_2^2 + \partial_1^2)\langle \psi, \alpha^2 \psi \rangle = -2\langle \psi, \alpha^2 \psi \rangle
\end{aligned}$$

and similarly

$$\partial_t A_2 = 2\langle \psi, \alpha^1 \psi \rangle.$$

In the same way in which we obtained (1.7) the Dirac equation (2.1) can be rewritten as

$$(-i\partial_t \pm |\nabla|)\psi_\pm = -m\beta\psi_\mp - \Pi_\pm(\alpha^j A_j \psi), \tag{2.8}$$

where $A_j = A_j^{df} + A_j^{cf}$, and in (2.6),(2.7) and (2.8) we replace ψ by $\psi_+ + \psi_-$.

3. Proof of Theorem 1.1

Taking the considerations of the previous section into account Theorem 1.1 reduces to the following proposition and its corollary.

Proposition 3.1. *Let $\epsilon > 0$ and $s > 3/8$. Then there exists $T > 0$ such that system (2.6), (2.7), (2.8) has a unique local solution $\psi_\pm \in X_\pm^{s,\frac{1}{2}+}[0,T]$, $A^{cf} \in X_{\tau=0}^{s+\frac{1}{8},\frac{1}{2}+}[0,T]$. Also A^{df} satisfies $|\nabla|^\epsilon A_j^{df} \in X_{|\tau|=|\xi|}^{s+\frac{3}{8}-\epsilon,\frac{1}{2}+}[0,T]$ and $\psi_\pm \in X_\pm^{s,1}[0,T]$.*

Corollary 3.2. *The solution satisfies* $\psi \in C^0([0,T], H^s)$, $A^{cf} \in C^0([0,T], H^{s+\frac{1}{8}})$, $|\nabla|^\epsilon A^{df} \in C^0([0,T], H^{s+\frac{3}{8}-\epsilon})$.

Proof of Proposition 3.1. We want to apply a Picard iteration. For the Cauchy problem for the Dirac equation

$$(-i\partial_t \pm |\nabla|)\psi_\pm = F_\pm, \quad \psi_\pm(0) = \psi_{\pm 0}$$

we use the well-known estimate (cf. e.g. [6])

$$\|\psi_\pm\|_{X_\pm^{s,b}[0,T]} \lesssim \|\psi_{\pm 0}\|_{H^s} + T^{1+b'-b}\|F_\pm\|_{X_\pm^{s,b'}[0,T]} ,$$

which holds for $0 < T \le 1$, $-\frac{1}{2} < b' \le 0 \le b \le b'+1$, $s \in \mathbb{R}$. Thus by standard arguments it suffices to show the following estimates for the right hand side of the Dirac equation (2.8):

$$\|A_j^{cf}\alpha^j\psi\|_{X_{|\tau|=|\xi|}^{s,-\frac{1}{2}++}} \lesssim \|A^{cf}\|_{X_{\tau=0}^{s+\frac{1}{8},\frac{1}{2}+}}\|\psi\|_{X_{|\tau|=|\xi|}^{s,\frac{1}{2}+}} , \tag{3.1}$$

$$\|A_j^{df}\alpha^j\psi\|_{X_{|\tau|=|\xi|}^{s,-\frac{1}{2}++}} \lesssim \||\nabla|^\epsilon A_j^{df}\|_{X_{|\tau|=|\xi|}^{s+\frac{3}{8}-\epsilon,\frac{1}{2}+}}\|\psi\|_{X_{|\tau|=|\xi|}^{s,\frac{1}{2}+}} , \tag{3.2}$$

$$\||\nabla|^\epsilon A_j^{df}\|_{X_{|\tau|=|\xi|}^{s+\frac{3}{8}-\epsilon,\frac{1}{2}+}} \lesssim \|\psi\|_{X_{|\tau|=|\xi|}^{s,\frac{1}{2}++}}^2 . \tag{3.3}$$

Similarly, for the right hand side of (2.7) we need

$$\|\langle\psi,\alpha^j\psi\rangle\|_{X_{\tau=0}^{s+\frac{1}{8},-\frac{1}{2}++}} \lesssim \|\psi\|_{X_{|\tau|=|\xi|}^{s,\frac{1}{2}+}}^2 . \tag{3.4}$$

Proof of (3.1): We even prove the estimate with $X_{|\tau|=|\xi|}^{s,-\frac{1}{2}++}$ replaced by $X_{|\tau|=|\xi|}^{s,0}$ on the left hand side. It reduces to

$$\int_* \frac{\widehat{u}_1(\tau_1,\xi_1)}{\langle\xi_1\rangle^{s+\frac{1}{8}}\langle\tau_1\rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\tau_2,\xi_2)}{\langle\xi_2\rangle^s\langle|\tau_2|-|\xi_2|\rangle^{\frac{1}{2}+}} \langle\xi_3\rangle^s\widehat{u}_3(\tau_3,\xi_3)d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2} ,$$

where * denotes integration over $\xi = (\xi_1,\xi_2,\xi_3)$, $\tau = (\tau_1,\tau_2,\tau_3)$ with $\xi_1+\xi_2+\xi_3 = 0$ and $\tau_1+\tau_2+\tau_3 = 0$. We assume here and in the following without loss of generality that the Fourier transforms are non-negative.

Case 1: $|\xi_1| \ge |\xi_2| \Rightarrow \langle\xi_3\rangle^s \lesssim \langle\xi_1\rangle^s$. It suffices to show

$$\int_* \frac{\widehat{u}_1(\tau_1,\xi_1)}{\langle\tau_1\rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\tau_2,\xi_2)}{\langle\xi_2\rangle^{s+\frac{1}{8}}\langle|\tau_2|-|\xi_2|\rangle^{\frac{1}{2}+}} \widehat{u}_3(\tau_3,\xi_3)d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2} .$$

This follows under the assumption $s > 3/8$ from the estimate

$$\begin{aligned}
\left|\int v_1 v_2 v_3 dx dt\right| &\lesssim \|v_1\|_{L_x^2 L_t^\infty}\|v_2\|_{L_x^\infty L_t^2}\|v_3\|_{L_x^2 L_t^2} \\
&\lesssim \|v_1\|_{X_{\tau=0}^{0,\frac{1}{2}+}}\|v_2\|_{X_{|\tau|=|\xi|}^{\frac{1}{2}+,\frac{1}{2}+}}\|v_3\|_{X_{|\tau|=|\xi|}^{0,0}} ,
\end{aligned} \tag{3.5}$$

where we used (1.12).

Case 2: $|\xi_2| \ge |\xi_1| \Rightarrow \langle\xi_3\rangle^s \lesssim \langle\xi_2\rangle^s$. In this case the desired estimate follows from

$$\int_* m(\xi_1,\xi_2,\xi_3,\tau_1,\tau_2,\tau_3)\widehat{u}_1(\xi_1,\tau_1)\widehat{u}_2(\xi_2,\tau_2)\widehat{u}_3(\xi_3,\tau_3)d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2} , \tag{3.6}$$

where

$$m = \frac{1}{\langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+} \langle \xi_1 \rangle^{\frac{1}{2}+} \langle \tau_1 \rangle^{\frac{1}{2}+}} .$$

The following argument is closely related to the proof of a similar estimate in [14]. By two applications of the averaging principle [13, Prop. 5.1], we may replace m by

$$m' = \frac{\chi_{||\tau_2|-|\xi_2||\sim 1} \chi_{|\tau_1|\sim 1}}{\langle \xi_1 \rangle^{\frac{1}{2}+}} .$$

Let now τ_2 be restricted to the region $\tau_2 = T + O(1)$ for some integer T. Then τ_3 is restricted to $\tau_3 = -T + O(1)$, because $\tau_1 + \tau_2 + \tau_3 = 0$, and ξ_2 is restricted to $|\xi_2| = |T| + O(1)$. The τ_3-regions are essentially disjoint for $T \in \mathbb{Z}$ and similarly the τ_2-regions. Thus by Schur's test [13, Lemma 3.11], we only have to show

$$\sup_{T\in\mathbb{Z}} \int_* \frac{\chi_{\tau_3=-T+O(1)} \chi_{\tau_2=T+O(1)} \chi_{|\tau_1|\sim 1} \chi_{|\xi_2|=|T|+O(1)}}{\langle \xi_1 \rangle^{\frac{1}{2}+}}$$

$$\times \widehat{u}_1(\xi_1,\tau_1)\widehat{u}_2(\xi_2,\tau_2)\widehat{u}_3(\xi_3,\tau_3)d\xi d\tau \lesssim \prod_{i=1}^{3} \|u_i\|_{L^2_{xt}} .$$

The τ-behaviour of the integral is now trivial, thus we reduce to

$$\sup_{T\in\mathbb{N}} \int_{\sum_{i=1}^{3}\xi_i=0} \frac{\chi_{|\xi_2|=T+O(1)}}{\langle \xi_1 \rangle^{\frac{1}{2}+}} \widehat{f}_1(\xi_1)\widehat{f}_2(\xi_2)\widehat{f}_3(\xi_3)d\xi \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2_x} . \qquad (3.7)$$

An elementary calculation shows that

$$L.H.S.\, of\, (3.7) \lesssim \sup_{T\in\mathbb{N}} \|\chi_{|\xi|=T+O(1)} * \langle\xi\rangle^{-1-}\|_{L^\infty(\mathbb{R}^2)}^{1/2} \prod_{i=1}^{3} \|f_i\|_{L^2_x} \lesssim \prod_{i=1}^{3} \|f_i\|_{L^2_x} ,$$

so that the desired estimate follows.

Proof of (3.4): This reduces to

$$\int_* \frac{\widehat{u}_1(\tau_1,\xi_1)}{\langle\xi_1\rangle^s \langle |\tau_1|-|\xi_1|\rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\tau_2,\xi_2)}{\langle\xi_2\rangle^s \langle |\tau_2|-|\xi_2|\rangle^{\frac{1}{2}+}} \frac{\langle\xi_3\rangle^{s+\frac{1}{8}}\widehat{u}_3(\tau_3,\xi_3)}{\langle\tau_3\rangle^{\frac{1}{2}-}} d\xi d\tau \lesssim \prod_{i=1}^{3} \|u_i\|_{L^2_{xt}} .$$

Assuming without loss of generality $|\xi_1| \le |\xi_2|$ we have to show

$$\int_* \frac{\widehat{u}_1(\tau_1,\xi_1)}{\langle\xi_1\rangle^s \langle |\tau_1|-|\xi_1|\rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\tau_2,\xi_2)}{\langle |\tau_2|-|\xi_2|\rangle^{\frac{1}{2}+}} \frac{\langle\xi_3\rangle^{1/8}\widehat{u}_3(\tau_3,\xi_3)}{\langle\tau_3\rangle^{\frac{1}{2}-}} d\xi d\tau \lesssim \prod_{i=1}^{3} \|u_i\|_{L^2_{xt}} .$$

Case 1: $|\tau_2| \ll |\xi_2|$. We reduce to

$$\int_* \frac{\widehat{u}_1(\tau_1,\xi_1)}{\langle\xi_1\rangle^s \langle |\tau_1|-|\xi_1|\rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\tau_2,\xi_2)}{\langle\xi_2\rangle^{\frac{3}{8}+}} \frac{\widehat{u}_3(\tau_3,\xi_3)}{\langle\tau_3\rangle^{\frac{1}{2}-}} d\xi d\tau \lesssim \prod_{i=1}^{3} \|u_i\|_{L^2_{xt}} .$$

This follows from

$$\left| \int v_1 v_2 v_3 dx dt \right| \lesssim \|v_1\|_{L^6_x L^{2+}_t} \|v_2\|_{L^3_x L^2_t} \|v_3\|_{L^2_x L^{\infty-}_t}$$

$$\lesssim \|v_1\|_{X^{\frac{1}{6}+,\frac{1}{2}+}_{|\tau|=|\xi|}} \|v_2\|_{X^{\frac{1}{3},0}_{|\tau|=|\xi|}} \|v_3\|_{X^{0,\frac{1}{2}-}_{\tau=0}} ,$$

where we used (1.14) for the first factor and Sobolev for the others. Obviously here is some headroom left.

Case 2: $|\tau_2| \gtrsim |\xi_2|$. In this case we use $\tau_1 + \tau_2 + \tau_3 = 0$ to estimate

$$1 \lesssim \frac{\langle \tau_2 \rangle^{\frac{1}{2}-}}{\langle \xi_2 \rangle^{\frac{1}{2}-}} \lesssim \frac{\langle \tau_1 \rangle^{\frac{1}{2}-}}{\langle \xi_2 \rangle^{\frac{1}{2}-}} + \frac{\langle \tau_3 \rangle^{\frac{1}{2}-}}{\langle \xi_2 \rangle^{\frac{1}{2}-}} \,.$$

2.1: If the second term on the right hand side is dominant we have to show, using also $\langle \xi_3 \rangle^{1/8} \lesssim \langle \xi_2 \rangle^{1/8}$:

$$\int_* \frac{\widehat{u}_1(\tau_1,\xi_1)}{\langle \xi_1 \rangle^s \langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\tau_2,\xi_2)}{\langle \xi_2 \rangle^{\frac{3}{8}-} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \widehat{u}_3(\tau_3,\xi_3) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L^2_{xt}} \,,$$

which follows for $s > \frac{3}{8}$ by Prop. 1.5.

2.2: If the first term on the right hand side is dominant we consider two subcases.

2.2.1: $|\tau_1| \lesssim |\xi_1|$. We reduce to

$$\int_* \frac{\widehat{u}_1(\tau_1,\xi_1) \langle \xi_1 \rangle^{\frac{1}{2}-s-}}{\langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\tau_2,\xi_2)}{\langle \xi_2 \rangle^{\frac{3}{8}-} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\tau_3,\xi_3)}{\langle \tau_3 \rangle^{\frac{1}{2}-}} d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L^2_{xt}} \,.$$

Using $|\xi_2| \ge |\xi_1|$ and $s > \frac{3}{8}$ it suffices to show

$$\int_* \frac{\widehat{u}_1(\tau_1,\xi_1)}{\langle \xi_1 \rangle^{\frac{1}{8}+} \langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\tau_2,\xi_2)}{\langle \xi_2 \rangle^{\frac{1}{8}+} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\tau_3,\xi_3)}{\langle \tau_3 \rangle^{\frac{1}{2}-}} d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L^2_{xt}} \,.$$

This follows from

$$\left| \int v_1 v_2 v_3 \, dx dt \right| \lesssim \|v_1\|_{L^4_x L^{2+}_t} \|v_2\|_{L^4_x L^{2+}_t} \|v_3\|_{L^2_x L^{\infty-}_t}$$

$$\lesssim \|v_1\|_{X^{\frac{1}{8}+,\frac{3}{8}+}_{|\tau|=|\xi|}} \|v_2\|_{X^{\frac{1}{8}+,\frac{3}{8}+}_{|\tau|=|\xi|}} \|v_3\|_{X^{0,\frac{1}{2}-}_{\tau=0}} \,,$$

where we used (1.15).

2.2.2: $|\tau_1| \gg |\xi_1| \Rightarrow \langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{2}-} \sim \langle \tau_1 \rangle^{\frac{1}{2}-}$. We have to show

$$\int_* \frac{\widehat{u}_1(\tau_1,\xi_1)}{\langle \xi_1 \rangle^s} \frac{\widehat{u}_2(\tau_2,\xi_2)}{\langle \xi_2 \rangle^{\frac{3}{8}-} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\tau_3,\xi_3)}{\langle \tau_3 \rangle^{\frac{1}{2}-}} d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L^2_{xt}} \,.$$

This follows from

$$\left| \int v_1 v_2 v_3 \, dx dt \right| \lesssim \|v_1\|_{L^3_x L^2_t} \|v_2\|_{L^6_x L^{2+}_t} \|v_3\|_{L^2_x L^{\infty-}_t}$$

$$\lesssim \|v_1\|_{X^{\frac{1}{3},0}_{|\tau|=|\xi|}} \|v_2\|_{X^{\frac{1}{6}+,\frac{1}{2}+}_{|\tau|=|\xi|}} \|v_3\|_{X^{0,\frac{1}{2}-}_{\tau=0}} \,,$$

where we used Sobolev for the first and last factor and (1.14) for the second one. This completes the proof of (3.4).

Proof of (3.3): We distinguish between low and high frequencies of A_j^{df}. For high frequencies, i.e. , $\operatorname{supp}(\mathcal{F} A_j^{df}) \subset \{|\xi| \ge 1\}$, we obtain by (2.6) and Prop. 1.5 for $s > 3/8$:

$$\||\nabla|^\epsilon A_j^{df}\|_{X^{s+\frac{3}{8}-\epsilon,\frac{1}{2}+}_{|\tau|=|\xi|}} \lesssim \|\langle \psi, \psi \rangle\|_{X^{s-\frac{5}{8},\frac{1}{2}+}_{|\tau|=|\xi|}} \lesssim \|\psi\|^2_{X^{s,\frac{1}{2}++}_{|\tau|=|\xi|}} \,.$$

In the low frequency case $|\xi_3| \le 1$, where $\langle \xi_1 \rangle \sim \langle \xi_2 \rangle$, it suffices to show

$$\int_* \frac{\widehat{u}_1(\tau_1,\xi_1)}{\langle \xi_1 \rangle^s \langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{2}++}} \frac{\widehat{u}_2(\tau_2,\xi_2)}{\langle \xi_2 \rangle^s \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}++}}$$

$$\times \frac{\widehat{u}_3(\tau_3,\xi_3)\langle|\tau_3|-|\xi_3|\rangle^{\frac{1}{2}+}\langle\xi_3\rangle^{s+\frac{3}{8}-\epsilon}\chi_{\{|\xi_3|\leq 1\}}}{|\xi_3|^{1-\epsilon}}d\xi d\tau \lesssim \prod_{i=1}^{3}\|u_i\|_{L_{xt}^2}.$$

Assuming without loss of generality $\langle\tau_2\rangle \leq \langle\tau_1\rangle$, we obtain $\langle|\tau_3|-|\xi_3|\rangle^{\frac{1}{2}+} \sim \langle\tau_3\rangle^{\frac{1}{2}+} \lesssim \langle\tau_1\rangle^{\frac{1}{2}+} + \langle\tau_2\rangle^{\frac{1}{2}+} \lesssim \langle\tau_1\rangle^{\frac{1}{2}+}$.

If $|\tau_1| \gg |\xi_1|$ or $|\tau_1| \ll |\xi_1|$, it suffices to show

$$\int_* \frac{\widehat{u}_1(\tau_1,\xi_1)}{\langle\xi_1\rangle^s}\frac{\widehat{u}_2(\tau_2,\xi_2)}{\langle\xi_2\rangle^s\langle|\tau_2|-|\xi_2|\rangle^{\frac{1}{2}++}}\frac{\widehat{u}_3(\tau_3,\xi_3)\chi_{\{|\xi_3|\leq 1\}}}{|\xi_3|^{1-\epsilon}}d\xi d\tau \lesssim \prod_{i=1}^{3}\|u_i\|_{L_{xt}^2}.$$

This follows from

$$\left|\int v_1 v_2 v_3 dx dt\right| \lesssim \|v_1\|_{L_{xt}^2}\|v_2\|_{L_t^\infty L_x^2}\|v_3\|_{L_t^2 L_x^\infty},$$

which gives the desired result using $\dot{H}_x^{1-\epsilon} \hookrightarrow L_x^\infty$ for low frequencies.

If $|\tau_1| \sim |\xi_1|$, we use $\langle\xi_1\rangle \sim \langle\xi_2\rangle$ and reduce to

$$\int_* \widehat{u}_1(\tau_1,\xi_1)\frac{\widehat{u}_2(\tau_2,\xi_2)}{\langle\xi_2\rangle^{2s-\frac{1}{2}-}\langle\langle|\tau_2|-|\xi_2|\rangle^{\frac{1}{2}++}}\frac{\widehat{u}_3(\tau_3,\xi_3)\chi_{\{|\xi_3|\leq 1\}}}{|\xi_3|^{1-\epsilon}} \lesssim \prod_{i=1}^{3}\|u_i\|_{L_{xt}^2},$$

which can be shown as before. We remark that we only used $s > 1/4$ in the low frequency case.

Proof of (3.2): We even prove the estimate with $X_{|\tau|=|\xi|}^{s,-\frac{1}{2}++}$ replaced by $X_{|\tau|=|\xi|}^{s,0}$ on the left hand side. For high frequencies of A_j^{df} we have to show

$$\|A_j^{df}\alpha^j\psi\|_{X_{|\tau|=|\xi|}^{s,0}} \lesssim \|A^{df}\|_{X_{|\tau|=|\xi|}^{s+\frac{3}{8},\frac{1}{2}+}}\|\psi\|_{X_{|\tau|=|\xi|}^{s,\frac{1}{2}+}},$$

which follows by Proposition 1.5. For the low frequency case of A_j^{df} it suffices to show

$$\int_* \frac{\widehat{u}_1(\tau_1,\xi_1)}{\langle\xi_1\rangle^s\langle|\tau_1|-|\xi_1|\rangle^{\frac{1}{2}+}}\widehat{u}_2(\tau_2,\xi_2)\langle\xi_2\rangle^s\frac{\widehat{u}_3(\tau_3,\xi_3)\chi_{\{|\xi_3|\leq 1\}}}{|\xi_3|^\epsilon\langle|\tau_3|-|\xi_3|\rangle^{\frac{1}{2}+}\langle\xi_3\rangle^{s+\frac{3}{8}-\epsilon}}d\xi d\tau$$
$$\lesssim \prod_{i=1}^{3}\|u_i\|_{L_{xt}^2}.$$

Using $\langle\xi_1\rangle \sim \langle\xi_2\rangle$ and $\langle\xi_3\rangle \sim 1$ and $\dot{H}_x^\epsilon \hookrightarrow L_x^\infty$ for low frequencies this easily follows from the estimate

$$\left|\int v_1 v_2 v_3 dx dt\right| \lesssim \|v_1\|_{L_t^\infty L_x^2}\|v_2\|_{L_t^2 L_x^2}\|v_3\|_{L_t^2 L_x^\infty}.$$

This completes the proof of (3.1)-(3.4). The property $\psi_\pm \in X_\pm^{s,1}[0,T]$ follows immediately from the proof of (3.1) and (3.2). $\square$

4. Proof of Theorem 1.2

Proof. Assume $s > 19/40$, say $s = \frac{19}{40} + \delta$ with $1 \gg \delta > 0$. Let $\psi \in C^0([0,T],H^s)$, $A_j \in C^0([0,T],H^{s+\frac{1}{8}})$.

Claim 1: $\psi_\pm \in X_\pm^{\frac{1}{4}+\alpha,\frac{1}{2}+}[0,T]$, where $\alpha = \frac{1}{40} + \frac{3}{2}\delta-$. By Sobolev's multiplication law we obtain

$$\|A_j\alpha^j\psi_\pm\|_{L^2([0,T],H^{2s-\frac{7}{8}})} \lesssim \|A\|_{C^0([0,T],H^{s+\frac{1}{8}})}\|\psi\|_{C^0([0,T],H^s)}T^{1/2} < \infty.$$

Thus $\psi_\pm \in X_\pm^{2s-\frac{7}{8},1}[0,T]$. Interpolation with $\psi_\pm \in X_\pm^{s,0}[0,T] \subset C^0([0,T],H^s)$ gives $\psi_\pm \in X_\pm^{\frac{1}{4}+\frac{3}{2}s-\frac{11}{16}-,\frac{1}{2}+}[0,T] = X_\pm^{\frac{1}{4}+\alpha,\frac{1}{2}+}[0,T]$.

We now iteratively improve the regularity of $\psi_\pm$, A^{cf} and A^{df} in order to end up in a class where uniqueness holds by Theorem 1.1. Let us assume that $\psi_\pm \in X_\pm^{\min(\frac{1}{4}+\alpha_k,s),\frac{1}{2}+}[0,T]$ with $\alpha_k = \frac{1}{40} + \left(\frac{3}{2}\right)^k \delta-$ for some $k \in \mathbb{N}$. This was just shown for $k = 1$. If $\frac{1}{4} + \alpha_k \geq s$, we obtain by (3.3) and (3.4) $|\nabla|^\epsilon A_j^{df} \in X_{|\tau|=|\xi|}^{s+\frac{3}{8}-\epsilon,\frac{1}{2}+}$ and $A_j^{cf} \in X_{\tau=0}^{s+\frac{1}{8},\frac{1}{2}+}[0,T]$, so that uniqueness follows from Theorem 1.1.

Otherwise we now prove

Claim 2: $A_j^{cf} \in X_{\tau=0}^{\min(\frac{1}{4}+2\alpha_k-,s+\frac{1}{8}),\frac{1}{2}+}[0,T]$. This reduces to

$$\|\langle \psi, \alpha^j \psi \rangle\|_{X_{\tau=0}^{\frac{1}{4}+2\alpha_k-,-\frac{1}{2}++}} \lesssim \|\psi\|_{X_{|\tau|=|\xi|}^{\frac{1}{4}+\alpha_k,\frac{1}{2}+}}^2,$$

which is equivalent to

$$\int_* \frac{\widehat{u}_1(\tau_1,\xi_1)}{\langle\xi_1\rangle^{\frac{1}{4}+\alpha_k}\langle|\tau_1|-|\xi_1|\rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\tau_2,\xi_2)}{\langle\xi_2\rangle^{\frac{1}{4}+\alpha_k}\langle|\tau_2|-|\xi_2|\rangle^{\frac{1}{2}+}} \frac{\langle\xi_3\rangle^{\frac{1}{4}+2\alpha_k-}\widehat{u}_3(\tau_3,\xi_3)}{\langle\tau_3\rangle^{\frac{1}{2}-}} d\xi d\tau$$
$$\lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2}.$$

Assuming without loss of generality $|\xi_1| \leq |\xi_2|$ we reduce to

$$\int_* \frac{\widehat{u}_1(\tau_1,\xi_1)}{\langle\xi_1\rangle^{\frac{1}{4}+\alpha_k}\langle|\tau_1|-|\xi_1|\rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\tau_2,\xi_2)}{\langle|\tau_2|-|\xi_2|\rangle^{\frac{1}{2}+}} \frac{\langle\xi_3\rangle^{\alpha_k-}\widehat{u}_3(\tau_3,\xi_3)}{\langle\tau_3\rangle^{\frac{1}{2}-}} d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2}.$$

Case 1: $|\tau_2| \ll |\xi_2|$. The left hand side is bounded by

$$\int_* \frac{\widehat{u}_1(\tau_1,\xi_1)}{\langle\xi_1\rangle^{\frac{1}{4}+\alpha_k}\langle|\tau_1|-|\xi_1|\rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\tau_2,\xi_2)}{\langle\xi_2\rangle^{\frac{1}{2}-\alpha_k+}} \frac{\widehat{u}_3(\tau_3,\xi_3)}{\langle\tau_3\rangle^{\frac{1}{2}-}} d\xi d\tau$$
$$\lesssim \int_* \frac{\widehat{u}_1(\tau_1,\xi_1)}{\langle\xi_1\rangle^{\frac{3}{4}}\langle|\tau_1|-|\xi_1|\rangle^{\frac{1}{2}+}} \widehat{u}_2(\tau_2,\xi_2)\frac{\widehat{u}_3(\tau_3,\xi_3)}{\langle\tau_3\rangle^{\frac{1}{2}-}} d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2},$$

because by (1.13)

$$\left|\int v_1 v_2 v_3 \, dx dt\right| \lesssim \|v_1\|_{L_x^\infty L_t^{2+}}\|v_2\|_{L_{xt}^2}\|v_3\|_{L_x^2 L_t^{\infty-}} \lesssim \|v_1\|_{X_{|\tau|=|\xi|}^{\frac{1}{2}+,\frac{1}{2}+}}\|v_2\|_{X_{|\tau|=|\xi|}^{0,0}}\|v_3\|_{0,\frac{1}{2}-\atop\tau=0}$$

Case 2: $|\tau_2| \gtrsim |\xi_2|$. In this case we obtain

$$1 \lesssim \frac{\langle\tau_2\rangle^{\frac{1}{2}-}}{\langle\xi_2\rangle^{\frac{1}{2}-}} \lesssim \frac{\langle\tau_1\rangle^{\frac{1}{2}-}}{\langle\xi_2\rangle^{\frac{1}{2}-}} + \frac{\langle\tau_3\rangle^{\frac{1}{2}-}}{\langle\xi_2\rangle^{\frac{1}{2}-}}.$$

2.1: Concerning the second term we use $\langle\xi_3\rangle \lesssim \langle\xi_2\rangle$ and reduce to

$$\int_* \frac{\widehat{u}_1(\tau_1,\xi_1)}{\langle\xi_1\rangle^{\frac{1}{4}+\alpha_k}\langle|\tau_1|-|\xi_1|\rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\tau_2,\xi_2)}{\langle\xi_2\rangle^{\frac{1}{2}-\alpha_k+}\langle|\tau_2|-|\xi_2|\rangle^{\frac{1}{2}+}} \widehat{u}_3(\tau_3,\xi_3) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{xt}^2},$$

which follows from Proposition 1.5.

2.2: Concerning the first term we consider two subcases.

2.2.1: $|\tau_1| \lesssim |\xi_1|$. This follows from

$$\int_* \frac{\widehat{u}_1(\tau_1,\xi_1)\langle\xi_1\rangle^{\frac{1}{4}-\alpha_k-}}{\langle|\tau_1|-|\xi_1|\rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\tau_2,\xi_2)}{\langle\xi_2\rangle^{\frac{1}{2}-\alpha_k}\langle|\tau_2|-|\xi_2|\rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\tau_3,\xi_3)}{\langle\tau_3\rangle^{\frac{1}{2}-}} d\xi d\tau$$

$$\lesssim \int_* \frac{\widehat{u}_1(\tau_1,\xi_1)}{\langle \xi_1 \rangle^{\frac{1}{8}+}\langle |\tau_1|-|\xi_1| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\tau_2,\xi_2)}{\langle \xi_2 \rangle^{\frac{1}{8}+}\langle |\tau_2|-|\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\tau_3,\xi_3)}{\langle \tau_3 \rangle^{\frac{1}{2}-}} d\xi d\tau$$

$$\lesssim \prod_{i=1}^3 \|u_i\|_{L^2_{xt}}.$$

Here we used $|\xi_1| \le |\xi_2|$ and $\alpha_k < 1/4$, and the last step uses (1.15) just as in the proof of (3.4).

2.2.2: $|\tau_1| \gg |\xi_1| \Longrightarrow \langle |\tau_1|-|\xi_1| \rangle^{\frac{1}{2}-} \sim \langle \tau_1 \rangle^{\frac{1}{2}-}$. We reduce to

$$\int_* \frac{\widehat{u}_1(\tau_1,\xi_1)}{\langle \xi_1 \rangle^{\frac{1}{4}+\alpha_k}} \frac{\widehat{u}_2(\tau_2,\xi_2)}{\langle \xi_2 \rangle^{\frac{1}{2}-\alpha_k}\langle |\tau_2|-|\xi_2| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_3(\tau_3,\xi_3)}{\langle \tau_3 \rangle^{\frac{1}{2}-}} d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L^2_{xt}}.$$

This is implied by

$$\Big| \int v_1 v_2 v_3 \, dx \, dt \Big| \lesssim \|v_1\|_{L^{\frac{8}{3}}_x L^2_t} \|v_2\|_{L^8_x L^{2+}_t} \|v_3\|_{L^2_x L^{\infty-}_t}$$

$$\lesssim \|v_1\|_{X^{\frac{1}{4},0}_{|\tau|=|\xi|}} \|v_2\|_{X^{\frac{1}{4}+,\frac{1}{2}+}_{|\tau|=|\xi|}} \|v_3\|_{X^{0,\frac{1}{2}-}_{\tau=0}},$$

where we used Sobolev, (1.16) and $\alpha_k < 1/4$.

Claim 3: $|\nabla|^\epsilon A_j^{df} \in X^{\frac{1}{2}+2\alpha_k-\epsilon,\frac{1}{2}+}_{|\tau|=|\xi|}$. For high frequencies we obtain

$$\||\nabla|^\epsilon A_j^{df}\|_{X^{\frac{1}{2}+2\alpha_k-\epsilon,\frac{1}{2}+}} \lesssim \|\langle \psi,\psi \rangle\|_{X^{-\frac{1}{2}+2\alpha_k-,\frac{1}{2}+}} \lesssim \|\psi\|^2_{X^{\frac{1}{4}+\alpha_k,\frac{1}{2}++}}$$

by use of Proposition 1.5. The low frequency case can be handled as in the proof of (3.3).

If after such an iteration step we obtained an α_k such that $\alpha_k > \frac{1}{8}$, we obtain by (3.1) and (3.2) combined with claim 2 and claim 3 the regularity $\psi_\pm \in X^{\frac{1}{4}+\alpha_k,\frac{1}{2}+}_\pm [0,T] \subset X^{\frac{3}{8}+,\frac{1}{2}+}_\pm [0,T]$, $|\nabla|^\epsilon A_j^{df} \in X^{\frac{1}{2}+2\alpha_k-\epsilon,\frac{1}{2}+}_{|\tau|=|\xi|} \subset X^{\frac{3}{4}-\epsilon,\frac{1}{2}+}_{|\tau|=|\xi|}[0,T]$ and $A_j^{cf} \in X^{\frac{1}{4}+2\alpha_k-,\frac{1}{2}+}_{\tau=0}[0,T] \subset X^{\frac{1}{2}+,\frac{1}{2}+}_{\tau=0}[0,T]$, where uniqueness holds by Theorem 1.1 and we are done.

If however $\alpha_k \le \frac{1}{8}$ we need a further iteration step.

Claim 4: The following estimate holds:

$$\|A_j^{cf}\alpha^j\psi_\pm\|_{L^2_t(H^{3\alpha_k--}_x)} \lesssim \|A^{cf}\|_{X^{\frac{1}{4}+2\alpha_k-,\frac{1}{2}+}_{\tau=0}} \|\psi_\pm\|_{X^{\frac{1}{4}+\alpha_k,\frac{1}{2}+}_\pm}.$$

This reduces to

$$\int_* \frac{\widehat{u}_1(\tau_1,\xi_1)}{\langle \xi_1 \rangle^{\frac{1}{4}+2\alpha_k-}\langle \tau_1 \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\tau_2,\xi_2)}{\langle \xi_2 \rangle^{\frac{1}{4}+\alpha_k}\langle |\tau_2|-|\xi_2| \rangle^{\frac{1}{2}+}} \widehat{u}_3(\tau_3,\xi_3)\langle \xi_3 \rangle^{3\alpha_k--} d\xi d\tau$$

$$\lesssim \prod_{i=1}^3 \|u_i\|_{L^2_{xt}}.$$

Case 1: $|\xi_1| \ge |\xi_2| \Rightarrow \langle \xi_3 \rangle \lesssim \langle \xi_1 \rangle$. Using $\alpha_k \le 1/4$ it suffices to show

$$\int_* \frac{\widehat{u}_1(\tau_1,\xi_1)}{\langle \tau_1 \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\tau_2,\xi_2)}{\langle \xi_2 \rangle^{\frac{1}{2}+}\langle |\tau_2|-|\xi_2| \rangle^{\frac{1}{2}+}} \widehat{u}_3(\tau_3,\xi_3) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L^2_{xt}},$$

which holds by (3.5).

Case 2: $|\xi_2| \geq |\xi_1| \Rightarrow \langle \xi_3 \rangle \lesssim \langle \xi_2 \rangle$. Here we use $\alpha_k \leq \frac{1}{8}$. We only have to show

$$\int_* \frac{\widehat{u}_1(\tau_1, \xi_1)}{\langle \xi_1 \rangle^{\frac{1}{2}+} \langle \tau_1 \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\tau_2, \xi_2)}{\langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \widehat{u}_3(\tau_3, \xi_3) d\xi d\tau \lesssim \prod_{i=1}^{3} \|u_i\|_{L^2_{xt}},$$

which follows from (3.6).

Claim 5: The following estimate holds:

$$\|A_j^{df} \alpha^j \psi_\pm\|_{L^2_t(H_x^{3\alpha_k--})} \lesssim \||\nabla|^\epsilon A^{df}\|_{X^{\frac{1}{2}+2\alpha_k-\epsilon-,\frac{1}{2}+}_{|\tau|=|\xi|}} \|\psi_\pm\|_{X^{\frac{1}{4}+\alpha_k,\frac{1}{2}+}_\pm} \cdot$$

In the case of high frequencies of A_j^{df} this follows from Proposition 1.5, where we have to use our assumption $\alpha_k \leq \frac{1}{8}$. In the case of low frequencies we can reduce to

$$\int_* \frac{\widehat{u}_1(\tau_1, \xi_1)\chi_{\{|\xi_1|\leq 1\}}}{|\xi_1|^\epsilon \langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\tau_2, \xi_2)}{\langle \xi_2 \rangle^{\frac{1}{4}+\alpha_k} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \widehat{u}_3(\tau_3, \xi_3) \langle \xi_3 \rangle^{3\alpha_k--} d\xi d\tau$$

$$\lesssim \int_* \frac{\widehat{u}_1(\tau_1, \xi_1)\chi_{\{|\xi_1|\leq 1\}}}{|\xi_1|^\epsilon \langle |\tau_1| - |\xi_1| \rangle^{\frac{1}{2}+}} \frac{\widehat{u}_2(\tau_2, \xi_2)}{\langle \xi_2 \rangle^{\frac{1}{4}-2\alpha_k} \langle |\tau_2| - |\xi_2| \rangle^{\frac{1}{2}+}} \widehat{u}_3(\tau_3, \xi_3) d\xi d\tau$$

$$\lesssim \prod_{i=1}^{3} \|u_i\|_{L^2_{xt}},$$

which easily follows from the estimate

$$\left| \int v_1 v_2 v_3 dx dt \right| \lesssim \|v_1\|_{L^\infty_{xt}} \|v_2\|_{L^2_{xt}} \|v_3\|_{L^2_{xt}}$$

for low frequencies of v_1, where we used again $\alpha_k \leq 1/8$.

We recall that $\alpha_k = \frac{1}{40} + \left(\frac{3}{2}\right)^k \delta \to \infty$ ($k \to \infty$) and $s = \frac{19}{40} + \delta$ with $1 \gg \delta > 0$. Thus for some $k \in \mathbb{N}$ we have $\alpha_k \leq \frac{1}{8}$ and $\alpha_{k+1} > \frac{1}{8}$. Claim 4 and claim 5 imply that $\psi_\pm \in X^{\min(3\alpha_k-,s),1}_\pm[0,T]$. Interpolation with $\psi_\pm \in X^{s,0}_\pm[0,T] \supset C^0([0,T], H^s)$ gives $\psi_\pm \in X^{\min(\frac{3}{2}\alpha_k+\frac{s}{2}-,s),\frac{1}{2}+}_\pm[0,T]$. We notice that $\frac{3}{2}\alpha_k + \frac{s}{2} = \frac{1}{4} + (\frac{1}{40} + (\frac{3}{2})^{k+1}\delta) + \frac{\delta}{2} > \frac{1}{4} + \alpha_{k+1}$. Therefore $\psi_\pm \in X^{\min(\frac{1}{4}+\alpha_{k+1},s),\frac{1}{2}+}_\pm[0,T] \subset X^{\frac{3}{8}+,\frac{1}{2}+}_\pm[0,T]$, and by (3.3) and (3.4) we obtain $A_j^{cf} \in X^{\frac{1}{2}+,\frac{1}{2}+}_{\tau=0}[0,T]$ and $|\nabla|^\epsilon A^{df} \in X^{\frac{3}{4}-\epsilon+,\frac{1}{2}+}_\pm[0,T]$. In these spaces however uniqueness holds by Theorem 1.1. $\square$

5. Proof of Theorem 1.3 and Theorem 1.4

Proof of Theorem 1.3. By standard arguments we only have to show

$$\|N(\psi_1, \psi_2, \psi_3)\|_{X^{s,-\frac{1}{2}++}_{\pm_4}} \lesssim \prod_{i=1}^{3} \|\psi_i\|_{X^{s,\frac{1}{2}+}_{\pm_i}},$$

where $\pm_i$ ($i = 1, 2, 3, 4$) denote independent signs. By duality this is reduced to the estimates

$$J := \int \langle N(\psi_1, \psi_2, \psi_3), \psi_4 \rangle dt\, dx \lesssim \prod_{i=1}^{3} \|\psi_i\|_{X^{s,\frac{1}{2}+}_{\pm_i}} \|\psi_4\|_{X^{-s,\frac{1}{2}--}_{\pm_4}} \cdot$$

By Fourier-Plancherel we obtain

$$J = \int_* q(\xi_1, \dots, \xi_4) \prod_{j=1}^{4} \widehat{\psi}_j(\xi_j, \tau_j) d\xi_1\, d\tau_1 \dots d\xi_4\, d\tau_4,$$

where * denotes integration over $\xi_1 - \xi_2 = \xi_4 - \xi_3 =: \xi_0$ and $\tau_1 - \tau_2 = \tau_4 - \tau_3$ and

$$q = \frac{1}{|\xi_0|^2}[(\xi_{0_1}(\langle \widehat{\psi}_1, \alpha_2 \widehat{\psi}_2 \rangle \langle \widehat{\psi}_3, \widehat{\psi}_4 \rangle - \langle \widehat{\psi}_1, \widehat{\psi}_2 \rangle \langle \alpha_2 \widehat{\psi}_3, \widehat{\psi}_4 \rangle)$$
$$- \xi_{0_2}(\langle \widehat{\psi}_1, \alpha_1 \widehat{\psi}_2 \rangle \langle \widehat{\psi}_3, \widehat{\psi}_4 \rangle - \langle \widehat{\psi}_1, \widehat{\psi}_2 \rangle \langle \alpha_1 \widehat{\psi}_3, \widehat{\psi}_4 \rangle)].$$

The specific structure of this term, namely the form of the matrices α_j plays no role in the following, thus the null structure is completely ignored.

We first consider the case $|\xi_0| \leq 1$. In this case we estimate J as follows:

$$\| \langle \nabla \rangle^{-s-1} |\nabla|^{-\frac{1}{2}} \langle \psi_1, \alpha_i \psi_2 \rangle \|_{L^2_{xt}} \lesssim \| \langle \psi_1, \alpha_i \psi_2 \rangle \|_{L^2_x H^{-s-1,\frac{4}{3}}_x} \lesssim \| \psi_1 \|_{L^4_t H^s_x} \| \psi_2 \|_{L^4_t H^{-s}_x}.$$

In the last step we used

$$\| fg \|_{H^s_x} \lesssim \| f \|_{H^s_x} \| g \|_{L^\infty_x} + \| f \|_{L^2_x} \| g \|_{H^{s,\infty}_x} \lesssim \| f \|_{H^s_x} \| g \|_{H^{s+1,4}_x}$$

which holds by the Leibniz rule for fractional derivatives and Sobolev's embedding theorem, and which is by duality equivalent to the required estimate

$$\| fg \|_{H^{-s-1,\frac{4}{3}}_x} \lesssim \| f \|_{H^s_x} \| g \|_{H^{-s}_x}.$$

The same estimate holds for $\alpha_i = I$. Similarly we obtain

$$\| \langle \nabla \rangle^{-s-1} |\nabla|^{-\frac{1}{2}} \langle \alpha_i \psi_3, \psi_4 \rangle \|_{L^2_{xt}} \lesssim \| \psi_3 \|_{L^4_t H^s_x} \| \psi_4 \|_{L^4_t H^{-s}_x}$$

for arbitrary matrices α_i, so that we obtain

$$J \lesssim \| \psi_1 \|_{X^{s,\frac{1}{4}}_{\pm_1}} \| \psi_2 \|_{X^{-s,\frac{1}{4}}_{\pm_2}} \| \psi_3 \|_{X^{s,\frac{1}{4}}_{\pm_3}} \| \psi_4 \|_{X^{-s,\frac{1}{4}}_{\pm_4}},$$

which is more than enough.

From now on we assume $|\xi_0| \geq 1$. We obtain

$$|J| \lesssim \sum_{j=1}^{2} \left(\| \langle \psi_1, \alpha_j \psi_2 \rangle \|_{X^{s-\frac{1}{2},\frac{1}{4}}_{|\tau|=|\xi|}} \| \langle \psi_3, \psi_4 \rangle \|_{X^{-s-\frac{1}{2},-\frac{1}{4}}_{|\tau|=|\xi|}} \right.$$
$$\left. + \| \langle \psi_1, \psi_2 \rangle \|_{X^{s-\frac{1}{2},\frac{1}{4}}_{|\tau|=|\xi|}} \| \langle \alpha_j \psi_3, \psi_4 \rangle \|_{X^{-s-\frac{1}{2},-\frac{1}{4}}_{|\tau|=|\xi|}} \right).$$

By Proposition 1.5 with $s_0 = \frac{1}{2} - s$, $b_0 = -1/4$, $s_1 = s_2 = s$, $b_1 = b_2 = \frac{1}{2} + \epsilon$ for the first factors and $s_0 = s + \frac{1}{2}$, $b_0 = 1/4$, $s_1 = s$, $s_2 = -s$, $b_1 = \frac{1}{2} + \epsilon$, $b_2 = \frac{1}{2} - 2\epsilon$ for the second factors we obtain

$$|J| \lesssim \prod_{j=1}^{3} \| \psi_j \|_{X^{s,\frac{1}{2}+\epsilon}_{|\tau|=|\xi|}} \| \psi_4 \|_{X^{-s,\frac{1}{2}-2\epsilon}_{|\tau|=|\xi|}}.$$

Using the embedding $X^{s,b}_\pm \subset X^{s,b}_{|\tau|=|\xi|}$ for $s \in \mathbb{R}$ and $b \geq 0$ we obtain the desired estimate. $\qquad \square$

Remark: The potentials are completely determined by ψ and (1.4). We have $A_\mu \sim |\nabla|^{-1} \langle \psi, \psi \rangle$, so that for $s \leq \frac{1}{2}$:

$$\| A_\mu \|_{\dot{H}^{2s}} \lesssim \| \langle \psi, \psi \rangle \|_{\dot{H}^{2s-1}} \lesssim \| \langle \psi, \psi \rangle \|_{L^{\frac{1}{1-s}}} \lesssim \| \psi \|^2_{L^{\frac{2}{1-s}}} \lesssim \| \psi \|^2_{H^s} < \infty$$

and for $\frac{1}{2} < s < 1$:

$$\| A_\mu \|_{\dot{H}^{2s}} \lesssim \| \langle \psi, \psi \rangle \|_{\dot{H}^{2s-1}} \lesssim \| \psi \|_{\dot{H}^{2s-1,\frac{2}{s}}} \| \psi \|_{L^{\frac{2}{1-s}}} \lesssim \| \psi \|^2_{H^s} < \infty$$

as well as

$$\| A_\mu \|_{\dot{H}^\epsilon} \lesssim \| \langle \psi, \psi \rangle \|_{\dot{H}^{\epsilon-1}} \lesssim \| \psi \|^2_{L^{\frac{4}{2-\epsilon}}} \lesssim \| \psi \|^2_{H^s} < \infty,$$

thus we obtain for $0 < \epsilon \ll 1$ and $s < 1$:

$$A_\mu \in C^0([0,T], \dot{H}^{2s} \cap \dot{H}^\epsilon).$$

Proof of Theorem 1.4. We first show $\psi_\pm \in X_\pm^{0,1}[0,T]$. We have to prove

$$\|N(\psi_1, \psi_2, \psi_3)\|_{L_t^2([0,T], L_x^2)} \lesssim \prod_{j=1}^3 \|\psi_j\|_{L_t^\infty([0,T], H_x^{\frac{1}{3}})},$$

where the implicit constant may depend on T. This follows from the estimate

$$
\begin{aligned}
\||\nabla|^{-1}\langle\psi_j, \alpha_i\psi_k\rangle\psi_3\|_{L_x^2} &\lesssim \||\nabla|^{-1}\langle\psi_j, \alpha_i\psi_k\rangle\|_{L_x^6}\|\psi_3\|_{L_x^3} \\
&\lesssim \|\langle\psi_j, \alpha_i\psi_k\rangle\|_{L_x^{\frac{3}{2}}}\|\psi_3\|_{L_x^3} \\
&\lesssim \|\psi_j\|_{L_x^3}\|\psi_k\|_{L_x^3}\|\psi_3\|_{L_x^3} \\
&\lesssim \|\psi_j\|_{H_x^{\frac{1}{3}}}\|\psi_k\|_{H_x^{\frac{1}{3}}}\|\psi_3\|_{H_x^{\frac{1}{3}}},
\end{aligned}
$$

and a similar estimate for the term $\||\nabla|^{-1}\langle\psi_j, \psi_k\rangle\alpha_i\psi_3\|_{L_x^2}$. Assume now $\psi \in C^0([0,T], H^{\frac{1}{3}+\epsilon})$, $\epsilon > 0$. Then we have shown that $\psi_\pm \in X_\pm^{\frac{1}{3}+\epsilon,0}[0,T] \cap X_\pm^{0,1}[0,T]$. By interpolation we get $\psi_\pm \in X_\pm^{\frac{1}{4}+\frac{\epsilon}{4},\frac{1}{4}+\epsilon}[0,T]$ for $\epsilon \ll 1$. Assume now that $\psi, \psi' \in C^0([0,T], H^{\frac{1}{3}+\epsilon})$ are two solutions of (1.5),(1.6), Then we have

$$
\begin{aligned}
&\sum_\pm \|\psi_\pm - \psi'_\pm\|_{X_\pm^{0,\frac{1}{2}+}[0,T]} \\
&\lesssim T^{0+}\sum_\pm \|N(\psi,\psi,\psi) - N(\psi',\psi',\psi')\|_{X_\pm^{0,-\frac{1}{2}++}[0,T]} \\
&\lesssim T^{0+}\sum_{\pm,\pm_1,\pm_2,\pm_3}\big(\|N(\psi_{\pm_1} - \psi'_{\pm_1}, \psi_{\pm_2}, \psi_{\pm_3})\|_{X_\pm^{0,-\frac{1}{2}++}[0,T]} \\
&\quad + \|N(\psi'_{\pm_1}, \psi_{\pm_2} - \psi'_{\pm_2}, \psi_{\pm_3})\|_{X_\pm^{0,-\frac{1}{2}++}[0,T]} \\
&\quad + \|N(\psi'_{\pm_1}, \psi'_{\pm_2}, \psi_{\pm_3} - \psi'_{\pm_3})\|_{X_\pm^{0,-\frac{1}{2}++}[0,T]}\big)
\end{aligned}
\tag{5.1}
$$

Here $\pm$, $\pm_j$ ($j = 1, 2, 3$) denote independent signs. We want to show that for the first term the following estimate holds:

$$
\begin{aligned}
J &:= \int \langle N(\psi_{\pm_1} - \psi'_{\pm_1}, \psi_{\pm_2}, \psi_{\pm_3}), \psi_4\rangle dx\, dt \\
&\lesssim \|\psi_{\pm_1} - \psi'_{\pm_1}\|_{X_{\pm_1}^{0,\frac{1}{2}+}}\|\psi_{\pm_2}\|_{X_{\pm_2}^{\frac{1}{4}+\frac{\epsilon}{4},\frac{1}{4}+\epsilon}}\|\psi_{\pm_3}\|_{X_{\pm_3}^{\frac{1}{4}+\frac{\epsilon}{4},\frac{1}{4}+\epsilon}}\|\psi_4\|_{X_{\pm_4}^{0,\frac{1}{2}--}}.
\end{aligned}
\tag{5.2}
$$

We consider the case $|\xi_0| \le 1$ first. Similarly as in the proof of Theorem 1.3 we obtain

$$|J| \lesssim \|\psi_{\pm_1} - \psi'_{\pm_1}\|_{X_{\pm_1}^{-\frac{1}{4}-\frac{\epsilon}{4},\frac{1}{4}}}\|\psi_{\pm_2}\|_{X_{\pm_2}^{\frac{1}{4}+\frac{\epsilon}{4},\frac{1}{4}}}\|\psi_{\pm_3}\|_{X_{\pm_3}^{\frac{1}{4}+\frac{\epsilon}{4},\frac{1}{4}}}\|\psi_4\|_{X_{\pm_4}^{-\frac{1}{4}-\frac{\epsilon}{4},\frac{1}{4}}},$$

which is more than sufficient. For $|\xi_0| \ge 1$ we obtain

$$
\begin{aligned}
|J| \lesssim \sum_{j=1}^2 \big(&\|\langle(\psi_{\pm_1} - \psi'_{\pm_1}), \alpha_j\psi_{\pm_2}\rangle\|_{X_{|\tau|=|\xi|}^{-\frac{1}{2},0}}\|\langle\psi_{\pm_3}, \psi_4\rangle\|_{X_{|\tau|=|\xi|}^{-\frac{1}{2},0}} \\
&+ \|\langle\psi_{\pm_1} - \psi'_{\pm_1}, \psi_{\pm_2}\rangle\|_{X_{|\tau|=|\xi|}^{-\frac{1}{2},0}}\|\langle\alpha_j\psi_{\pm_3}, \psi_4\rangle\|_{X_{|\tau|=|\xi|}^{-\frac{1}{2},0}}\big)
\end{aligned}
$$

$$\lesssim \|\psi_{\pm_1} - \psi'_{\pm_1}\|_{X^{0,\frac{1}{2}+}_{|\tau|=|\xi|}} \|\psi_{\pm_2}\|_{X^{\frac{1}{4}+\frac{\epsilon}{4},\frac{1}{4}+\epsilon}_{|\tau|=|\xi|}} \|\psi_{\pm_3}\|_{X^{\frac{1}{4}+\frac{\epsilon}{4},\frac{1}{4}+\epsilon}_{|\tau|=|\xi|}} \|\psi_4\|_{X^{0,\frac{1}{2}--}_{|\tau|=|\xi|}},$$

where we used Proposition 1.5 for the first factor with the choice $s_0 = \frac{1}{2}$, $b_0 = 0$, $s_1 = 0$, $b_1 = \frac{1}{2}+$, $s_2 = \frac{1}{4} + \frac{\epsilon}{4}$, $b_2 = \frac{1}{4} + \epsilon$ and for the second factor with $s_0 = \frac{1}{2}$, $b_0 = 0$, $s_1 = \frac{1}{4} + \frac{\epsilon}{4}$, $b_1 = \frac{1}{4} + \epsilon$, $s_2 = 0$, $b_2 = \frac{1}{2} - -$. The embedding $X^{s,b}_{\pm} \subset X^{s,b}_{|\tau|=|\xi|}$ for $b \geq 0$ gives (5.2). The other terms in (5.1) are treated similarly. We obtain

$$\sum_{\pm} \|\psi_{\pm} - \psi'_{\pm}\|_{X^{0,\frac{1}{2}+}_{\pm}[0,T]}$$

$$\lesssim T^{0+} \sum_{j=1}^{2} \left(\|\psi_{\pm_j}\|^2_{X^{\frac{1}{4}+\frac{\epsilon}{4},\frac{1}{4}+\epsilon}_{\pm_j}[0,T]} + \|\psi'_{\pm_j}\|^2_{X^{\frac{1}{4}+\frac{\epsilon}{4},\frac{1}{4}+\epsilon}_{\pm_j}[0,T]} \right) \sum_{\pm} \|\psi_{\pm} - \psi'_{\pm}\|_{X^{0,\frac{1}{2}+}_{\pm}[0,T]}.$$

We recall that $\psi_{\pm}$, $\psi'_{\pm} \in X^{\frac{1}{4}+\frac{\epsilon}{4},\frac{1}{4}+\epsilon}_{\pm}[0,T]$, so that for sufficiently small T this implies $\|\psi_{\pm} - \psi'_{\pm}\|_{X^{0,\frac{1}{2}+}_{\pm}[0,T]} = 0$, thus local uniqueness. By iteration T can be chosen arbitrarily. $\square$

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