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MULTIPLE SOLUTIONS FOR IMPULSIVE HAMILTONIAN SYSTEMS

GUANGHUI ZHOU

Abstract. In this article, we study second-order impulsive Hamiltonian systems, We obtain some existence and multiplicity results, by using a variational method and critical point theorem. An example illustrate the feasibility of our results.

1. Introduction

In this article, we study the second-order Hamiltonian systems with impulsive effects

$$\begin{gathered}
-\ddot{u} + A(t)u = \lambda b(t)\nabla G(u), \quad \text{a.e. } t \in [0,T], \\
\Delta(\dot{u}^i(t_j)) = \dot{u}^i(t_j^+) - \dot{u}^i(t_j^-) = I_{ij}(u^i(t_j)), \quad i = 1,2,\dots,N, j = 1,2,\dots,l, \\
u(0) - u(T) = u'(0) - u'(T) = 0,
\end{gathered} \tag{1.1}$$

where $A : [0,T] \to \mathbb{R}^{N\times N}$ is a continuous map from $[0,T]$ to the set of N-order symmetric matrices, T is a real positive number, $u(t) = (u^1(t), u^2(t), \dots, u^N(t))$, t_j, $j = 1,2,\dots,l$, are the instants where the impulses occur and $0 = t_0 < t_1 < t_2 < \dots < t_l < t_{l+1} = T$, $I_{ij} : \mathbb{R} \to \mathbb{R}$ $(i = 1,2,\dots,N; j = 1,2,\dots,l)$ are continuous.

Recently, with the development of theory and applications of impulsive differential systems, there have been some results considering the existence and multiplicity of solutions for impulsive problems, by using variational method (see [1, 2, 3, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]). To obtain the existence and multiplicity of solutions, impulsive functions $I_{ij}(\cdot)$ of all theorems in [14, 18], are required to satisfy the following conditions

$$I_{ij}(y)y \ge 0 \quad \text{for all } i \in \mathcal{A} = \{1,2,\dots,N\},]; j \in \mathcal{B} = \{1,2,\dots,l\},\ y \in \mathbb{R}, \tag{1.2}$$

or

$$I_{ij}(y)y \le 0 \quad \text{for all } i \in \mathcal{A},\ j \in \mathcal{B},\ y \in \mathbb{R}. \tag{1.3}$$

However, as Dai pointed out in [3], there are many functions which do not satisfy (1.2) or (1.3). For example, when $N = 3$ and $l = 2$, impulsive functions of (1.1) are

$$I_{ij}(y) = -y + 1 \quad \text{for } i = 1,2,3; j = 1,2, \tag{1.4}$$

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or a more complicated case such as

$$I_{ij}(y) = \begin{cases} \frac{y}{2} + 1, & i = 1, 2, 3; j = 1, \\ -y & i = 1, 2; j = 2, \\ \frac{y}{2} & i = 3, j = 2. \end{cases} \tag{1.5}$$

So it is important to consider such case. Motivated by [3, 5] and the above facts, we will reconsider problem (1.1) and study the existence of solutions without assumption (1.2) or (1.3), which show that suitable impulses won't influence the existence of of solutions.

The organization of this article is as follows. In Section 2, we introduce some definitions and lemmas. In Section 3, by using critical point theorem [6, 7], we obtain some existence and multiplicity result of solutions for (1.1).

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts. $A : [0, T] \to \mathbb{R}^{N\times N}$ is a matrix-valued function fulfilling the following technical assumptions:

- (A1) $A(t) = (a_{ij}(t))$ is a symmetric matrix with $a_{ij} \in L^\infty([0, T], \mathbb{R}^+)$ for every $t \in [0, T]$.
- (A2) There exists a positive constant μ such that $A(t)\xi \cdot \xi \geq \mu|\xi|^2$ for every $\xi \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

The set

$$\begin{gathered} H^1_T = \big\{u : [0, T] \to \mathbb{R}^N : u \text{ is absolutely continuous}, \\ u(0) = u(T) \text{ and } \dot{u} \in L^2([0, T], \mathbb{R}^N)\big\} \end{gathered}$$

is a Hilbert space with the usual norm

$$\|u\|_{H^1_T} = \Big(\int_0^T (|\dot{u}(t)|^2 + |u(t)|^2)dt\Big)^{1/2}.$$

For every $u, v \in H^1_T$, by $(A1), (A_2)$, we define an inner product

$$\langle u, v\rangle = \int_0^T [(\dot{u}(t), \dot{v}(t)) + (A(t)u(t), v(t))]dt, \quad \forall u, v \in H^1_T,$$

which induces the norm

$$\|u\| = \Big(\int_0^T (|\dot{u}(t)|^2 + A(t)|u(t)|^2)dt\Big)^{1/2}.$$

As in [4, 17], we have

$$\begin{gathered} A(t)\xi \cdot \xi = \sum_{j=1}^N \sum_{i=1}^N a_{ij}(t)\xi_i\xi_j \leq \sum_{j=1}^N \sum_{i=1}^N \|a_{ij}\|_\infty |\xi|^2, \\ \sqrt{m}\|u\|_{H^1_T} \leq \|u\| \leq \sqrt{M}\|u\|_{H^1_T}, \end{gathered} \tag{2.1}$$

where $m = \min\{1, \mu\}, M = \max\{1, \Sigma^N_{i,j=1}\|a_{ij}\|_\infty\}, \|a_{ij}\|_\infty = \max_{t\in[0,T]} |a_{ij}(t)|$. Since $(H^1_T, \|\cdot\|_{H^1_T})$ is compactly embedded in $C([0, T], \mathbb{R}^N)$, then there is a positive number $\bar{k}$ such that for every $u \in H^1_T$,

$$\|u\|_\infty \leq \bar{k}\|u\|, \tag{2.2}$$

Thus
$$\bar{k} \le k = \sqrt{\frac{2}{m}} \max\{\sqrt{T}, \frac{1}{\sqrt{T}}\}. \tag{2.3}$$
For any $u, v \in H_T^1$, let
$$\Phi(u) = \frac{1}{2}\|u\|^2 + \sum_{j=1}^{l}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(s)ds, \quad \Psi(u) = \int_0^T b(t)G(u(t))dt. \tag{2.4}$$
By standard argument, we see that Φ, Ψ are Gâteaux differentiable at any $u \in H_T^1$ and
$$\begin{gathered}\langle \Phi'(u), v\rangle = \int_0^T [(\dot{u}(t), \dot{v}(t)) + (A(t)u(t), v(t))]dt + \sum_{j=1}^{l}\sum_{i=1}^{N} I_{ij}(u^i(t_j))v^i(t_j), \\ \langle \Psi(u), v\rangle = \int_0^T b(t)(\nabla G(u(t)), v(t))dt.\end{gathered} \tag{2.5}$$
A critical point of the functional $\Phi - \lambda\Psi$ is a function $u \in H_T^1$ such that $\Phi'(u)(v) - \lambda\Psi'(u)(v) = 0$ for every $v \in H_T^1$, i.e.

Definition 2.1. A function $u \in H_T^1$ is a weak solution of (1.1) if
$$\begin{aligned}&\int_0^T [(\dot{u}(t), \dot{v}(t)) + (A(t)u(t), v(t))]dt + \sum_{j=1}^{l}\sum_{i=1}^{N} I_{ij}(u^i(t_j))v^i(t_j) \\ &= \lambda \int_0^T b(t)(G(u(t)), v(t))dt\end{aligned} \tag{2.6}$$
holds for any $v \in H_T^1$.

Hence, we can claim that each critical point of the functional $\Phi - \lambda\Psi$ is a weak solution to problem (1.1).

Theorem 2.2 ([4, Theorem 3.3]). *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to R$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semi-continuous and coercive and Ψ is sequentially weakly upper semi-continuous. Assume that*

(i) *Φ is convex;*
(ii) *For every $x_1, x_2 \in X$ such that $\Psi(x_1) \ge 0$ and $\Psi(x_2) \ge 0$, one has $\inf_{s\in[0,1]} \Psi(sx_1 + (1-s)x_2) \ge 0$;*
(iii) *$\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;*
(iv) *there are three positive constants r_1, r_2, r_3 with $\inf_X \Phi < r_1 < r_2$ such that, if we put*
$$\begin{gathered}\varphi^{(1)}(r_i) = \inf_{u\in\Phi^{-1}((-\infty,r_i))} \frac{\sup_{v\in\Phi^{-1}((-\infty,r_i))}\Psi(v) - \Psi(u)}{r_i - \Phi(u)}, \\ \varphi_2(r_1, r_2) = \inf_{u\in\Phi^{-1}((-\infty,r_1))} \sup_{v\in\Phi^{-1}([r_1,r_2))} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)}, \\ \varphi^{(3)}(r_2, r_3) = \frac{\sup_{u\in\Phi^{-1}((-\infty,r_2+r_3))}\Psi(u)}{r_3}, \\ \varphi_3(r_1, r_2, r_3) = \max\{\varphi^{(1)}(r_1), \varphi^{(1)}(r_2), \varphi^{(3)}(r_2, r_3)\},\end{gathered}$$
one has $\varphi_3(r_1, r_2, r_3) < \varphi_2(r_1, r_2)$.

(v) *For each* $\lambda \in \Lambda_{r_1,r_2,r_3} := (\frac{1}{\varphi_2(r_1,r_2)}, \frac{1}{\varphi_3(r_1,r_2,r_3)})$*, if we put*

$$\Psi_{\frac{r_3}{\lambda}}(u) = \begin{cases} \Psi(u) & \text{if } \Psi(u) \le \frac{r_3}{\lambda}, \\ \frac{r_3}{\lambda} & \text{if } \Psi(u) > \frac{r_3}{\lambda}, \end{cases}$$

then $\Phi - \lambda\Psi_{\frac{r_3}{\lambda}}$ *satisfies the condition* $(PS)_c$*, with* $c \in \mathbb{R}$*.*

Then for each $\lambda \in \Lambda_{r_1,r_2,r_3}$*, the functional* $\Phi - \lambda\Psi$ *admits at least three critical points* $u_1, u_2, u_3 \in X$ *such that* $u_1 \in \Phi^{-1}((-\infty, r_1)), u_2 \in \Phi^{-1}([r_1, r_2))$ *and* $u_3 \in \Phi^{-1}((-\infty, r_2 + r_3))$*.*

Theorem 2.3 ([6, Theorem 2.6])**.** *Let* X *be a reflexive real Banach space, let* $\Phi : X \to R$ *be a be a sequentially weakly lower semi-continuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on* X^**, and let* $\Psi : X \to R$ *be a sequentially weakly upper semi-continuous and continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exist* $r \in \mathbb{R}$ *and* $x_0, \bar{x} \in X$*, with* $\Phi(x_0) < r < \Phi(\bar{x})$ *and* $\Psi(x_0) = 0$*, such that*

(i) $\sup_{x\in\Phi^{-1}((-\infty,r])} \Psi(x) < (r - \Phi(x_0))\frac{\Psi(\bar{x})}{\Phi(\bar{x})-\Phi(x_0)}$;

(ii) *for each*

$$\lambda \in \Lambda_r := (\frac{\Phi(\bar{x}) - \Phi(x_0)}{\Psi(\bar{x})}, \frac{r - \Phi(x_0)}{\sup_{x\in\Phi^{-1}((-\infty,r])} \Psi(x)}),$$

the functional $\Phi - \lambda\Psi$ *is coercive.*

Then for each $\lambda \in \Lambda_r$*, the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical points* $u_1, u_2, u_3 \in X$*.*

Theorem 2.4 ([7, Theorem 2.1])**.** *Let* X *be a reflexive real Banach space, let* $\Phi, \Psi : X \to R$ *be two Gâteaux differentiable functionals such that* Φ *is sequentially weakly lower semicontinuous and coercive and* Ψ *is sequentially weakly upper semicontinuous. For every* $r > \inf_X \Phi$*, let us put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}((-\infty,r))} \frac{\sup_{v\in\Phi^{-1}((-\infty,r))} \Psi(v) - \Psi(u)}{r - \Phi(u)},$$

and

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then one has

(a) *For every* $r > \inf_X \Phi$ *and every* $\lambda \in (0, \frac{1}{\varphi(r)})$*, the restriction of the functional* $I_\lambda = \Phi - \lambda\Psi$ *to* $\Phi^{-1}((-\infty, r))$ *admits a global minimum, which is a critical point (local minimum) of* I_λ *in* X*.*

(b) *If* $\gamma < +\infty$ *then, for each* $\lambda \in (0, \frac{1}{\gamma})$*, the following alternative holds: Either*
  - (b1) I_λ *possesses a global minimum, or*
  - (b2) *there is a sequence* $\{u_n\}$ *of critical points (local minima) of* I_λ *such that* $\lim_{n\to\infty} \Phi(u_n) = +\infty$*.*

(c) *If* $\delta < +\infty$ *then, for each* $\lambda \in (0, \frac{1}{\delta})$*, the following alternative holds: Either*
  - (c1) *there is a global minimum of* Φ *which is a local minimum of* I_λ*, or*
  - (c2) *there is a sequence of pairwise distinct critical points (local minima) of* I_λ *which weakly converges to a global minimum of* Φ*.*

As in the proof of [3, Lemma 5], we have the following result.

Lemma 2.5. *Suppose that $I_{ij}(y)$ is nondecreasing in $y \in \mathbb{R}$ for all $i = 1, 2, \dots, N$, $j = 1, 2, \dots, l$. Then $\phi(u) = \sum_{j=1}^{l} \sum_{i=1}^{N} \int_0^{u^i(t_j)} I_{ij}(s)ds$ is convex in $u \in \mathbb{R}^N$.*

3. Existence and multiplicity of solutions

For convenience, we introduce the assumption

(A3) There exist constants $c_{ij} > 0$, $d_{ij} > 0$, $\gamma_{ij} \in [0, 1)$, $i = 1, 2, \dots, N$, $j = 1, 2, \dots, l$, such that
$$|I_{ij}(y)| \le c_{ij} + d_{ij}|y|^{\gamma_{ij}} \quad \text{for all } y \in \mathbb{R}.$$

Theorem 3.1. *Assume that* (A1)–(A3) *hold. $I_{ij}(y)$ is nondecreasing for $y \in \mathbb{R}$, for any $i = 1, 2, \dots, N$, $j = 1, 2, \dots, l$ with $I_{ij}(0) = 0$. Let $G \in C^1(\mathbb{R}^N, R)$ be such that*

(1) *$G(\xi) \ge G(0) = 0$ for any $\xi \in \mathbb{R}^N$,*
(2) *there exist $\rho_2 \gg \rho_1 > 0$ and $\bar{\xi} \in \mathbb{R}^N$ such that*
(3) *$\rho_1 < |\bar{\xi}| < \sqrt{\frac{4M}{\mu}}\rho_2$, where $\mu < M$ defined as (A_2) and* (2.1) *respectively;*
(4)
$$(1 + \frac{4M}{\mu})\frac{\max_{|\xi| \le \rho_1} G(\xi)}{\rho_1^2} + 4\frac{\max_{|\xi| \le \rho_2} G(\xi)}{\rho_2^2} < \frac{G(\bar{\xi})}{|\bar{\xi}|^2}.$$

Then for every $b \in L^1([0, T]) \setminus \{0\}$ and for every λ in
$$\Lambda_{\rho_1,\rho_2,\rho_3} := \Big(\frac{MT}{\|b\|_{L^1}\big(1 - \frac{\mu}{M}\big)}\frac{|\bar{\xi}|^2}{G(\bar{\xi})}, \frac{T}{4\|b\|_{L^1}}\frac{1}{\max\Big\{\frac{1}{\mu}\frac{\max_{|\xi| \le \rho_1} G(\xi)}{\rho_1^2}, \frac{1}{M}\frac{\max_{|\xi| \le \rho_2} G(\xi)}{\rho_2^2}\Big\}}\Big),$$
Equation (1.1) *has at least three nontrivial solutions u_1, u_2, u_3 such that $\|u_i\|_\infty \le \rho_2$, $i = 1, 2$.*

Proof. Let Φ, Ψ be as (2.4). Since $I_{ij}(y)$ is nondecreasing in $y \in \mathbb{R}$ for any $i = 1, 2, \dots, N$, $j = 1, 2, \dots, l$, we have
$$\phi(u) = \sum_{j=1}^{l} \sum_{i=1}^{N} \int_0^{u^i(t_j)} I_{ij}(s)ds$$
which is convex for $u \in \mathbb{R}^N$, from Lemma 2.5. It is obvious that $\|u\|$ is convex in $u \in \mathbb{R}^N$. Thus $\Phi(u)$ is convex in $u \in \mathbb{R}^N$. We also know $\Phi(u)$ is continuously Gâteaux differentiable and sequentially weakly lower semi-continuous. By (A3) and (2.2), we have
$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\|u\|^2 + \sum_{j=1}^{l} \sum_{i=1}^{N} \int_0^{u^i(t_j)} I_{ij}(s)ds \\
&\ge \frac{1}{2}\|u\|^2 - \sum_{j=1}^{l} \sum_{i=1}^{N} (c_{ij}|u| + d_{ij}|u|^{1+\gamma_{ij}}) \\
&\ge \frac{1}{2}\|u\|^2 - \sum_{j=1}^{l} \sum_{i=1}^{N} (c_{ij}\|u\|_\infty + d_{ij}\|u\|_\infty^{1+\gamma_{ij}}) \\
&\ge \frac{1}{2}\|u\|^2 - \sum_{j=1}^{l} \sum_{i=1}^{N} (c_{ij}\bar{k}\|u\| + d_{ij}\bar{k}^{1+\gamma_{ij}}\|u\|^{1+\gamma_{ij}}) \to +\infty
\end{aligned}$$

as $\|u\| \to +\infty$, i.e. Φ is obviously coercive. Ψ is continuously Gâteaux differentiable with compact derivative, hence it is sequentially weakly continuous. In addition, assumption (1) and for every $b \in L^1([0,T]) \setminus \{0\}$ imply that $\Psi(u) \geq 0$ for every $u \in H_T^1$, hence (ii) of Theorem 2.2 holds. Again from assumption (1), we get (iii).

Choose $\bar{v}(t) = \bar{\xi}, t \in [0,T]$. Let

$$r_1 = \frac{\mu T \rho_1^2}{4} > 0, \quad r_2 = r_3 = MT\rho_2^2 > 0,$$

for sufficiently small

$$\rho_1 = \Big(\frac{2\bar{\gamma}\sum_{j=1}^{l}\sum_{i=1}^{N} d_{ij}}{\mu T}\Big)^{\frac{1}{1-\bar{\gamma}}} > 0,$$
$$\bar{\rho}_1 = \min\Big\{\frac{\mu T\rho_1^2}{8\sum_{j=1}^{l}\sum_{i=1}^{N} c_{ij}}, \Big(\frac{\mu T\rho_1^2}{8\sum_{j=1}^{l}\sum_{i=1}^{N} d_{ij}}\Big)^{\frac{1}{1+\bar{\gamma}}}\Big\},$$

and sufficiently big

$$\rho_2 \gg \Big(\frac{2\sum_{j=1}^{l}\sum_{i=1}^{N}(c_{ij}+d_{ij})}{MT}\Big)^{\frac{1}{1-\bar{\gamma}}} > 0,$$

where $\bar{\gamma} = \max_{i,j}\{\gamma_{ij}\}$. Then from Section 2, we have

$$\Phi^{-1}((-\infty, r_1)) \subseteq \{u \in C([0,T],\mathbb{R}^N) : \|u\|_\infty \leq \rho_1\}, \tag{3.1}$$
$$\Phi^{-1}((-\infty, r_2)) \subseteq \Phi^{-1}((-\infty, r_2+r_3)) \subseteq \{u \in C([0,T],\mathbb{R}^N) : \|u\|_\infty \leq \rho_2\}, \tag{3.2}$$

then by (3.1), (3.2), we have

$$\begin{aligned}
\varphi^{(1)}(r_i) &= \inf_{u\in\Phi^{-1}((-\infty,r_i))} \frac{\sup_{v\in\Phi^{-1}((-\infty,r_i))}\Psi(v) - \Psi(u)}{r_i - \Phi(u)} \\
&\leq \frac{\sup_{v\in\Phi^{-1}((-\infty,r_i))}\Psi(v)}{r_i} \\
&\leq \begin{cases} \frac{4\|b\|_{L^1}}{\mu T}\frac{\max_{|\xi|\leq\rho_1} G(\xi)}{\rho_1^2}, & \text{if } i=1, \\ \frac{4\|b\|_{L^1}}{MT}\frac{\max_{|\xi|\leq\rho_2} G(\xi)}{\rho_2^2}, & \text{if } i=2. \end{cases}
\end{aligned} \tag{3.3}$$

and

$$\varphi^{(3)}(r_2,r_3) = \frac{\sup_{u\in\Phi^{-1}((-\infty,r_2+r_3))}\Psi(u)}{r_3} \leq \frac{4\|b\|_{L^1}}{MT}\frac{\max_{|\xi|\leq\rho_2} G(\xi)}{\rho_2^2}, \tag{3.4}$$

By (3.3), (3.4), we have

$$\begin{aligned}
\varphi_3(r_1,r_2,r_3) &= \max\{\varphi^{(1)}(r_1), \varphi^{(1)}(r_2), \varphi^{(3)}(r_2,r_3)\} \\
&\leq \frac{4\|b\|_{L^1}}{T}\max\Big\{\frac{1}{\mu}\frac{\max_{|\xi|\leq\rho_1} G(\xi)}{\rho_1^2}, \frac{1}{M}\frac{\max_{|\xi|\leq\rho_2} G(\xi)}{\rho_2^2}\Big\}.
\end{aligned} \tag{3.5}$$

Taking into account (A2), (2) in the assumptions and (A3), we have

$$
\begin{aligned}
\Phi(\bar v) &= \frac{1}{2}\|\bar v\|^2 + \sum_{j=1}^{l}\sum_{i=1}^{N}\int_0^{\bar v^i(t_j)} I_{ij}(s)ds \\
&\geq \frac{\mu T}{2}|\bar\xi|^2 - \sum_{j=1}^{l}\sum_{i=1}^{N}(c_{ij}|\bar\xi| + d_{ij}|\bar\xi|^{1+\gamma_{ij}}) \\
&\geq \frac{\mu T}{2}|\bar\xi|^2 - \sum_{j=1}^{l}\sum_{i=1}^{N}(c_{ij}|\bar\xi| + d_{ij}|\bar\xi|^{1+\bar\gamma}) \\
&\geq \inf_{\bar\rho_1\geq x\geq\rho_1}\{\frac{\mu T}{2}\rho_1^2 - \sum_{j=1}^{l}\sum_{i=1}^{N}(c_{ij}x + d_{ij}x^{1+\bar\gamma})\} \\
&= \frac{\mu T\rho_1^2}{4} = r_1 > 0,
\end{aligned} \tag{3.6}
$$

for sufficiently small $\rho_1 > 0$.

In view of (2), (H1) and (2.1), we have

$$
\begin{aligned}
\Phi(\bar v) &= \frac{1}{2}\|\bar v\|^2 + \sum_{j=1}^{l}\sum_{i=1}^{N}\int_0^{\bar v^i(t_j)} I_{ij}(s)ds \\
&\leq \frac{T}{2}\sum_{j=1}^{N}\sum_{i=1}^{N}\|a_{ij}\|_\infty|\bar\xi|^2 + \sum_{j=1}^{l}\sum_{i=1}^{N}(c_{ij}|\bar\xi| + d_{ij}|\bar\xi|^{1+\gamma_{ij}}) \\
&\leq \frac{TM}{2}\rho_2^2 + \sum_{j=1}^{l}\sum_{i=1}^{N}(c_{ij}\rho_2 + d_{ij}\rho_2^{1+\bar\gamma}) \leq MT\rho_2^2 = r_2,
\end{aligned} \tag{3.7}
$$

for sufficiently big

$$
\rho_2 \gg \Big(\frac{2\sum_{j=1}^{l}\sum_{i=1}^{N}(c_{ij}+d_{ij})}{MT}\Big)^{\frac{1}{1-\bar\gamma}} > 0,
$$

so by (2), we have

$$
r_1 < \Phi(\bar v) < r_2. \tag{3.8}
$$

Since

$$
\sup_{v\in\Phi^{-1}([r_1,r_2))}\frac{\Psi(v)-\Psi(u)}{\Phi(v)-\Phi(u)} \geq \frac{\Psi(\bar v)-\Psi(u)}{\Phi(\bar v)-\Phi(u)}, \tag{3.9}
$$

for $u \in \Phi^{-1}((-\infty, r_1))$, it follows that

$$
\varphi_2(r_1, r_2) \geq \inf_{u\in\Phi^{-1}((-\infty,r_1))}\frac{\Psi(\bar v)-\Psi(u)}{\Phi(\bar v)-\Phi(u)}. \tag{3.10}
$$

Fix $u \in \Phi^{-1}((-\infty, r_1))$, from (3.1) and $(g_1)-(g_3)$, we have

$$
\Psi(\bar v) - \Psi(u) \geq \|b\|_{L^1}\big(G(\bar\xi) - \max_{|\xi|\leq\rho_1} G(\xi)\big) > 0.
$$

From (3.7), we have

$$
0 < \Phi(\bar v) - \Phi(u) \leq \Phi(\bar v) \leq TM|\bar\xi|^2. \tag{3.11}
$$

Then by (3.5), (3.10)-(3.11) and (2), we have

$$\begin{aligned}\varphi_2(r_1,r_2) &\geq \frac{\|b\|_{L^1}}{TM}\frac{\big(G(\bar\xi)-\max_{|\xi|\leq\rho_1}G(\xi)\big)}{|\bar\xi|^2}\\ &> \frac{\|b\|_{L^1}}{MT}\Big(1-\frac{1}{\max\{1,\frac{M}{\mu}\}}\Big)\frac{G(\bar\xi)}{|\bar\xi|^2}\\ &= \frac{\|b\|_{L^1}}{MT}\Big(1-\frac{\mu}{M}\Big)\frac{G(\bar\xi)}{|\bar\xi|^2}\\ &> \varphi_3(r_1,r_2,r_3),\end{aligned}\tag{3.12}$$

for $M>\mu>0$;i.e. that (iv) of Theorem 2.2 holds. Moreover $\Lambda_{\rho_1,\rho_2}\subseteq\Lambda_{r_1,r_2,r_3}$ and for every $\lambda\in\Lambda_{\rho_1,\rho_2}$. Assumption (v) of Theorem 2.2 is verified as a simple consequence of the regularity of Φ,Ψ (see [7, Remark 3.10]). Then from Theorem 2.2, the proof is complete. □

As in the proof of Theorem 3.1, by Theorem 2.3, we have the following result.

Theorem 3.2. *Assume that the assumptions* $(A1),(A_2)$ *and the hypotheses (A3) hold.* $I_{ij}(y)$ *is nondecreasing in* $y\in\mathbb{R}$ *for any* $i=1,2,\dots,N$, $j=1,2,\dots,l$ *with* $I_{ij}(0)=0$. *Let* $G\in C^1(\mathbb{R}^N,R)$ *be such that*

1. $G(0)=0$;
2. *there exist* $\rho>0$ *and* $\bar\xi\in\mathbb{R}^N$ *such that*
$$\frac{\max_{|\xi|\leq\rho}G(\xi)}{\rho^2}<\frac{\mu(M-\mu)}{4M^2}\frac{G(\bar\xi)}{MT|\bar\xi|^2};$$
3. $\limsup_{|\xi|\to\infty}\frac{G(\xi)}{|\xi|^2}<\frac{\max_{|\xi|\leq\rho}G(\xi)}{\rho^2}$.

Then for every $b\in L^1([0,T])\setminus\{0\}$ *and for every* λ *in*

$$\Lambda:=\Big(\frac{MT}{\|b\|_{L^1}(1-\frac{\mu}{M})}\frac{|\bar\xi|^2}{G(\bar\xi)},\frac{T\mu}{4\|b\|_{L^1}}\frac{\rho^2}{\max_{|\xi|\leq\rho}G(\xi)}\Big),$$

(1.1) *has at least three nontrivial solutions* u_1,u_2,u_3, *where* $k, 0<\mu<M$ *are defined as* (2.3), *(A2),* (2.1), *respectively.*

Theorem 3.3. *Assume that* (A1)-(A3) *hold.* $I_{ij}(y)$ *is nondecreasing in* $y\in\mathbb{R}$ *for any* $i=1,2,\dots,N, j=1,2,\dots,l$ *with* $I_{ij}(0)=0$. *Let*

$$\alpha=\liminf_{\rho\to+\infty}\frac{\max_{|\xi|\leq\rho}G(\xi)\xi}{\rho^2},\quad \beta=\limsup_{\rho\to+\infty}\frac{G(\xi)\xi}{|\xi|^2},$$

and assume that $\alpha<4\beta$. *Then for every* $b\in L^1([0,T])\setminus\{0\}$ *and for every* λ *in* $\Lambda:=\big(\frac{\mu T}{4\|b\|_{L^1}\beta},\frac{\mu T}{4\|b\|_{L^1}\alpha}\big)$, (1.1) *has an unbounded sequence nontrivial solutions.*

Proof. For every $b\in L^1([0,T])\setminus\{0\}$, let Φ,Ψ be as (2.4), using Theorem 2.4, from the proof of Theorem 3.1, we have that the functionals Φ,Ψ satisfy the regularity assumptions required in Theorem 2.4. Let us now verify that $\gamma=\liminf_{\rho\to+\infty}\varphi(\rho)<+\infty$.

Let $\{\rho_n\}$ be a sequence of positive numbers such that $\rho_n\to+\infty$ as $n\to+\infty$ and

$$\lim_{n\to+\infty}\frac{\max_{|\xi|\leq\rho_n}G(\xi)}{\rho_n^2}=\liminf_{\rho\to+\infty}\frac{\max_{|\xi|\leq\rho}G(\xi)}{\rho^2}.\tag{3.13}$$

Let $r_n = \frac{\mu T \rho_n^2}{4}, \forall\, n \in N$, similar to the reasoning of (3.3), we have

$$
\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}((-\infty,r_n))} \frac{\big(\sup_{v\in\Phi^{-1}((-\infty,r_n))}\Psi(v)\big)-\Psi(u)}{r_n-\Phi(u)} \\
&\le \frac{\sup_{v\in\Phi^{-1}((-\infty,r_n))}\Psi(v)-G(0)\|b\|L^1}{r_n} \\
&\le \frac{4\|b\|_{L^1}}{\mu T}\frac{\max_{|\xi|\le\rho_n}G(\xi)-G(0)}{\rho_n^2} \\
&= \frac{4\|b\|_{L^1}}{\mu T}\frac{\max_{|\xi|\le\rho_n}G(\xi)}{\rho_n^2}.
\end{aligned} \tag{3.14}
$$

Then

$$
0 \le \gamma := \liminf_{r\to+\infty}\varphi(r) \le \frac{4\|b\|_{L^1}}{\mu T}\liminf_{\rho\to+\infty}\frac{\max_{|\xi|\le\rho}G(\xi)}{\rho^2} = \frac{4\|b\|_{L^1}}{\mu T}\alpha < +\infty, \tag{3.15}
$$

i.e. $\gamma = \liminf_{\rho\to+\infty}\varphi(\rho) = \frac{4\|b\|_{L^1}}{\mu T}\alpha < +\infty$. In view of $\alpha < 4\beta$ and (3.15), we get that $\Lambda \subseteq (0, \frac{1}{\gamma})$. Now, we verify that the functional $\Phi - \lambda\Psi$ is unbounded from below for $\lambda \in \Lambda$. In fact, by the choice of λ and the positivity of β, one has that there exists a sequence $\{\xi_n\} \subseteq \mathbb{R}^N$ with $|\xi_n| \to +\infty$ such that for any $n \in N$,

$$
\liminf_{n\to+\infty}\frac{G(\xi_n)}{|\xi_n|^2} > \frac{1}{\frac{4\|b\|_{L^1}}{\mu T}\lambda}, \tag{3.16}
$$

i.e. $|\xi_n| \to +\infty$ for any $n \in N$, as $|\xi_n| \to +\infty$,

$$
\frac{T}{2}\sum_{j=1}^{N}\sum_{i=1}^{N}\|a_{ij}\|_\infty + \sum_{j=1}^{l}\sum_{i=1}^{N}\Big(\frac{c_{ij}}{|\xi_n|}+\frac{d_{ij}}{|\xi_n|^{\bar\gamma}}\Big) - \lambda\|b\|_{L^1}\frac{G(\xi_n)}{|\xi_n|^2} < 0, \tag{3.17}
$$

If we let $v_n(t) = \xi_n$, for all $n \in N$, then for $v_n \in H_T^1$, using the first equality of (2.1), we have

$$
\begin{aligned}
I_\lambda &= \Phi(v_n) - \lambda\Psi(v_n) \\
&= \frac{1}{2}\|v_n\|^2 + \sum_{j=1}^{l}\sum_{i=1}^{N}\int_0^{v_n^i(t_j)} I_{ij}(s)ds - \int_0^T b(t)G(v_n(t))dt \\
&\le \frac{T}{2}\sum_{j=1}^{N}\sum_{i=1}^{N}\|a_{ij}\|_\infty|\xi_n|^2 + \sum_{j=1}^{l}\sum_{i=1}^{N}(c_{ij}|\xi_n| + d_{ij}|\xi_n|^{1+\bar\gamma}) - \lambda\|b\|_{L^1}G(\xi_n) \\
&= |\xi_n|^2\Big[\frac{T}{2}\sum_{j=1}^{N}\sum_{i=1}^{N}\|a_{ij}\|_\infty + \sum_{j=1}^{l}\sum_{i=1}^{N}\Big(\frac{c_{ij}}{|\xi_n|}+\frac{d_{ij}}{|\xi_n|^{\bar\gamma}}\Big) - \lambda\|b\|_{L^1}\frac{G(\xi_n)}{|\xi_n|^2}\Big].
\end{aligned} \tag{3.18}
$$

By (3.17) and (3.18), we can conclude that $I_\lambda = \Phi - \lambda\Psi$ is unbounded from below. Applying (b) of Theorem 2.4, the proof of Theorem 3.3 is complete. $\square$

Example 3.4. Let $G \in C^1(R,R)$ be as

$$G(x)=\begin{cases} 0, & \text{if } \ x\le 0;\\ \frac{\frac12 e^{e^x}-e(x+1)}{16}, & \text{if } 0<x<e^2;\\ \frac{e^{e^x}-\frac12 e^{e^{e^2}}-ex-e}{16}, & \text{if } e^2\le x<e^6;\\ \frac{e^{(e^{e^2}+2)}-e}{4e^6}x^3-\frac{e^{(e^{e^2}+2)}-e}{4}e^{12}+\frac{e^{e^{e^6}}-\frac12 e^{e^{e^2}}-e^7-e}{16}, & \text{if } x\ge e^6. \end{cases}$$

Then for suitable $b\in L^1([0,1])\setminus\{0\}$ and for

$$\lambda\in\Big(\frac{M}{\|b\|_{L^1}\big(1-\frac{\mu}{M}\big)}\frac{|\bar\xi|^2}{G(\bar\xi)},\frac{1}{4\|b\|_{L^1}}\frac{1}{\max\big\{\frac1\mu\frac{\max_{|\xi|\le\rho_1}G(\xi)}{\rho_1^2},\frac1M\frac{\max_{|\xi|\le\rho_2}G(\xi)}{\rho_2^2}\big\}}\Big),$$

the problem

$$\begin{gathered} -\ddot u+A(t)u=\lambda b(t)\nabla G(u),\quad \text{a.e. } t\in[0,1],\\ \Delta(\dot u^i(t_j))=\dot u^i(t_j^+)-\dot u^i(t_j^-)=I_{ij}(u^i(t_j)),\quad i=1,\ j=1,2, \\ u(0)-u(1)=u'(0)-u'(1)=0, \end{gathered}\tag{3.19}$$

has at least two nontrivial solutions u_1,u_2 with $\|u\|_\infty\le 2e^6$, where $N=1$, $T=1$, $A(t)=1+\frac{\sin(2\pi t)}{20}$, $\mu=0.95$, $M=1.05$, $\rho_1=1$, $\rho_2=2e^6$, $\bar\xi=\frac{99e^6}{100}$, and

$$I_{ij}(y)=\begin{cases}\frac y4+1, i=1, j=1,\\ -\frac y8, i=1, j=2.\end{cases}\tag{3.20}$$

We easily have

$$\begin{gathered} \big(1+\frac{4M}{\mu}\big)\frac{\max_{|\xi|\le\rho_1}G(\xi)}{\rho_1^2}=\frac{103}{19}G(1)=\frac{103}{304}(e^e-2),\\ 4\frac{\max_{|\xi|\le\rho_2}G(\xi)}{\rho_2^2}=4\frac{G(e^6)}{e^{12}}=\frac{e^{e^{e^6}}-\frac12 e^{e^{e^2}}-e^7-e}{16e^{12}},\\ \frac{G(\bar\xi)}{|\bar\xi|^2}=\frac{e^{e^{\frac{99}{100}e^6}}-\frac12 e^{e^{e^2}}-ex-e}{16\times\frac{9801e^{12}}{10000}}=\frac{625}{9801e^{12}}(e^{e^{\frac{99}{100}e^6}}-\frac12 e^{e^{e^2}}-\frac23 e^7-e), \end{gathered}$$

then the assumptions of Theorem 3.1 are satisfied.

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Guanghui Zhou
School of Mathematical Sciences, Information College, Huaibei Normal University, Huaibei 235000, China
E-mail address: `163zgh@163.com`