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WELL-POSEDNESS OF NON-AUTONOMOUS DEGENERATE PARABOLIC EQUATIONS UNDER SINGULAR PERTURBATIONS

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Abstract. This article concerns the asymptotic behavior of the following non-autonomous degenerate parabolic equation with singular perturbations defined on a bounded domain in $\mathbb{R}^n$,

$$\frac{\partial u}{\partial t} + \lambda u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) - \varepsilon \operatorname{div}\Big(|\nabla \frac{\partial u}{\partial t}|^{p-2}\nabla \frac{\partial u}{\partial t}\Big) + f(x,t,u) = g(x,t),$$

where λ is a positive constant, $p > 2$ and $\varepsilon \in (0,1]$. The well-posedness and upper semicontinuity of pullback attractors are established for the problem without the uniqueness of solutions under singular perturbations.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with sufficiently regular boundary $\partial\Omega$. Consider the following non-autonomous degenerate parabolic equation under singular perturbations defined in Ω for $t > \tau$ with $\tau \in \mathbb{R}$,

$$\frac{\partial u}{\partial t} + \lambda u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) - \varepsilon \operatorname{div}\Big(|\nabla \frac{\partial u}{\partial t}|^{p-2}\nabla \frac{\partial u}{\partial t}\Big) + f(x,t,u) = g(x,t), \quad (1.1)$$

with boundary condition

$$u(x,t) = 0, \quad x \in \partial\Omega \text{ and } t > \tau, \quad (1.2)$$

and initial condition

$$u(x,\tau) = u_\tau(x), \quad x \in \Omega, \quad (1.3)$$

where $\lambda > 0$ and $p > 2$ are constants and $\varepsilon \in (0,1]$.

Nonclassical diffusion equations have been used to model physical phenomena, for instance non-Newtonian flows, soil mechanics, heat conduction, etc (see, e.g., [1, 12, 17]). In the case of $p = 2$, the upper semicontinuity of global attractors of (1.1)–(1.3) has been studied by several authors in [2, 3, 20, 21, 24] and the references therein as well as [5] for some interesting results on the attractors for delay systems. The stability result of pullback attractors for multi-valued processes was established in [23], and the upper semicontinuity of pullback attractors for nonclassical diffusion equations without the uniqueness of solutions under singular perturbations was addressed.

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Recently, the existence and upper semicontinuity of pullback attractors have been proved in [15] for multi-valued processes generated by non-autonomous lattice nonclassical diffusion delay systems, in particular, the operator $(-1)^p\triangle^p = (-1)^p\triangle\circ\cdots\circ\triangle$, p times, has been considered instead of $-\triangle$, where p is any positive integer and $\triangle$ denotes the discrete one-dimensional Laplace operator. For the continuous case, the p-Laplace operator was defined as

$$\begin{aligned}\Delta_p u &= \operatorname{div}(|\nabla u|^{p-2}\nabla u)\\ &= |\nabla u|^{p-4}\big\{|\nabla u|^2\triangle u + (p-2)\sum_{i,j=1}^n \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i\partial x_j}\big\}.\end{aligned}$$

The well-posedness and continuity of attractors for p-laplacian problems have been investigated in [14] when the diffusion increases to infinity. For the non-autonomous equation, the existence of uniform attractors and pullback attractors of non-autonomous degenerate parabolic equations have been proved in [7, 19]. The existence of random attractors for p-Laplace equations driven by deterministic and stochastic forcing was studied in [18], in addition, the upper semicontinuity of random attractors was presented as the intensity of noise approaches zero.

In this article, we assume that the nonlinearity $f \in C(\Omega\times\mathbb{R}\times\mathbb{R};\mathbb{R})$ and the external force g satisfy the following conditions:

(H1) the function $F(x,t,s) = \int_0^s f(x,t,\omega)d\omega$ satisfies

$$F(x,t,s) \geqslant \gamma_1|s|^q - \varphi_1(x), \tag{1.4}$$

$$|F_t'(x,t,s)| \leqslant \alpha_0 F(x,t,s) + \varphi_2(x,t), \tag{1.5}$$

where $\gamma_1 > 0$ and $q \geqslant 2$ are constants, α_0 is sufficiently small, the functions $\varphi_1 \in L^1(\Omega)$ and $\varphi_2 \in L^1_{\rm loc}(\mathbb{R};L^1(\Omega))$ satisfies

$$\int_{-\infty}^t\int_\Omega e^{\alpha r}|\varphi_2(x,r)|\,dx\,dt < \infty, \quad \forall t\in\mathbb{R};$$

(H2) there exist positive constants γ_2, γ_3 and functions $\varphi_3 \in L^1_{\rm loc}(\mathbb{R};L^1(\Omega))$, $\varphi_4 \in L^{q_1}(\Omega)$ such that

$$f(x,t,s)s \geqslant \gamma_2 F(x,t,s) - \varphi_3(x,t), \tag{1.6}$$

$$|f(x,t,s)| \leqslant \gamma_3|s|^{q-1} + \varphi_4(x), \tag{1.7}$$

$$\int_{-\infty}^t\int_\Omega e^{\alpha r}|\varphi_3(x,r)|\,dx\,dt < \infty, \quad \forall t\in\mathbb{R},$$

where $\frac{1}{q}+\frac{1}{q_1} = 1$;

(H3) the external force $g \in L^2_{\rm loc}(\mathbb{R};L^2(\Omega))$ satisfies

$$\int_{-\infty}^t\int_\Omega e^{\alpha r}|g(x,r)|^2\,dx\,dt < \infty, \quad \forall t\in\mathbb{R}, \tag{1.8}$$

where α is a fixed number given in Lemma 3.2.

The main goal of this paper is to establish the well-posedness and upper semicontinuity of pullback attractors for (1.1)–(1.3) under singular perturbations. Because of the lack of the uniqueness of solutions, in order to obtain the pullback attractor we use the general theory of attractors for multi-valued processes developed in [4, 22]. Comparing with the case of $p = 2$ the main new difficulty which appears is to deal with the forth term in (1.1).

This article is organized as follows. In Section 2, we recall basic concepts and some necessary results concerning multi-valued processes and pullback attractors. Section 3 is devoted to the asymptotic behavior of (1.1)–(1.3). The well-posedness and upper semicontinuity of pullback attractors for (1.1)–(1.3) under singular perturbations are given in Section 4.

The following notation will be used throughout the paper. Let $H = L^2(\Omega)$ with the norm $\|\cdot\|_2$ and inner product $(\cdot,\cdot)$, and let $V = W_0^{1,p}(\Omega)$. The norm of $L^p(\Omega)$ is written as $\|\cdot\|_p$. The letter C is a generic positive constant which may change their values from line to line or even in the same line.

2. Preliminaries

Let X be a Banach space with norm $\|\cdot\|_X$, and let 2^X be the collection of all subsets of X. Denote by $H_X^*(\cdot,\cdot)$ the Hausdorff semidistance between two nonempty subsets of a Banach space $(X, \|\cdot\|_X)$, which are defined by

$$H_X^*(A,\, B) = \sup_{a\in A} \operatorname{dist}_X(a, B),$$

where $\operatorname{dist}_X(a,\, B) = \inf_{b\in B} \|a-b\|_X$. Finally, denote by $\mathcal{N}(A, r)$ the open neighborhood $\{y \in X : \operatorname{dist}_X(y, A) < r\}$ of radius $r > 0$ of a subset A of a Banach space X.

Definition 2.1. A family of mappings $U(t,\tau) : X \to 2^X, t \geqslant \tau, \tau \in \mathbb{R}$, is called to be a multi-valued process (MVP in short) if it satisfies:

1. $U(\tau,\tau)x = \{x\}$ for all $\tau \in \mathbb{R}, ; x \in X$;
2. $U(t,s)U(s,\tau)x = U(t,\tau)x$ for all $t \geqslant s \geqslant \tau,\ \tau \in \mathbb{R},\ x \in X$.

Let $\mathscr{D}$ be a nonempty class of parameterized sets $\mathcal{D} = \{D(t)\}_{t\in\mathbb{R}} \subset 2^X$.

Definition 2.2. A collection $\mathscr{D}$ of some families of nonempty subsets of X is said to be inclusion-closed if for each $\mathcal{D} \in \mathscr{D}$,

$$\{\tilde{D}(t) : \tilde{D}(t) \text{ is a nonempty subset of } D(t), \forall t \in \mathbb{R}\} \tag{2.1}$$

also belongs to $\mathscr{D}$, see, e.g., [9].

Definition 2.3. Let $\{U(t,\tau)\}$ be a multi-valued process on X.

(1) $\mathcal{Q} = \{Q(t)\}_{t\in\mathbb{R}} \in \mathscr{D}$ is called a $\mathscr{D}$-pullback absorbing set for $\{U(t,\tau)\}$ if for any $\mathcal{B} = \{B(t)\}_{t\in\mathbb{R}} \in \mathscr{D}$ and each $t \in \mathbb{R}$, there exists a $t_0 = t_0(\mathcal{B}, t) \in \mathbb{R}^+$ such that

$$U(t, t-s)B(t-s) \subset Q(t), \quad \forall s \geqslant t_0.$$

(2) $\{U(t,\tau)\}$ is said to be $\mathscr{D}$-pullback asymptotically upper-semicompact in X with respect to $\mathcal{B}$ if for any fixed $t \in \mathbb{R}$, any sequence $y_n \in U(t, t-s_n)x_n$ has a convergent subsequence in X whenever $s_n \to +\infty$ $(n \to \infty)$, $x_n \in B(t-s_n)$ with $\mathcal{B} = \{B(t)\}_{t\in\mathbb{R}} \in \mathscr{D}$.

Definition 2.4. A family of nonempty compact subsets $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}} \in \mathscr{D}$ is called to be a $\mathscr{D}$-pullback attractor for the multi-valued process $\{U(t,\tau)\}$, if it satisfies

(1) $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}}$ is invariant, i.e.,

$$U(t,\tau)A(\tau) = A(t), \quad \forall t \geqslant \tau,\ \tau \in \mathbb{R};$$

(2) $\mathcal{A}$ attracts every member of $\mathscr{D}$; that is, for every $\mathcal{B} = \{B(t)\}_{t\in\mathbb{R}} \in \mathscr{D}$ and any fixed $t \in \mathbb{R}$,

$$\lim_{s\to+\infty} H_X^*(U(t,t-s)B(t-s), A(t)) = 0.$$

Definition 2.5. A mapping $\psi : \mathbb{R} \to X$ is called a complete orbit of the multi-valued process $\{U(t,\tau)\}$ if for every $\tau \in \mathbb{R}$ and $t \geqslant \tau$, the following holds:

$$\psi(t) \in U(t,\tau)\psi(\tau).$$

If, in addition, there exists $\mathcal{D} = \{D(t)\}_{t\in\mathbb{R}} \in \mathscr{D}$ such that $\psi(t)$ belongs to $D(t)$ for every $t \in \mathbb{R}$, then ψ is called a $\mathscr{D}$-complete orbit of $\{U(t,\tau)\}$.

The following existence result of pullback attractors for multi-valued processes can be found in [4, 22].

Theorem 2.6. *Let $\mathscr{D}$ be a inclusion-closed collection of some families of nonempty subsets of X and $\{U(t,\tau)\}$ be a multi-valued process on X. Also U has closed values and let $U(t,\tau)x$ is norm-to-weak upper-semicontinuous in x for fixed $t \geqslant \tau$, $\tau \in \mathbb{R}$ (i.e., if $x_n \to x$ in X, then for any $y_n \in U(t,\tau)x_n$, there exists a $y \in U(t,\tau)x$ such that $y_n \rightharpoonup y$ (weak convergence)). Suppose that $\{U(t,\tau)\}$ is $\mathscr{D}$-pullback asymptotically upper-semicompact in X and $\{U(t,\tau)\}$ has a $\mathscr{D}$-pullback absorbing set $\mathcal{Q} = \{Q(t)\}_{t\in\mathbb{R}}$ in $\mathscr{D}$. Then, the $\mathscr{D}$-pullback attractor $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}}$ is unique and is given by, for each $t \in \mathbb{R}$,*

$$\begin{aligned} A(t) &= \cap_{T\geqslant 0}\overline{\cup_{s\geqslant T} U(t,t-s)Q(t-s)} \\ &= \{\psi(t) : \psi \text{ is a } \mathscr{D}\text{-complete orbit of } \{U(t,\tau)\}\}. \end{aligned} \tag{2.2}$$

Let X be a reflexive and separable Banach space, and let X_w be the space X endowed with the weak topology. Since bounded closed and convex subsets in the strong topology are compact in the weak topology (due to Mazur's lemma), the $\mathscr{D}$-pullback absorbing set $\mathcal{Q} = \{Q(t)\}_{t\in\mathbb{R}}$ obtained through ultimately boundedness is compact in X_w. Then in the same way as in Theorem 2.6 we have the following result needed to multi-valued processes without further compactness assumptions.

Theorem 2.7. *Let $\mathscr{D}$ be a inclusion-closed collection of some families of nonempty subsets of X and $\{U(t,\tau)\}$ be a multi-valued process on X. Also for any fixed $t \geqslant \tau$, $\tau \in \mathbb{R}$, U has weakly closed values and $U(t,\tau)$ is weakly upper-semicontinuous in bounded sets. Assume that $\{U(t,\tau)\}$ has a $\mathscr{D}$-pullback absorbing set $\mathcal{Q} = \{Q(t)\}_{t\in\mathbb{R}}$ in $\mathscr{D}$, and for all $t \in \mathbb{R}$, $Q(t)$ is a weakly closed nonempty subset of X. Then $\{U(t,\tau)\}$ has a unique $\mathscr{D}$-pullback attractor $\mathcal{A}_w = \{A_w(t)\}_{t\in\mathbb{R}}$ with weakly compact component sets determined by*

$$A_w(t) = \cap_{T\geqslant 0}\overline{\cup_{s\geqslant T} U(t,t-s)Q(t-s)}^{w} \quad \text{for each } t \in \mathbb{R}. \tag{2.3}$$

Note that the component subsets $A_w(t)$ of the pullback attractor $\mathcal{A}_w$ are weakly compact in X, hence they are closed and bounded in the strong norm topology.

3. Existence of solutions and their long time behavior

In this section, we firstly establish the existence of solutions for (1.1)–(1.3), and then give uniform estimates of solutions which are useful for obtaining the existence of pullback attractors.

Theorem 3.1. *Suppose* (H1), (H2) *hold and let* $g \in L^2_{\rm loc}(\mathbb{R}; L^2(\Omega))$. *Then for any fixed* $\varepsilon \in (0,1]$, *every* $\tau \in \mathbb{R}$ *and any* $u_\tau \in V \cap L^q(\Omega)$, *there exists a solution* $u(t)$ *to problem* (1.1)–(1.3), *and* $u(t)$ *satisfies*

$$u \in C([\tau, T]; V) \cap L^\infty(\tau, T; L^q(\Omega)), \quad \forall T > \tau.$$

Proof. We divide the proof into two steps.

Step 1. Multiplying (1.1) by $u + \frac{\partial u}{\partial t}$ and then integrating on Ω, we obtain

$$\begin{aligned} &\frac{d}{dt}\Big((\frac{1}{2}+\frac{\lambda}{2})\|u\|_2^2 + \frac{1}{p}\|\nabla u\|_p^p\Big) + \|\frac{\partial u}{\partial t}\|_2^2 + \lambda\|u\|_2^2 + \|\nabla u\|_p^p + \varepsilon\|\nabla\frac{\partial u}{\partial t}\|_p^p \\ &+ \varepsilon\int_\Omega |\nabla\frac{\partial u}{\partial t}|^{p-2}\nabla\frac{\partial u}{\partial t}\nabla u dx + \int_\Omega f(x,t,u)(u+\frac{\partial u}{\partial t})dx \\ &= \int_\Omega g(x,t)(u+\frac{\partial u}{\partial t})dx. \end{aligned} \tag{3.1}$$

It follows from (1.5) and (1.6) that

$$\begin{aligned} &\int_\Omega f(x,t,u)(u+\frac{\partial u}{\partial t})dx \\ &\geqslant \frac{d}{dt}\int_\Omega F(x,t,u)dx + (\gamma_2-\alpha_0)\int_\Omega F(x,t,u)dx - \|\varphi_2(t)\|_1 - \|\varphi_3(t)\|_1. \end{aligned} \tag{3.2}$$

By (3.2) and Young's inequality, we deduce from (3.1) that

$$\begin{aligned} &\frac{d}{dt}((\frac{1}{2}+\frac{\lambda}{2})\|u\|_2^2 + \frac{1}{p}\|\nabla u\|_p^p + \int_\Omega F(x,t,u)dx) + \frac{\lambda}{2}\|u\|_2^2 \\ &+ (1-\frac{\varepsilon}{2})\|\nabla u\|_p^p + \frac{\varepsilon}{2}\|\nabla\frac{\partial u}{\partial t}\|_p^p + (\gamma_2-\alpha_0)\int_\Omega F(x,t,u)dx + \frac{1}{2}\|\frac{\partial u}{\partial t}\|_2^2 \\ &\leqslant C\|g(t)\|_2^2 + \|\varphi_2(t)\|_1 + \|\varphi_3(t)\|_1. \end{aligned} \tag{3.3}$$

We choose α_0 sufficiently small such that $\alpha_0 < \gamma_2$. Using (1.4), we have

$$\begin{aligned} &\frac{d}{dt}\Big((\frac{1}{2}+\frac{\lambda}{2})\|u\|_2^2 + \frac{1}{p}\|\nabla u\|_p^p + \int_\Omega F(x,t,u)dx\Big) + \frac{\lambda}{2}\|u\|_2^2 \\ &+ (1-\frac{\varepsilon}{2})\|\nabla u\|_p^p + \frac{1}{2}\|\frac{\partial u}{\partial t}\|_2^2 + \gamma_1(\gamma_2-\alpha_0)\|u\|_q^q + \frac{\varepsilon}{2}\|\nabla\frac{\partial u}{\partial t}\|_p^p \\ &\leqslant (\gamma_2-\alpha_0)\|\varphi_1\|_1 + \|\varphi_2(t)\|_1 + \|\varphi_3(t)\|_1 + C\|g(t)\|_2^2. \end{aligned} \tag{3.4}$$

By (1.4), (1.7) and Young' inequality, it yields

$$\int_\Omega F(x,t,u(t))dx \geqslant \gamma_1\|u(t)\|_q^q - \|\varphi_1\|_1, \tag{3.5}$$

and

$$\begin{aligned} \int_\Omega F(x,\tau,u(\tau))dx &= \int_\Omega\int_0^{u(\tau)} f(x,t,\omega)d\omega dx \\ &\leqslant \int_\Omega\int_0^{u(\tau)} (\gamma_3|\omega|^{q-1} + \varphi_4(x))d\omega dx \\ &\leqslant C\|u(\tau)\|_q^q + C\|\varphi_4\|_{q_1}^{q_1}. \end{aligned} \tag{3.6}$$

Then, integrating (3.4) from τ to t, in view of (3.5) and (3.6), we obtain

$$\begin{aligned}
&(\frac{1}{2}+\frac{\lambda}{2})\|u(t)\|_2^2+\frac{1}{p}\|\nabla u(t)\|_p^p+\gamma_1\|u(t)\|_q^q+\frac{\lambda}{2}\int_\tau^t\|u(r)\|_2^2dr\\
&+\frac{1}{2}\int_\tau^t\|\frac{\partial u(r)}{\partial r}\|_2^2dr+(1-\frac{\varepsilon}{2})\int_\tau^t\|\nabla u(r)\|_p^pdr\\
&+\gamma_1(\gamma_2-\alpha_0)\int_\tau^t\|u(r)\|_q^qdr+\frac{\varepsilon}{2}\int_\tau^t\|\nabla\frac{\partial u(r)}{\partial r}\|_p^pdr\\
&\leqslant(\frac{1}{2}+\frac{\lambda}{2})\|u(\tau)\|_2^2+\frac{1}{p}\|\nabla u(\tau)\|_p^p+C\|u(\tau)\|_q^q+\int_\tau^t\|\varphi_2(r)\|_1dr\\
&\quad+C\|\varphi_4\|_{q_1}^{q_1}+\|\varphi_1\|_1+(\gamma_2-\alpha_0)\|\varphi_1\|_1(t-\tau)+\int_\tau^t\|\varphi_3(r)\|_1dr\\
&\quad+C\int_\tau^t\|g(r)\|_2^2dr.
\end{aligned}\tag{3.7}$$

Step 2. Let $A:V\to V^*$ be the operator defined by

$$(A(u_1),u_2)_{(V^*,V)}=\int_\Omega|\nabla u_1|^{p-2}\nabla u_1\cdot\nabla u_2dx,\quad\text{for all } u_1,u_2\in V,\tag{3.8}$$

where $(\cdot,\cdot)_{(V^*,V)}$ is the duality pairing of V^* and V. Note that A is a monotone operator as in [13]. Let $\{e_j\}_{j=1}^\infty\subseteq V\cap L^q(\Omega)$ be an orthonormal basis of H such that the span $\{e_j:j\in\mathbb{N}\}$ is dense in $V\cap L^q(\Omega)$. Given $n\in\mathbb{N}$, let X_n be the space spanned by $\{e_j:j=1,\dots,n\}$ and $P_n:H\to X_n$ be the projection given by

$$P_nu=\sum_{j=1}^n(u,e_j)e_j,\quad\forall u\in H.$$

Note that P_n can be extended to V^* and $(L^q(\Omega))^*$ by

$$P_n\phi=\sum_{j=1}^n(\phi(e_j))e_j,\quad\text{for }\phi\in V^*\text{or }\phi\in(L^q(\Omega))^*.$$

Consider the following system for $u^n\in X_n$ defined for $t>\tau$:

$$\frac{du^n}{dt}+\lambda u^n+P_nA(u^n)-\varepsilon P_nA(\frac{du^n}{dt})+P_nf(\cdot,t,u^n)=P_ng(\cdot,t),\tag{3.9}$$

with initial condition

$$u^n(\tau)=P_nu_\tau.\tag{3.10}$$

Then it follows from (3.7) that for any $T>\tau$,

$$\{u^n\}_{n=1}^\infty\text{ is bounded in }L^\infty(\tau,T;V)\cap L^\infty(\tau,T;L^q(\Omega)),\tag{3.11}$$

$$\{\frac{du^n}{dt}\}_{n=1}^\infty\quad\text{is bounded in }L^p(\tau,T;V).\tag{3.12}$$

Analogous to the proof of [8, Theorem 3.1, Section XV.3] and the argument in [16, Section IV4.4], by a standard argument we obtain that for any fixed $\varepsilon\in(0,1]$, every $\tau\in\mathbb{R}$ and any $u_\tau\in V\cap L^q(\Omega)$, system (1.1)–(1.3) has a solution $u\in C([\tau,T];V)\cap L^\infty(\tau,T;L^q(\Omega))$ for any $T>\tau$. $\square$

Based on Theorem 3.1, we can define a family of multi-valued mappings $U^\varepsilon(t,\tau):V\cap L^q(\Omega)\to V\cap L^q(\Omega)$ for each $\varepsilon>0$ by setting

$$U^\varepsilon(t,\tau)u_\tau=\big\{u(t):u(\cdot)\text{ is a solution of (1.1)–(1.3) with }u_\tau\in V\cap L^q(\Omega)\big\}.$$

Then we can verify that $\{U^\varepsilon(t,\tau)\}$ be a multi-valued process on $V\cap L^q(\Omega)$.

Let S be a nonempty bounded subset of the Banach space X, and let $\|S\|_X=\sup_{u\in S}\|u\|_X$. We consider a family $\mathcal{D}=\{D(t)\}_{t\in\mathbb{R}}$ of bounded nonempty subsets of $V\cap L^q(\Omega)$ such that for every $t\in\mathbb{R}$,

$$\lim_{s\to-\infty}e^{\alpha s}(\|D(t+s)\|_H^2+\|D(t+s)\|_V^p+\|D(t+s)\|_{L^q(\Omega)}^q)=0, \tag{3.13}$$

where $\alpha>0$ will be given in the proof of Lemma 3.2. In the sequel, we will use $\mathscr{D}$ to denote the collection of all families with property (3.13):

$$\mathscr{D}=\{\mathcal{D}=\{D(t)\}_{t\in\mathbb{R}}:\mathcal{D}\text{ satisfies (3.13)}\}.$$

It is obvious that $\mathscr{D}$ is inclusion-closed.

To consider the asymptotic behavior of problem (1.1)–(1.3), we need the following uniform estimates of solutions.

Lemma 3.2. *Suppose* (H1)–(H3) *hold. Then the multi-valued process* $\{U^\varepsilon(t,\tau)\}$ *corresponding to problem* (1.1)–(1.3) *possesses a closed uniformly (with respect to* $\varepsilon\in(0,1]$*)* $\mathscr{D}$*-pullback absorbing set* $\mathcal{Q}=\{Q(t)\}_{t\in\mathbb{R}}$ *in* $\mathscr{D}$*, i.e., for each* $t\in\mathbb{R}$ *and any* $\mathcal{B}=\{B(t)\}_{t\in\mathbb{R}}\in\mathscr{D}$*, there exists* $T=T(\mathcal{B},t)>0$ *which is independent of* ε *such that for all* $\varepsilon\in(0,1]$*,*

$$U^\varepsilon(t,t-s)B(t-s)\subseteq Q(t),\quad\forall s\geqslant T.$$

Proof. We choose α and α_0 sufficiently small, such that

$$\begin{aligned}&\frac{\lambda}{2}\|u\|_2^2+(1-\frac{\varepsilon}{2})\|\nabla u\|_p^p+(\gamma_2-\alpha_0)\int_\Omega F(x,t,u)dx\\&\geqslant\alpha\Big((\frac12+\frac{\lambda}{2})\|u\|_2^2+\frac1p\|\nabla u\|_p^p+\int_\Omega F(x,t,u)dx\Big).\end{aligned}$$

Then it follows from (3.3) that

$$\begin{aligned}&\frac{d}{dt}\Big((\frac12+\frac{\lambda}{2})\|u\|_2^2+\frac1p\|\nabla u\|_p^p+\int_\Omega F(x,t,u)dx\Big)+\frac12\|\frac{\partial u}{\partial t}\|_2^2\\&+\alpha\Big((\frac12+\frac{\lambda}{2})\|u\|_2^2+\frac1p\|\nabla u\|_p^p+\int_\Omega F(x,t,u)dx\Big)+\frac{\varepsilon}{2}\|\nabla\frac{\partial u}{\partial t}\|_p^p\\&\leqslant C\|g(t)\|_2^2+\|\varphi_2(t)\|_1+\|\varphi_3(t)\|_1.\end{aligned}\tag{3.14}$$

Using Gronwall's lemma, we deduce that

$$
\begin{aligned}
&(\frac{1}{2}+\frac{\lambda}{2})\|u(t)\|_2^2+\frac{1}{p}\|\nabla u(t)\|_p^p+\int_\Omega F(x,t,u(t))dx \\
&+\frac{1}{2}e^{-\alpha t}\int_{t-s}^t e^{\alpha r}\|\frac{\partial u(r)}{\partial r}\|_2^2dr+\frac{\varepsilon}{2}e^{-\alpha t}\int_{t-s}^t e^{\alpha r}\|\nabla\frac{\partial u(r)}{\partial r}\|_p^pdr \\
&\leqslant(\frac{1}{2}+\frac{\lambda}{2})e^{-\alpha s}\|u(t-s)\|_2^2+\frac{1}{p}e^{-\alpha s}\|\nabla u(t-s)\|_p^p \\
&\quad+e^{-\alpha s}\int_\Omega F(x,t-s,u(t-s))dx+Ce^{-\alpha t}\int_{t-s}^t e^{\alpha r}\|g(r)\|_2^2dr \\
&\quad+e^{-\alpha t}\int_{t-s}^t e^{\alpha r}(\|\varphi_2(r)\|_1+\|\varphi_3(r)\|_1)dr.
\end{aligned}
\tag{3.15}
$$

By a similar arguments as in (3.5) and (3.6), we have

$$
\begin{aligned}
&(\frac{1}{2}+\frac{\lambda}{2})\|u(t)\|_2^2+\frac{1}{p}\|\nabla u(t)\|_p^p+\gamma_1\|u(t)\|_q^q \\
&+\frac{1}{2}e^{-\alpha t}\int_{t-s}^t e^{\alpha r}\|\frac{\partial u(r)}{\partial r}\|_2^2dr+\frac{\varepsilon}{2}e^{-\alpha t}\int_{t-s}^t e^{\alpha r}\|\nabla\frac{\partial u(r)}{\partial r}\|_p^pdr \\
&\leqslant Ce^{-\alpha s}(\|u(t-s)\|_2^2+\|\nabla u(t-s)\|_p^p+\|u(t-s)\|_q^q)+Ce^{-\alpha s}\|\varphi_4\|_{q_1}^{q_1} \\
&\quad+C\|\varphi_1\|_1+Ce^{-\alpha t}\int_{t-s}^t e^{\alpha r}\|g(r)\|_2^2dr \\
&\quad+Ce^{-\alpha t}\int_{t-s}^t e^{\alpha r}(\|\varphi_2(r)\|_1+\|\varphi_3(r)\|_1)dr,
\end{aligned}
\tag{3.16}
$$

where C is independent of $\varepsilon\in(0,1]$. Denote by $R(t)$ the nonnegative number given for each $t\in\mathbb{R}$ by

$$
(R(t))^2=C+Ce^{-\alpha t}\int_{-\infty}^t e^{\alpha r}\|g(r)\|_2^2dr, \tag{3.17}
$$

and consider the family of closed bounded balls $\mathcal{Q}=\{Q(t)\}_{t\in\mathbb{R}}$ in $V\cap L^q(\Omega)$ defined by

$$
Q(t)=\{\psi\in V\cap L^q(\Omega):\|\psi\|_2^2+\|\nabla\psi\|_p^p+\|\psi\|_q^q\leqslant(R(t))^2\}. \tag{3.18}
$$

It is straightforward to check that $\mathcal{Q}\in\mathscr{D}$, and moreover, by (3.13) and (3.16), the family of $\mathcal{Q}$ is uniformly (with respect to $\varepsilon\in(0,1]$) $\mathscr{D}$-pullback absorbing for the family of multi-valued processes $\{U^\varepsilon(t,\tau)\}$, $\varepsilon\in(0,1]$ and thus the proof is complete. $\square$

We recall that X_w be the Banach space X endowed with the weak topology. We say that $u^n\to u\in C([\tau,T];X_w)$ in $C([\tau,T];X_w)$ if

$$
u^n(s^n)\to u(s)\quad\text{in } X_w \text{ for all } s^n\to s\in[\tau,T].
$$

Lemma 3.3. *Let $\{u_\tau^n\}_{n=1}^\infty$ be a bounded subset of $V\cap L^q(\Omega)$, $u_\tau\in V\cap L^q(\Omega)$ and let $u_\tau^n\to u_\tau$ weakly in $V\cap L^q(\Omega)$ as $n\to\infty$. Suppose* (H1)–(H3) *hold and fix $T>\tau$. Then for any fixed $\varepsilon\in(0,1]$ and any sequence $u^n(t)\in U^\varepsilon(t,\tau)u_\tau^n$, there exist $u(t)\in U^\varepsilon(t,\tau)u_\tau$ and a subsequence $\{u^{n_k}\}_{k=1}^\infty$ satisfying*

$$
\begin{gathered}
u^{n_k}\to u\quad\text{weakly in } C([\tau,T];V),\\
u^{n_k}\to u\quad\text{weak-star in } L^\infty(\tau,T;L^q(\Omega)).
\end{gathered}
$$

Proof. Inequality (3.7) implies that

$$\{u^n\}_{n=1}^{\infty} \text{ is bounded in } L^{\infty}(\tau,T;V)\cap L^{\infty}(\tau,T;L^q(\Omega)). \tag{3.19}$$

Hence, there exist a function $u\in L^{\infty}(\tau,T;V)\cap L^{\infty}(\tau,T;L^q(\Omega))$ and a subsequence $\{u^n\}_{n=1}^{\infty}$ (relabeled as $\{u^n\}_{n=1}^{\infty}$) such that

$$u^n\to u \quad \text{weak-star in } L^{\infty}(\tau,T;V)\cap L^{\infty}(\tau,T;L^q(\Omega)). \tag{3.20}$$

On the other hand, integrating (3.4) from s_1 to s_2 with $s_1,s_2\in[\tau,T]$ and $s_1<s_2$, then it follows from the similar argument of (3.7) that

$$\begin{aligned}
&\varepsilon\int_{s_1}^{s_2}\|\nabla\frac{\partial u^n(r)}{\partial r}\|_p^p dr\\
&\leqslant (1+\lambda)\|u^n(s_1)\|_2^2+\frac{2}{p}\|\nabla u^n(s_1)\|_p^p+C\|u^n(s_1)\|_q^q+2\int_{s_1}^{s_2}\|\varphi_2(r)\|_1dr\\
&\quad +C\|\varphi_4\|_{q_1}^{q_1}+2\|\varphi_1\|_1+2(\gamma_2-\alpha_0)\|\varphi_1\|_1(s_2-s_1)+2\int_{s_1}^{s_2}\|\varphi_3(r)\|_1dr\\
&\quad +C\int_{s_1}^{s_2}\|g(r)\|_2^2dr \qquad\qquad (3.21)\\
&\leqslant (1+\lambda)\|u^n(\tau)\|_2^2+\frac{2}{p}\|\nabla u^n(\tau)\|_p^p+C\|u^n(\tau)\|_q^q\\
&\quad +4\int_{\tau}^{T}\|\varphi_2(r)\|_1dr+C\|\varphi_4\|_{q_1}^{q_1}+4\|\varphi_1\|_1+4(\gamma_2-\alpha_0)\|\varphi_1\|_1(T-\tau)\\
&\quad +4\int_{\tau}^{T}\|\varphi_3(r)\|_1dr+C\int_{\tau}^{T}\|g(r)\|_2^2dr,
\end{aligned}$$

and thus by Hölder's inequality, we have

$$\begin{aligned}
\|\nabla u^n(s_2)-\nabla u^n(s_1)\|_p^p &\leqslant \int_{\Omega}(s_2-s_1)^{\frac{p}{p_1}}\Big(\int_{s_1}^{s_2}|\nabla\frac{\partial u^n(r)}{\partial r}|^p dr\Big)dx\\
&=(s_2-s_1)^{p-1}\int_{s_1}^{s_2}\|\nabla\frac{\partial u^n(r)}{\partial r}\|_p^p dr \\
&\leqslant \frac{C}{\varepsilon}(s_2-s_1)^{p-1},
\end{aligned}\tag{3.22}$$

where $\frac{1}{p}+\frac{1}{p_1}=1$. From (3.19) we deduce that for any $t\in[\tau,T]$, the sequence $\{u^n(t)\}_{n=1}^{\infty}$ is relatively weakly compact in $V\cap L^q(\Omega)$. Arguing as in the proof of [6, Theorem 4], by the diagonal method and (3.22) we obtain the existence of a continuous function $v(\cdot)$ and a subsequence of $\{u^n\}_{n=1}^{\infty}$ (denoted again $\{u^n\}_{n=1}^{\infty}$) such that

$$u^n\to v \quad \text{weakly in } C([\tau,T];V). \tag{3.23}$$

By the similar argument of the existence of solutions in Theorem 3.1, in view of (3.20) and (3.23), we conclude that $u=v$ is a solution of (1.1)–(1.3) and $u(\tau)=v(\tau)=u_{\tau}$, which completes the proof. □

Lemma 3.3 implies that for any fixed $t\geqslant\tau$, $\tau\in\mathbb{R}$, U has weakly closed values and $U(t,\tau)$ is weakly upper-semicontinuous in bounded sets. Thanks to Theorem 2.7 and Lemma 3.2, we obtain the following result.

Theorem 3.4. *Suppose* (H1)–(H3) *hold. Then for any $\varepsilon \in (0,1]$, the multi-valued process $\{U^\varepsilon(t,\tau)\}$ associated to problem* (1.1)–(1.3) *possesses a unique $\mathscr{D}$-pullback attractor $\mathcal{A}_w^\varepsilon = \{A_w^\varepsilon(t)\}_{t\in\mathbb{R}}$, which is invariant and pullback attracts every member of $\mathscr{D}$ in the weak topology of $V \cap L^q(\Omega)$, and its component sets are weakly compact in $V \cap L^q(\Omega)$.*

Hence, the component sets are compact in the topology of H, and $\mathcal{A}_w^\varepsilon$ pullback attracts every member of $\mathscr{D}$ in the topology of H.

4. Limit problem and convergence properties

In this section, we study the asymptotic dynamic of the problem (1.1)–(1.3) as $\varepsilon \to 0$. When $\varepsilon = 0$, we need to consider the following system defined in Ω for $t > \tau$ with $\tau \in \mathbb{R}$,

$$\frac{\partial u}{\partial t} + \lambda u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(x,t,u) = g(x,t), \tag{4.1}$$

with boundary condition

$$u(x,t) = 0, \quad x \in \partial\Omega \text{ and } t > \tau, \tag{4.2}$$

and initial condition

$$u(x,\tau) = u_\tau(x), \quad x \in \Omega. \tag{4.3}$$

Analogous to the arguments in [19] and Theorem 3.4, we obtain the existence of solutions and pullback attractors for (4.1)–(4.3).

Theorem 4.1. *Suppose* (H1)–(H3) *hold and let $g \in L^2_{\rm loc}(\mathbb{R}; L^2(\Omega))$. Then for every $\tau \in \mathbb{R}$ and any $u_\tau \in H$, there exists a solution $u(t)$ to problem* (4.1)–(4.3)*, and $u(t)$ satisfies*

$$u \in C([\tau,T]; H) \cap L^p(\tau,T;V) \cap L^q(\tau,T;L^q(\Omega)), \quad \forall T > \tau.$$

We consider a family $\mathcal{D} = \{D(t)\}_{t\in\mathbb{R}}$ of bounded nonempty subsets of H such that for every $t \in \mathbb{R}$,

$$\lim_{s\to-\infty} e^{\alpha s}\|D(t+s)\|_H^2 = 0, \tag{4.4}$$

and we will use $\mathscr{D}_H$ to denote the collection of all families with property (4.4):

$$\mathscr{D}_H = \{\mathcal{D} = \{D(t)\}_{t\in\mathbb{R}} : \mathcal{D} \text{ satisfies } (4.4)\}.$$

It is clear that $\mathscr{D}_H$ is inclusion-closed.

Theorem 4.2. *Suppose* (H1)–(H3) *hold. Then*

(1) *there exists a unique $\mathscr{D}_H$-pullback attractor $\mathcal{A}^0 = \{A^0(t)\}_{t\in\mathbb{R}}$ for the multi-valued process $\{U^0(t,\tau)\}$ on H generated by problem* (4.1)–(4.3)*;*
(2) *the multi-valued process $\{U^0(t,\tau)\}$ possesses pullback attractors $\mathcal{A}^0_{V,w} = \{A^0_{V,w}(t)\}_{t\in\mathbb{R}}$ and $\mathcal{A}^0_{L^q,w} = \{A^0_{L^q,w}(t)\}_{t\in\mathbb{R}}$ in the weak topology, their component sets are weakly compact in V and $L^q(\Omega)$ and hence closed and bounded in the topology of V and $L^q(\Omega)$, $\mathcal{A}^0_{V,w}$ and $\mathcal{A}^0_{L^q,w}$ pullback attract every member of $\mathcal{D}_H$ in the weak topology of V and $L^q(\Omega)$, respectively.*

Now we present the equi-continuity of solutions of problem (1.1)–(1.3), which will be used in the proof of Theorem 4.4.

Lemma 4.3. *Suppose* (H1)–(H3) *hold. Then for any fixed* $T > \tau$*, every* $u_\tau \in V \cap L^q(\Omega)$ *and any* $s_1, s_2 \in [\tau, T]$ *with* $s_1 < s_2$*, any solution* u *of* (1.1)–(1.3) *satisfies*

$$\|u(s_2) - u(s_1)\|_2 \leqslant C(s_2 - s_1)^{1/2},$$

where C *is independent of* $\varepsilon \in (0, 1]$.

Proof. Integrating (3.4) from s_1 to s_2 with $s_1, s_2 \in [\tau, T]$ and $s_1 < s_2$, then it follows from the similar argument of (3.7) that

$$\begin{aligned}
&\int_{s_1}^{s_2} \|\frac{\partial u(r)}{\partial r}\|_2^2 dr \\
&\leqslant (1+\lambda)\|u(s_1)\|_2^2 + \frac{2}{p}\|\nabla u(s_1)\|_p^p + C\|u(s_1)\|_q^q + 2\int_{s_1}^{s_2} \|\varphi_2(r)\|_1 dr \\
&+ C\|\varphi_4\|_{q_1}^{q_1} + 2\|\varphi_1\|_1 + 2(\gamma_2 - \alpha_0)\|\varphi_1\|_1(s_2 - s_1) + 2\int_{s_1}^{s_2} \|\varphi_3(r)\|_1 dr \\
&+ C\int_{s_1}^{s_2} \|g(r)\|_2^2 dr \\
&\leqslant (1+\lambda)\|u(\tau)\|_2^2 + \frac{2}{p}\|\nabla u(\tau)\|_p^p + C\|u(\tau)\|_q^q + 4\int_{\tau}^{T} \|\varphi_2(r)\|_1 dr \\
&\quad + C\|\varphi_4\|_{q_1}^{q_1} + 4\|\varphi_1\|_1 + 4(\gamma_2 - \alpha_0)\|\varphi_1\|_1(T - \tau) \\
&\quad + 4\int_{\tau}^{T} \|\varphi_3(r)\|_1 dr + C\int_{\tau}^{T} \|g(r)\|_2^2 dr,
\end{aligned} \tag{4.5}$$

and by Hölder's inequality, we have

$$\begin{aligned}
\|u(s_2) - u(s_1)\|_2^2 &\leqslant \int_\Omega (s_2 - s_1)\Big(\int_{s_1}^{s_2} |\frac{\partial u(r)}{\partial r}|^2 dr\Big) dx \\
&= (s_2 - s_1)\int_{s_1}^{s_2} \|\frac{\partial u(r)}{\partial r}\|_2^2 dr.
\end{aligned} \tag{4.6}$$

Then the conclusion follows immediately from (4.5) and (4.6). □

Theorem 4.4. *Suppose* (H1)–(H3) *hold, let* $\{u_\tau^\varepsilon : \varepsilon \in (0, 1]\}$ *is a bounded subset of* $V \cap L^q(\Omega)$*,* $u_\tau^0 \in H$ *and let* $u_\tau^\varepsilon \to u_\tau^0$ *in the topology of* H *as* $\varepsilon \to 0$*. Then for any fixed* $T > \tau$ *and any sequence* u^ε *of* (1.1)–(1.3) *with initial data* u_τ^ε*, we can find a solution* u^0 *of* (4.1)–(4.3) *with initial data* u_τ^0 *and a subsequence of* $\{u^\varepsilon\}$ *which converges to* u^0 *in* $C([\tau, T]; H)$ *and weakly to* u^0 *in* $L^p(\tau, T; V) \cap L^q(\tau, T; L^q(\Omega))$.

Proof. We divide the proof into two steps.

Step 1. Let $\varepsilon_n \in (0, 1]$ be a sequence of positive numbers with $\varepsilon_n \to 0$ $(n \to \infty)$, and let u^{ε_n} be the solution of (1.1)–(1.3) with $u^{\varepsilon_n}(\tau) = u_\tau^{\varepsilon_n}$. It follows from (3.7) that for any $t \in [\tau, T]$,

$$\{u^{\varepsilon_n}(t)\}_{n=1}^\infty \text{ is bounded in } V \cap L^q(\Omega), \tag{4.7}$$

$$\{u^{\varepsilon_n}\}_{n=1}^\infty \text{ is bounded in } L^\infty(\tau, T; H) \cap L^\infty(\tau, T; V) \cap L^\infty(\tau, T; L^q(\Omega)), \tag{4.8}$$

consequently

$$\begin{gathered}
\{u^{\varepsilon_n}\}_{n=1}^\infty \text{ is bounded in } L^p(\tau, T; V) \cap L^q(\tau, T; L^q(\Omega)), \\
\{\frac{\partial u^{\varepsilon_n}}{\partial t}\}_{n=1}^\infty \text{ is bounded in } L^2(\tau, T; H),
\end{gathered} \tag{4.9}$$

$$\{Au^{\varepsilon_n}\}_{n=1}^{\infty} \text{ is bounded in } L^{p_1}(\tau,T;V^*) \text{ with } \frac{1}{p}+\frac{1}{p_1}=1, \tag{4.10}$$

$$\{f(t,x,u^{\varepsilon_n})\}_{n=1}^{\infty} \text{ is bounded in } L^{q_1}(\tau,T;L^{q_1}(\Omega)) \text{ with } \frac{1}{q}+\frac{1}{q_1}=1, \tag{4.11}$$

$$\varepsilon_n^{1/p}\|\nabla\frac{\partial u^{\varepsilon_n}}{\partial t}\|_{L^p(\tau,T;L^p(\Omega))} \leqslant C_0, \tag{4.12}$$

for some constant $C_0>0$. Hence, there exist a function $u^0\in L^\infty(\tau,T;H)\cap L^p(\tau,T;V)\cap L^q(\tau,T;L^q(\Omega))$ and a subsequence of $\{u^{\varepsilon_n}\}_{n=1}^{\infty}$ (relabeled as $\{u^{\varepsilon_n}\}_{n=1}^{\infty}$) such that

$$u^{\varepsilon_n}\to u^0 \quad \text{weak-star in } L^\infty(\tau,T;H), \tag{4.13}$$

$$u^{\varepsilon_n}\to u^0 \quad \text{weakly in } L^p(\tau,T;V) \text{ and } L^q(\tau,T;L^q(\Omega)), \tag{4.14}$$

$$A(u^{\varepsilon_n})\to \chi_1 \quad \text{weakly in } L^{p_1}(\tau,T;V^*), \tag{4.15}$$

$$f(t,x,u^{\varepsilon_n})\to \chi_2 \quad \text{weakly in } L^{q_1}(\tau,T;L^{q_1}(\Omega)). \tag{4.16}$$

Thanks to Lemma 4.3, (4.8) and the compactness of embedding $V\hookrightarrow H$, by the Ascoli-Arzelà theorem we deduce that there exists a subsequence of $\{u^{\varepsilon_n}\}_{n=1}^{\infty}$ (denoted again $\{u^{\varepsilon_n}\}_{n=1}^{\infty}$) such that

$$u^{\varepsilon_n}\to u^0 \quad \text{strongly in } C([\tau,T];H). \tag{4.17}$$

Step 2. It remains to show that u^0 is a solution of (4.1)–(4.3) with $u^0(\tau)=u^0_\tau$. Noticing that u^{ε_n} is a solution of (1.1)–(1.3) with $u^{\varepsilon_n}(\tau)=u^{\varepsilon_n}_\tau$, i.e., u^{ε_n} satisfies

$$\begin{aligned} &\frac{\partial u^{\varepsilon_n}}{\partial t}+\lambda u^{\varepsilon_n}-\operatorname{div}(|\nabla u^{\varepsilon_n}|^{p-2}\nabla u^{\varepsilon_n}) \\ &-\varepsilon_n\operatorname{div}(|\nabla\frac{\partial u^{\varepsilon_n}}{\partial t}|^{p-2}\nabla\frac{\partial u^{\varepsilon_n}}{\partial t})+f(x,t,u^{\varepsilon_n})=g(x,t), \end{aligned} \tag{4.18}$$

from (4.12) and Hölder's inequality, we obtain that for any $\xi\in V\cap L^q(\Omega)$,

$$\begin{aligned} &-\varepsilon_n\int_\tau^T\Big(\operatorname{div}(|\nabla\frac{\partial u^{\varepsilon_n}(t)}{\partial t}|^{p-2}\nabla\frac{\partial u^{\varepsilon_n}(t)}{\partial t}),\xi\Big)_{(V^*,V)}dt \\ &\leqslant \varepsilon_n\|\nabla\frac{\partial u^{\varepsilon_n}(t)}{\partial t}\|^{p-1}_{L^p(\tau,T;L^p(\Omega))}\|\nabla\xi\|_{L^p(\tau,T;L^p(\Omega))} \\ &\leqslant \varepsilon_n^{1/p}C_0^{p-1}\|\nabla\xi\|_{L^p(\tau,T;L^p(\Omega))}\to 0 \quad \text{as } n\to\infty. \end{aligned} \tag{4.19}$$

By (4.12)–(4.17) and (4.19), one can show that

$$\frac{d}{dt}(u^0,\xi)+\lambda(u^0,\xi)+(\chi_1,\xi)_{(V^*,V)}+(\chi_2,\xi)_{(L^{q_1},L^q)}=(g(t),\xi). \tag{4.20}$$

Since $V\hookrightarrow H$ is compact, in view of (4.8) and (4.9), up to a subsequence we have

$$u^{\varepsilon_n}\to u^0 \quad \text{in } L^2(\tau,T;L^2(\Omega)), \tag{4.21}$$

which implies

$$u^{\varepsilon_n}\to u^0 \quad \text{for almost every } (t,x)\in[\tau,T]\times\Omega. \tag{4.22}$$

From this and the continuity of f, we obtain

$$f(x,t,u^{\varepsilon_n})\to f(x,t,u^0) \text{ for almost every } (t,x)\in[\tau,T]\times\Omega. \tag{4.23}$$

By (4.16) and (4.23), we have

$$\chi_2=f(x,t,u^0). \tag{4.24}$$

Finally, by the similar argument of (4.19), in view of (4.8) and (4.12), we find that

$$\begin{aligned} &- \varepsilon_n \int_\tau^T \Big(\operatorname{div}(|\nabla \frac{\partial u^{\varepsilon_n}(t)}{\partial t}|^{p-2} \nabla \frac{\partial u^{\varepsilon_n}(t)}{\partial t}), u^{\varepsilon_n}(t) \Big)_{(V^*,V)} dt \\ &\leqslant \varepsilon_n \|\nabla \frac{\partial u^{\varepsilon_n}(t)}{\partial t}\|^{p-1}_{L^p(\tau,T;L^p(\Omega))} \|\nabla u^{\varepsilon_n}(t)\|_{L^p(\tau,T;L^p(\Omega))} \\ &\leqslant \varepsilon_n^{1/p} C \to 0 \end{aligned} \tag{4.25}$$

as $n \to \infty$. By (4.13)–(4.17) and (4.25), we can argue as in [18] to show that

$$\chi_1 = A(u^0). \tag{4.26}$$

It follows from (4.20), (4.24) and (4.26) that u^0 is a solution of problem (4.1)–(4.3), and thus the proof of this theorem is complete. □

We obtain the upper semicontinuity of pullback attractors under singular perturbations.

Theorem 4.5. *Suppose* (H1)–(H3) *hold, and let* $\mathcal{A}_w^\varepsilon = \{A_w^\varepsilon(t)\}_{t\in\mathbb{R}}$ *and* $\mathcal{A}^0 = \{A^0(t)\}_{t\in\mathbb{R}}$ *be the pullback attractors for the multi-valued processes* $\{U^\varepsilon(t,\tau)\}$ *and* $\{U^0(t,\tau)\}$ *in* $\mathscr{D}$ *and* $\mathscr{D}_H$ *generated by* (1.1)–(1.3) *and* (4.1)–(4.3)*, respectively. Then for any* $\tau \in \mathbb{R}$*,*

$$\lim_{\varepsilon\to 0} H^*\big(A_w^\varepsilon(\tau), A^0(\tau)\big) = 0, \tag{4.27}$$

where $H^*(\cdot,\cdot)$ *is the Hausdorff semidistance between two nonempty subsets of* H*.*

Proof. We will use the argument of contradiction. Indeed, assume that (4.27) is not true, then there exist a $\tau \in \mathbb{R}$, a positive constant η, a sequence of positive numbers ε_n converging to zero, and a corresponding sequence $u_\tau^{\varepsilon_n} \in A^{\varepsilon_n}(\tau)$ such that

$$\operatorname{dist}_H(u_\tau^{\varepsilon_n}, A^0(\tau)) \geqslant \eta > 0, \quad \forall n \in \mathbb{N}. \tag{4.28}$$

Let u^{ε_n} be a solution of (4.1)–(4.3) with initial condition $u^{\varepsilon_n}(\tau) = u_\tau^{\varepsilon_n}$. It is clear that $u^{\varepsilon_n}(t)$ belongs to $A^{\varepsilon_n}(t)$ for all $t \geqslant \tau$. Note that $\mathcal{A}_w^\varepsilon = \{A_w^\varepsilon(t)\}_{t\in\mathbb{R}}$ is invariant, hence there exists $u_{\tau-1}^{\varepsilon_n} \in A^{\varepsilon_n}(\tau-1)$ such that $u_\tau^{\varepsilon_n} \in U^{\varepsilon_n}(\tau,\tau-1)u_{\tau-1}^{\varepsilon_n}$. If we now take $u^{\varepsilon_n}(s) \in U^{\varepsilon_n}(s,\tau-1)u_{\tau-1}^{\varepsilon_n}$ for $s \in [\tau-1,\tau]$, then we have $u^{\varepsilon_n}(s) \in A^{\varepsilon_n}(s)$ for all $s \geqslant \tau-1$. Applying the above procedure several times we can construct $u^{\varepsilon_n}(s) \in A^{\varepsilon_n}(s)$ for all $s \geqslant \tau - m$, $m \in \mathbb{N}$. Letting $m \to \infty$, we obtain a $\mathscr{D}$-complete orbit of $u^{\varepsilon_n}(s)$, $s \in \mathbb{R}$, of the multi-valued process $U^{\varepsilon_n}(t,\tau)$ such that $u^{\varepsilon_n}(s) \in A^{\varepsilon_n}(s)$ for all $s \in \mathbb{R}$.

For any $t \in \mathbb{R}$, since $\{u^{\varepsilon_n}(t)\}_{n=1}^\infty$ is a bounded subset of $V \cap L^q(\Omega)$, there exists a subsequence of $\{u^{\varepsilon_n}(t)\}_{n=1}^\infty$ (relabeled as $\{u^{\varepsilon_n}(t)\}_{n=1}^\infty$) such that $u^{\varepsilon_n}(t) \to u^0(t)$ in H as $n \to \infty$. Then, using Theorem 4.4 and the diagonal method, one can choose a subsequence of $\{u^{\varepsilon_n}(\cdot)\}$ and a $\mathscr{D}_H$-complete orbit u^0 of (4.1)–(4.3) such that

$$u^{\varepsilon_n}(t) \to u^0(t) \quad \text{in } C(J;H) \tag{4.29}$$

for any compact interval $J \subset \mathbb{R}$. Theorem 2.6 implies that $u^0(t) \in A^0(t)$ for all $t \in \mathbb{R}$, this and (4.29) lead to a contradiction with (4.28), hence the proof is completed. □

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