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WEAK SOLUTIONS FOR PARABOLIC EQUATIONS WITH $p(x)$-GROWTH

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Abstract. In this article we study nonlinear parabolic equations with $p(x)$-growth in the space $W^{1,x}L^{p(x)}(Q) \cap L^\infty(0,T;L^2(\Omega))$. By using the method of parabolic regularization, we prove the existence and uniqueness of weak solutions for the equation

$$\frac{\partial u}{\partial t} = \operatorname{div}(a(u)|\nabla u|^{p(x)-2}\nabla u) + f(x,t).$$

Also, we study the localization property of weak solutions for the above equation.

1. Introduction and statement of main results

Let $N \geq 2$ be an integer and Ω be a bounded simply connected domain in $\mathbb{R}^N$. Let Q be $\Omega \times (0,T)$ where $T > 0$ is given. We consider the parabolic initial boundary-value problem

$$\begin{gathered} \frac{\partial u}{\partial t} = \operatorname{div}(a(u)|\nabla u|^{p(x)-2}\nabla u) + f(x,t), \quad (x,t) \in Q, \\ u(x,t) = 0, \quad (x,t) \in \Gamma, \\ u(x,0) = u_0(x), \quad x \in \Omega. \end{gathered} \tag{1.1}$$

where Γ denotes the lateral boundary of the cylinder Q, and $a(u) = u^\sigma + d_0$ with σ and d_0 two positive constants to be defined later.

For the case p constant, there are many results about the existence, uniqueness and the qualitative properties of the solutions, we refer the reader to [6, 7, 19, 24].

In recent years, the research of variational problems with nonstandard growth conditions has been an interesting topic, see for examples [1, 2, 3, 4, 8, 9, 10, 13, 15, 16, 17, 20, 21, 23] and the references therein. In [4], the authors studied the nonlinear parabolic equations with nonstandard anisotropic growth conditions:

$$u_t - \sum_i \frac{d}{dx_i}[a_i(z,u)|D_i u|^{p_i(z)-2}D_i u + b_i(z,u)] + d(z,u) = 0 \tag{1.2}$$

where $z = (x,t)$. They proved the existence and uniqueness of weak solutions by applying Galerkin's method in the Orlicz-Sobolev spaces $W(Q)$ with the norm $\|u\|_{W(Q)} = \sum_i \|D_i u\|_{p_i(z),Q} + \|u\|_{2,Q}$. Note that the coefficient of nonlinearity in [4] is allowed to depend on x and t and is assumed to be the Caratheodory function,

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and so problem (1.2) is called the evolutional $p(x,t)$-Laplacian. In [13], the authors considered the quasilinear degenerate parabolic problem with nonstandard growth:

$$\begin{gathered} \frac{\partial u}{\partial t} = \operatorname{div}(a(u)|\nabla u|^{p(x,t)-2}\nabla u) + f(x,t), \quad (x,t) \in Q_T, \\ u(x,t) = 0, \quad (x,t) \in \Gamma_T, \\ u(x,0) = u_0(x), \quad x \in \Omega. \end{gathered} \tag{1.3}$$

and studied the existence, uniqueness and localization property of weak solutions for (1.3). It is worthy pointing out that they used the Banach spaces $L^{p(x,t)}(Q_T)$ and $W(Q_T)$ which appeared in [13] as solution space. Indeed, many authors dedicated to studying the variable exponent problems, in which $p(x,t)$ depends on x and t, see for instance [4, 13, 3, 20]. But for some important problems, the solution spaces only depending on variable x for parabolic equations are needed. Note that $p(x)$-growth problems can be regarded as a kind of problems with nonstandard growth, which appear in nonlinear elastic, electrorheological fluids and other physics phenomena. For a recent overview of variable exponent spaces with applications to nonlinear partial differential equations we refer to [16] and the references therein.

To illustrate the significance of variable exponent spaces independent of the time variable t, we would like to mention a paper [5], which has been an excellent reference as the applications of variable exponent spaces. More precisely, the authors in [5] studied the Dirichlet problem

$$\frac{\partial u}{\partial t} - \operatorname{div}(\phi_r(x,Du)) + \lambda(u-I) = 0, \quad (x,t) \in \Omega \times [0,T], \tag{1.4}$$

which is a model for image denoising, enhancement, and restoration, where $\lambda \geq 0$ is a constant,

$$\phi(x,r) = \begin{cases} \frac{1}{q(x)}|r|^{q(x)}, & |r| \leq \beta, \\ |r| - \frac{\beta q(x) - \beta^{q(x)}}{q(x)}, & |r| > \beta, \end{cases}$$

where $q(x)$ satisfies $1 \leq q(x) \leq 2$. They proved the existence and uniqueness of weak solutions and also discussed the behavior of weak solutions for (1.4) as $t \to \infty$. Notice that the direction and speed of diffusion at each location depend on the local behavior, hence $q(x)$ only depends on the location x in the image. Thanks to this fact, the authors gave the above model which can study the denoising, enhancement, and restoration for the image well. Based on the above reason, we thus seek for a kind of space in which the variable exponent only depend on x for problem (1.2). Considering that the space $W^{1,x}L^{p(x)}(Q)$, which is different from the space $W(Q_T)$ in [4, 13], can provide a suitable framework to discuss the similar physical problems in [5], which was introduced and discussed in [11, 18], so we take this space as our working space to discuss the problem (1.2), where $p(x)$ only depends on the space variable x, not on the time variable t.

In this article, we will the existence, uniqueness and localization property of solutions for (1.2) in the space $W^{1,x}L^{p(x)}(Q)$. Throughout this paper, unless special statement, we always suppose that the exponent $p(x)$ is continuous on $\overline{\Omega}$ with logarithmic module of continuity

$$1 < p^- = \inf_{x\in\Omega} p(x) \leq p(x) \leq \sup_{x\in\Omega} p(x) = p^+ < \infty. \tag{1.5}$$

$$\forall x \in \Omega,\ y \in \Omega, |x-y| < 1, \quad |p(x)-p(y)| \leq \omega(|x-y|), \tag{1.6}$$

where
$$\limsup_{\tau\to 0^+}\omega(\tau)\ln\frac{1}{\tau}=C<+\infty.$$
First we give the definition of (weak) solutions for problem (1.1).

Definition 1.1. A function $u(x,t)\in W^{1,x}L^{p(x)}(Q)\cap L^\infty(0,T;L^2(\Omega))$ is called a (weak) solution of (1.1) if
$$-\int_Q u\frac{\partial\varphi}{\partial t}\,dx\,dt+\int_\Omega u\varphi dx|_0^T+\int_Q(u^\sigma+d_0)|\nabla u|^{p(x)-2}\nabla u\nabla\varphi\,dx\,dt=\int_Q f(x,t)\varphi\,dx\,dt$$
for all $\varphi\in C^1(0,T;C_0^\infty(\Omega))$.

Now we are in a position to give results about the existence and uniqueness of solutions for problem (1.2).

Theorem 1.2. *Let $p(x)$ satisfy (1.5)–(1.6). If the following conditions hold*

(H1) $\max\{1,\frac{2N}{N+2}\}<p^-<N$, $2\le\sigma<\frac{2p^+}{p^+-1}$;

(H2) $u_0\ge 0$, $f\ge 0$, $\|u_0\|_{\infty,\Omega}+\int_0^T\|f(x,t)\|_{\infty,\Omega}dt:=K(T)<\infty$,

then (1.2) has at least one nontrivial weak solution in $W^{1,x}L^{p(x)}(Q)$.

Theorem 1.3. *Suppose that the conditions in Theorem 1.2 are fulfilled and $2<\sigma<\frac{2p^+}{p^+-1}$, $p^+\ge 2$. Then the nonnegative solution of (1.2) is unique within the class of all nonnegative weak solutions.*

Let us define
$$\operatorname{supp}\omega=\overline{\Big\{x\in G:\lim_{\rho\to 0}\frac{\operatorname{meas}(G\cap B_\rho(x))}{\operatorname{meas}(B_\rho(x))}>0\Big\}},$$
where $G=\{x\in\Omega:\omega>0\}$, $B_\rho(x)=\{y\in\Omega:|x-y|<\rho\}$. Hence we can present the localization property of solutions.

Theorem 1.4. *Assume that the hypotheses of Theorem 1.3 are satisfied and $2<\sigma<\frac{2(p^+-p^-)}{p^-(p^+-1)}$, $\operatorname{supp}u_0\in\Omega$. If u is a nonnegative solution of problem (1.2) and $f\equiv 0$, then $\operatorname{supp}u\subset\operatorname{supp}u_0$ a.e. in Q.*

This paper is organized as follows. In Section 2, we shall introduce the space $W^{m,x}L^{p(x)}(Q)$ and the necessary properties, which will be needed later. Section 3 and Section 4 are devoted to proving the existence and uniqueness of solutions for problem (1.2) respectively. In Section 5, we will discuss the localization property of solutions to problem (1.2).

2. Preliminaries

In this section we recall the basic knowledge of the general spaces $L^{p(x)}(\Omega)$, $W^{m,p(x)}(\Omega)$ and $W^{m,x}L^{p(x)}(Q)$ and the necessary results which will be useful in the sequel, we refer to [11, 18, 12, 14] for more details. Denote
$$E=\{\omega:\omega\text{ is a measurable function on }\Omega\},$$
where $\Omega\subset\mathbb{R}^N$ is an open subset.

Let $p(x):\Omega\to[1,\infty]$ be an element in E. Denote $\Omega_\infty=\{x\in\Omega:p(x)=\infty\}$. For $u\in E$, we define
$$\rho(u)=\int_{\Omega\setminus\Omega_\infty}|u(x)|^{p(x)}dx+\text{ess},\sup_{x\in\Omega_\infty}|u(x)|.$$

The space $L^{p(x)}(\Omega) = \{u \in E : \exists \lambda > 0, \rho(\lambda u) < \infty\}$ endowed with the norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\{\lambda > 0 : \rho(\frac{u}{\lambda}) \le 1\}.$$

We define the conjugate function $p'(x)$ of $p(x)$ by

$$p'(x) = \begin{cases} \infty, & \text{if } p(x) = 1; \\ 1, & \text{if } p(x) = \infty; \\ \frac{p(x)}{p(x)-1}, & \text{if } 1 < p(x) < \infty. \end{cases}$$

Lemma 2.1 ([11]). *(a) The dual space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, if $1 \le p(x) < \infty$.*
(b) The space $L^{p(x)}(\Omega)$ is reflexive if and only if (1.5) is satisfied.

Lemma 2.2 ([11]). *If $1 \le p(x) < \infty$, $C_0^\infty(\Omega)$ is dense in the space $L^{p(x)}(\Omega)$ and $L^{p(x)}(\Omega)$ is separable.*

Lemma 2.3 ([11]). *If $1 \le p(x) \le \infty$, for every $u(x) \in L^{p(x)}(\Omega)$ and $v(x) \in L^{p'(x)}(\Omega)$, we have*

$$\int_\Omega |u(x)v(x)|dx \le C\|u(x)\|_{L^{p(x)}(\Omega)}\|v(x)\|_{L^{p'(x)}(\Omega)},$$

where C is only dependent on $p(x)$ and Ω, not dependent on $u(x), v(x)$.

Lemma 2.4 ([11]). *Let $1 \le p(x) < \infty$. The following conclusions hold:*

(1) $\|u\|_{L^{p(x)}(\Omega)} < 1\ (= 1, > 1) \Leftrightarrow \rho(u) < 1 (= 1, > 1)$.
(2) *If $\|u\|_{L^{p(x)}(\Omega)} \ge 1$, then $\|u\|^{p^-}_{L^{p(x)}(\Omega)} \le \rho(u) \le \|u\|^{p^+}_{L^{p(x)}(\Omega)}$.*
(3) *If $\|u\|_{L^{p(x)}(\Omega)} \le 1$, then $\|u\|^{p^+}_{L^{p(x)}(\Omega)} \le \rho(u) \le \|u\|^{p^-}_{L^{p(x)}(\Omega)}$.*

Let $m > 0$ be an integer. For each $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, α_i are nonnegative integers and $|\alpha| = \Sigma_{i=1}^n \alpha_i$, and denote by D^α the distributional derivative of order α with respect to the variable x.

We now introduce the generalized Lebesgue-Sobolev space $W^{m,p(x)}(\Omega)$ which is defined as

$$W^{m,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \le m\}.$$

Here $W^{m,p(x)}(\Omega)$ is a Banach space endowed with the norm

$$\|u\| = \sum_{|\alpha| \le m} \|D^\alpha u\|_{L^{p(x)}(\Omega)}.$$

The space $W_0^{m,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{m,p(x)}(\Omega)$. The dual space $(W_0^{m,p(x)}(\Omega))^*$ is denoted by $W^{-m,p'(x)}(\Omega)$ equipped with the norm

$$\|f\|_{W^{-m,p'(x)}(\Omega)} = \inf \Sigma_{|\alpha| \le m}\|f_\alpha\|_{L^{p'(x)}(\Omega)},$$

where infimum is taken on all possible decompositions

$$f = \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha f_\alpha, \quad f_\alpha \in L^{p'(x)}(\Omega).$$

Lemma 2.5 ([11]). *(i) $W^{m,p(x)}(\Omega)$ and $W_0^{m,p(x)}(\Omega)$ are separable if $1 \le p(x) < \infty$.*
(ii) $W^{m,p(x)}(\Omega)$ and $W_0^{m,p(x)}(\Omega)$ are reflexive if (1.5) holds.

We define the space

$$W^{m,x}L^{p(x)}(Q) = \{u \in L^{p(x)}(Q) : D^\alpha u \in L^{p(x)}(Q), |\alpha| \le m\}.$$

It is easy to see that $W^{m,x}L^{p(x)}(Q)$ is a Banach space with the norm $\|u\| = \sum_{|\alpha|\le m} \|D^\alpha u\|_{L^{p(x)}(Q)}$, where $p(x)$ is independent of t, see [8] for further discussions. The space $W_0^{m,x}L^{p(x)}(Q)$ is defined as the closure of $C_0^\infty(Q)$ in $W^{m,x}L^{p(x)}(Q)$ and $W_0^{m,x}L^{p(x)}(Q) \hookrightarrow L^{p(x)}(Q)$ is continuous embedding. Let $\bar{M}$ be the number of multiindexes α which satisfies $0 \le |\alpha| \le m$, then the space $W_0^{m,x}L^{p(x)}(Q)$ can be considered as a close subspace of the product space $\Pi_{i=1}^{\bar{M}} L^{p(x)}(Q)$. So if $1 < p(x) < \infty$, $\Pi_{i=1}^{\bar{M}} L^{p(x)}(Q)$ is reflexive and further we can get that the space $W_0^{m,x}L^{p(x)}(Q)$ is reflexive. The dual space $(W_0^{m,x}L^{p(x)}(Q))^*$ is denoted by $W^{-m,x}L^{p'(x)}(Q)$ equipped with the norm

$$\|f\|_{W^{-m,x}L^{p'(x)}(Q)} = \sup_{\|u\|_{W_0^{m,x}L^{p(x)}(Q)}\le 1} |< f, u >| = \inf \sum_{|\alpha|\le m} \|f_\alpha\|_{L^{p'(x)}(Q)},$$

where the infimum is taken on all possible decompositions

$$f = \sum_{|\alpha|\le m} (-1)^{|\alpha|} D_x^\alpha f_\alpha, \quad f_\alpha \in L^{p'(x)}(Q).$$

In what follows, we denote $\|u(x,t)\|_{k,\Omega} = \big(\int_\Omega |u(x,t)|^k dx\big)^{1/k}$, $\|u(x,t)\|_{\infty,Q} = \sup_{(x,t)\in Q} |u(x,t)|$.

3. Existence of solutions

Let us consider the auxiliary parabolic problem

$$\begin{gathered}\frac{\partial u}{\partial t} = \operatorname{div}(a_{n,H}(u)|\nabla u|^{p(x)-2}\nabla u) + f(x,t), \quad (x,t) \in Q,\\ u(x,t) = 0, \quad (x,t) \in \Gamma,\\ u(x,0) = u_0(x), \quad x \in \Omega,\end{gathered} \tag{3.1}$$

here H stands for a positive parameter to be chosen later and notice that

$$0 < d_0 \le a_{n,H}(u) = \Big(\min(|u|^2, H^2) + \frac{1}{n^2}\Big)^{\sigma/2} + d_0 \le (H^2+1)^{\sigma/2} + d_0,$$

and $1 < n < \infty$

Since $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(x)}(\Omega)$, we may construct the sequence of approximate solutions $u_n(x,t) = \sum_{k=1}^n c_k^n(t)\varphi_k(x)$, and with similar arguments as in [4], we obtain that problem (3.1) has a weak solution $u_n(x,t) \in W_0^{1,x}L^{p(x)}(Q) \cap L^2(Q)$ satisfying the identity

$$\int_{t_1}^{t_2}\int_\Omega [u_{nt}\xi + a_{n,H}(u_n)|\nabla u_n|^{p(x)-2}\nabla u_n \nabla\xi - f(x,t)\xi]\,dx\,dt = 0, \tag{3.2}$$

where $t_1 < t_2 \in (0,T)$. To prove the main result, we need the following a priori estimates.

Lemma 3.1. *The solution of* (3.1) *satisfies the estimate*

$$\|u_n\|_{\infty,\Omega} \le \|u_0\|_{\infty,\Omega} + \int_0^T \|f(x,t)\|_{\infty,\Omega} dt = K(T) < \infty.$$

Proof. First, we introduce the function

$$u_{nH} = \begin{cases} H, & \text{if } u_n > H; \\ u_n, & \text{if } |u_n| \le H; \\ -H, & \text{if } u_n < -H. \end{cases}$$

We choose the function u_{nH}^{2k-1} as a test function in (3.2) with $k \in N$. In (3.2), let $t_2 = t + h$, $t_1 = t$, with $t, t + h \in (0, T)$. Then

$$\begin{aligned} &\int_t^{t+h} \int_\Omega \Big[(u_{nH})_t u_{nH}^{2k-1} + a_{n,H}(u_{nH})|\nabla u_{nH}|^{p(x)-2}\nabla u_{nH}\nabla u_{nH}^{2k-1} \\ &- f(x,t)u_{nH}^{2k-1}\Big]\,dx\,dt = 0, \end{aligned}$$

i.e.

$$\begin{aligned} &\frac{1}{2k}\int_t^{t+h} \frac{d}{dt}\Big(\int_\Omega u_{nH}^{2k} dx\Big) dt \\ &+ \int_t^{t+h} \int_\Omega (2k-1)a_{n,H}(u_{nH})u_{nH}^{2(k-1)}|\nabla u_{nH}|^{p(x)}\,dx\,dt \\ &= \int_t^{t+h} \int_\Omega f(x,t)u_{nH}^{2k-1}\,dx\,dt. \end{aligned} \tag{3.3}$$

Dividing by h, letting $h \to 0$, and applying Lebesgue's dominated convergence theorem, we have that for all $t \in (0, T)$,

$$\begin{aligned} &\frac{1}{2k}\frac{d}{dt}\int_\Omega u_{nH}^{2k} dx + \int_\Omega (2k-1)a_{n,H}(u_{nH})u_{nH}^{2(k-1)}|\nabla u_{nH}|^{p(x)} dx \\ &= \int_\Omega f(x,t)u_{nH}^{2k-1} dx. \end{aligned} \tag{3.4}$$

By Lemma 2.3, the right-hand side of the above equality can be rewritten as

$$\Big|\int_\Omega f(x,t)u_{nH}^{2k-1} dx\Big| \le \|u_{nH}\|_{2k,\Omega}^{2k-1}\|f\|_{2k,\Omega}, \quad k = 1, 2, 3, \dots,$$

whence

$$\begin{aligned} &\|u_{nH}\|_{2k,\Omega}^{2k-1}\frac{d}{dt}(\|u_{nH}\|_{2k,\Omega}) + (2k-1)\int_\Omega a_{n,H}(u_{nH})u_{nH}^{2(k-1)}|\nabla u_{nH}|^{p(x)} dx \\ &\le \|u_{nH}\|_{2k,\Omega}^{2k-1}\|f\|_{2k,\Omega}, \quad k = 1, 2, 3, \dots \end{aligned} \tag{3.5}$$

Integrating over $(0, t)$ for the above inequality for all t, we obtain

$$\|u_{nH}(\cdot,t)\|_{2k,\Omega} \le \|u_{nH}(\cdot,0)\|_{2k,\Omega} + \int_0^T \|f\|_{2k,\Omega} dt, \quad \forall k \in N.$$

Letting $k \to \infty$, one gets

$$\|u_{nH}(\cdot,t)\|_{\infty,\Omega} \le \|u_{nH}(\cdot,0)\|_{\infty,\Omega} + \int_0^T \|f\|_{\infty,\Omega} dt \le \|u_0\|_{\infty,\Omega} + \int_0^T \|f\|_{\infty,\Omega} dt.$$

If we choose $H > K(T)$, then $u_{nH}(\cdot,t) \le \sup|u_{nH}(\cdot,t)| \le K(T) < H$, and hence $u_{nH}(\cdot,t) = u_n(\cdot,t)$. $\square$

Remark 3.2. According to the above arguments, we obtain $u_{nH}(\cdot,t) = u_n(\cdot,t)$, and

$$\min\{u_n^2, H^2\} = u_n^2, \quad a_{n,H}(u_{nH}) = a_{n,H}(u_n) = \Big(\frac{1}{n^2} + u_n^2\Big)^{\sigma/2} + d_0.$$

Corollary 3.3. *If $u_0 \geq 0$ and $f \geq 0$, then the solution $u_n(x,t)$ is nonnegative in Q.*

Proof. Set $u_n^- = \min\{u_n, 0\}$, then we obtain $u_n^-(\cdot,0) = 0$. By Remark 3.2, and let $k = 1$ in (3.4), we obtain

$$\frac{1}{2}\frac{d}{dt}(\|u_n^-(x,t)\|_{2,\Omega}^2) + \int_\Omega a_{n,H}(u_n^-)|\nabla u_n^-|^{p(x)}dx \leq 0,$$

since $\int_\Omega f(x,t)u_n^- dx \leq 0$. Then integrating over $(0,t)$ for the above inequality for all t, we obtain

$$\|u_n^-(x,t)\|_{2,\Omega} \leq \|u_n^-(\cdot,0)\|_{2,\Omega} = 0.$$

Then the required assertion follows. $\square$

Remark 3.4. From Corollary 3.3 we know that the constructed weak solution is nonnegative. But to our best knowledge, it still remains unknown whether any solution of problem (1.2) is nonnegative if the given data are nonnegative.

Lemma 3.5. *The solution of* (3.1) *satisfies the estimates*

$$\int_Q u_n^\sigma |\nabla u_n|^{p(x)}\,dx\,dt \leq HK(T)|\Omega|^{1/2}, \tag{3.6}$$

$$\frac{1}{n^\sigma}\int_Q |\nabla u_n|^{p(x)}\,dx\,dt \leq HK(T)|\Omega|^{1/2}, \tag{3.7}$$

$$d_0\int_Q |\nabla u_n|^{p(x)}\,dx\,dt \leq HK(T)|\Omega|^{1/2}. \tag{3.8}$$

Proof. We proceed as in the proof of Lemma 3.1. Take $k = 1$ in (3.5), it follows

$$\frac{d}{dt}(\|u_n\|_{2,\Omega}) + \frac{1}{\|u_n\|_{2,\Omega}}\int_\Omega a_{n,H}(u_n)|\nabla u_n|^{p(x)}dx \leq \|f\|_{2,\Omega}.$$

Furthermore, we integrate the above equation over $(0,t)$ for all $t \in (0,T)$,

$$\|u_n(\cdot,t)\|_{2,\Omega} + \frac{1}{\|u_n\|_{2,\Omega}}\int_0^t\int_\Omega a_{n,H}(u_n)|\nabla u_n|^{p(x)}\,dx\,dt \leq \|u_n(\cdot,0)\|_{2,\Omega} + \int_0^t \|f\|_{2,\Omega}dt,$$

i.e.

$$\begin{aligned}\int_0^t\int_\Omega a_{n,H}(u_n)|\nabla u_n|^{p(x)}\,dx\,dt &\leq \|u_n\|_{2,\Omega}(\|u_n(\cdot,0)\|_{2,\Omega} + \int_0^t \|f\|_{2,\Omega}dt)\\ &\leq H|\Omega|^{1/2}K(T).\end{aligned}$$

Since $a_{n,H}(u_n) \geq d_0$, we obtain (3.8); since $a_{n,H}(u_n) \geq \frac{1}{n^\sigma}$, we obtain (3.7); since $H > K(T)$, one gets $a_{n,H}(u_n) \geq u_n^\sigma$, hence we obtain (3.6). $\square$

Lemma 3.6. *The solution of* (3.1) *satisfies the estimate*

$$\|u_{nt}\|_{W^{-1,x}L^{p(x)}(Q)} \leq C(H,\sigma,p^\pm,K(T),|\Omega|).$$

Proof. From (3.2), for $\xi \in W_0^{1,x}L^{p(x)}(Q)$ we have

$$
\begin{aligned}
&\int_Q u_{nt}\xi\,dx\,dt \\
&= -\int_Q \Big[\Big(u_n^2+\frac{1}{n^2}\Big)^{\sigma/2}+d_0\Big]|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\xi\,dx\,dt+\int_Q f\xi\,dx\,dt \\
&\le \int_Q \Big[\Big(u_n^2+\frac{1}{n^2}\Big)^{\sigma/2}+d_0\Big]|\nabla u_n|^{p(x)-1}|\nabla\xi|\,dx\,dt+\int_Q |f||\xi|\,dx\,dt \\
&\le 2\|[(u_n^2+\frac{1}{n^2})^{\sigma/2}+d_0]|\nabla u_n|^{p(x)-1}\|_{p'(x)}\|\nabla\xi\|_{p(x)}+2\|f\|_{p'(x)}\|\xi\|_{p(x)} \\
&\le 2\max\Big\{\Big(\int_Q\Big\{\Big[(u_n^2+\frac{1}{n^2})^{\sigma/2}+d_0\Big]|\nabla u_n|^{p(x)-1}\Big\}^{\frac{p(x)}{p(x)-1}}\,dx\,dt\Big)^{\frac{1}{p'^+}}, \\
&\Big(\int_Q\Big\{\Big[\Big(u_n^2+\frac{1}{n^2}\Big)^{\sigma/2}+d_0\Big]|\nabla u_n|^{p(x)-1}\Big\}^{\frac{p(x)}{p(x)-1}}\,dx\,dt\Big)^{\frac{1}{p'^-}}\Big\}\|\nabla\xi\|_{p(x)} \\
&+2\max\Big\{\Big(\int_Q|f|^{p'(x)}\,dx\,dt\Big)^{\frac{1}{p'^+}},\Big(\int_Q|f|^{p'(x)}\,dx\,dt\Big)^{\frac{1}{p'^-}}\Big\}\|\xi\|_{p(x)} \\
&\le (2((K^2(T)+1)^{\sigma/2}+d_0)^{\frac{1}{p^\pm-1}}K(T)|\Omega|H+2|f|_\infty|T|)\|\xi\|_{W^{1,x}L^{p(x)}(Q)},
\end{aligned}
$$

which yields the desired conclusion. □

From the above conclusion and the uniform estimates in n, we obtain a subsequence, still denoted $\{u_n\}_n$, such that

$$
\begin{gathered}
u_n\to u\quad\text{a.e. in } Q;\\
\nabla u_n\rightharpoonup\nabla u\quad\text{weakly in } L^{p(x)}(Q);\\
u_n^\sigma|\nabla u_n|^{p(x)-2}D_iu_n\rightharpoonup A_i(x,t)\quad\text{weakly in } L^{p'(x)}(Q);\\
|\nabla u_n|^{p(x)-2}D_iu_n\rightharpoonup W_i(x,t)\quad\text{weakly in } L^{p'(x)}(Q),
\end{gathered}
\tag{3.9}
$$

for $u\in W^{1,x}L^{p(x)}(Q)$, $A_i(x,t)\in L^{p'(x)}(Q)$, $W_i(x,t)\in L^{p'(x)}(Q)$.

Lemma 3.7. *For almost all $(x,t)\in Q$,*

$$
\lim_{n\to\infty}\int_Q\Big(\Big(u_n^2+\frac{1}{n^2}\Big)^{\sigma/2}-u_n^\sigma\Big)|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\xi\,dx\,dt=0,\quad\forall\xi\in W_0^{1,x}L^{p(x)}(Q).
$$

Proof. By Young's inequality, we have

$$
\begin{aligned}
I &:= \int_Q\Big(\Big(u_n^2+\frac{1}{n^2}\Big)^{\sigma/2}-u_n^\sigma\Big)|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\xi\,dx\,dt \\
&= \frac{\sigma}{2}\frac{1}{n^2}\int_Q\Big(\int_0^1\Big(u_n^2+s\frac{1}{n^2}\Big)^{\frac{\sigma-2}{2}}ds\Big)|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\xi\,dx\,dt \\
&\le \sigma\frac{1}{n^2}\Big(K^2(T)+1\Big)^{\frac{\sigma-2}{2}}\||\nabla u_n|^{p(x)-1}\|_{p'(x)}\|\nabla\xi\|_{p(x)} \\
&\le C\frac{1}{n^2}\Big\{\Big(\int_Q|\nabla u_n|^{p(x)}\,dx\,dt\Big)^{\frac{p^+-1}{p^+}},\Big(\int_Q|\nabla u_n|^{p(x)}\,dx\,dt\Big)^{\frac{p^--1}{p^-}}\Big\}\|\nabla\xi\|_{p(x)}.
\end{aligned}
$$

By (3.7), we obtain

$$
I\le CH\big(\frac{1}{n}\big)^{2-\sigma\frac{p^+-1}{p^+}}\|\nabla\xi\|_{p(x)}.
$$

Letting $n \to \infty$, we obtain the desired conclusion. $\square$

Lemma 3.8. *For almost all* $(x,t) \in Q$,

$$A_i(x,t) = u^\sigma W_i(x,t), \quad i = 1,2,\dots,N.$$

Proof. In (3.9), letting $n \to \infty$, we have

$$\int_Q u_n^\sigma |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \xi \, dx\, dt \to \sum_{i=1}^N \int_Q A_i(x,t) D_i \xi \, dx\, dt; \tag{3.10}$$

$$\int_Q |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \xi \, dx\, dt \to \sum_{i=1}^N \int_Q W_i(x,t) D_i \xi \, dx\, dt. \tag{3.11}$$

By Lebesgue's dominated convergence theorem we have

$$\lim_{n\to\infty} \sum_{i=1}^N \int_Q (u_n^\sigma - u^\sigma) A_i(x,t) D_i \xi \, dx\, dt = 0. \tag{3.12}$$

From (3.9) it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{i=1}^N \int_Q [u_n^\sigma |\nabla u_n|^{p(x)-2} D_i u_n - u^\sigma W_i(x,t)] D_i \xi \, dx\, dt \\
&= \lim_{n\to\infty} \sum_{i=1}^N \int_Q [(u_n^\sigma - u^\sigma) |\nabla u_n|^{p(x)-2} D_i u_n \\
&\quad + u^\sigma (|\nabla u_n|^{p(x)-2} D_i u_n - W_i(x,t))] D_i \xi \, dx\, dt = 0.
\end{aligned}$$

By (3.10)–(3.12) and the above equalities, we complete the proof. $\square$

Lemma 3.9. *For almost all* $(x,t) \in Q$,

$$W_i(x,t) = |\nabla u|^{p(x)-2} D_i(u), \quad i = 1,2,\dots N.$$

Proof. In (3.2), choosing $\xi = (u_n - u)\Phi$ with $\Phi \in W_0^{1,x} L^{p(x)}(Q)$, $\Phi \geq 0$, we have

$$\begin{aligned}
&\int_Q [u_{nt}(u_n - u)\Phi + \Phi(u_n^\sigma + d_0)|\nabla u_n|^{p(x)-2} \nabla u_n \nabla (u_n - u)] \, dx\, dt \\
&+ \int_Q [(u_n - u)(u_n^\sigma + d_0)|\nabla u_n|^{p(x)-2} \nabla u_n \nabla \Phi - f(x,t)(u_n - u)\Phi] \, dx\, dt \\
&+ \int_Q ((u_n^2 + \frac{1}{n^2})^{\sigma/2} - u_n^\sigma)|\nabla u_n|^{p(x)-2} \nabla u_n \nabla \xi \, dx\, dt = 0.
\end{aligned}$$

It follows that

$$\int_Q \Phi(u_n^\sigma + d_0)|\nabla u_n|^{p(x)-2} \nabla u_n \nabla (u_n - u)] \, dx\, dt = 0. \tag{3.13}$$

On the other hand, by the fact that $u_n,\ u \in L^\infty(Q)$ and $|\nabla u| \in L^{p(x)}(Q)$, we have

$$\lim_{n\to\infty} \int_Q \Phi(u^\sigma + d_0)|\nabla u|^{p(x)-2} \nabla u \nabla (u_n - u) \, dx\, dt = 0. \tag{3.14}$$

$$\lim_{n\to\infty} \int_Q \Phi(u_n^\sigma - u^\sigma)|\nabla u|^{p(x)-2} \nabla u \nabla (u_n - u) \, dx\, dt = 0. \tag{3.15}$$

Note that

$$\begin{aligned} 0 &\le (|\nabla u|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)\nabla(u_n - u) \\ &\le \frac{1}{d_0}[(u_n^\sigma + d_0)|\nabla u_n|^{p(x)-2}\nabla u_n - (u_n^\sigma - u^\sigma)|\nabla u|^{p(x)-2}\nabla u]\nabla(u_n - u) \\ &\quad - \frac{1}{d_0}(u^\sigma + d_0)|\nabla u|^{p(x)-2}\nabla u\nabla(u_n - u). \end{aligned} \tag{3.16}$$

Bring (3.13)–(3.15) into (3.16), we obtain

$$\lim_{n\to\infty}\int_Q \Phi(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)\nabla(u_n - u)\,dx\,dt = 0.$$

The rest arguments are the same as those of [22, Theorem 2.1]. Thus the existence of weak solutions for problem (1.2) is obtained by a standard limiting process. ☐

4. Uniqueness of solutions

In this section, we study the uniqueness of the solutions to (1.1). To obtain the main conclusion of this section, we need the following lemma.

Lemma 4.1. *Let $M(s) = |s|^{p(x)-2}s$, then for all $\xi, \eta \in \mathbb{R}^N$,*

$$\begin{aligned} &(M(\xi) - M(\eta))(\xi - \eta) \\ &\ge \begin{cases} 2^{-p(x)}|\xi - \eta|^{p(x)}, & \text{if } 2 \le p(x) < \infty; \\ (p(x)-1)|\xi - \eta|^2(|\xi|^{p(x)} + |\eta|^{p(x)})^{\frac{p(x)-2}{p(x)}}, & \text{if } 1 < p(x) < 2. \end{cases} \end{aligned}$$

Now we shall prove Theorem 1.3 by contradiction. Suppose $u(x,t)$ and $v(x,t)$ are two nonnegative weak solutions of problem (1.1) and there is a $\delta > 0$ such that for some $0 < \tau \le T$, $w = u - v > \delta$ on the set $\Omega_\delta = \Omega \cap \{x : w(x,t) > \delta\}$ and $\mu(\Omega_\delta) > 0$. Let

$$F_\varepsilon(\xi) = \begin{cases} \frac{1}{\alpha-1}\varepsilon^{1-\alpha} - \frac{1}{\alpha-1}\xi^{1-\alpha}, & \text{if } \xi > \varepsilon; \\ 0, & \text{if } \xi \le \varepsilon. \end{cases}$$

where $\delta > 2\varepsilon > 0$ and $\alpha = \sigma/2$.

By the definition of weak solution, we take a test-function $\xi = F_\varepsilon(w)$,

$$\begin{aligned} 0 &= \int_{Q_\tau} [w_t F_\varepsilon(w) + (v^\sigma + d_0)(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\nabla F_\varepsilon(w)]\,dx\,dt \\ &\quad + \int_{Q_\tau} (u^\sigma - v^\sigma)|\nabla u|^{p(x)-2}\nabla u\nabla F_\varepsilon(w)\,dx\,dt \\ &= \int_{Q_{\varepsilon,\tau}} w_t F_\varepsilon(w)\,dx\,dt \\ &\quad + \int_{Q_{\varepsilon,\tau}} (v^\sigma + d_0)w^{-\alpha}(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\nabla w\,dx\,dt \\ &\quad + \int_{Q_{\varepsilon,\tau}} (u^\sigma - v^\sigma)w^{-\alpha}|\nabla u|^{p(x)-2}\nabla u\nabla w\,dx\,dt \\ &= J_1 + J_2 + J_3, \end{aligned} \tag{4.1}$$

with $Q_{\varepsilon,\tau} = Q_\tau \cap \{(x,t) \in Q_\tau : w > \varepsilon\}$.

Now, let $t_0 = \inf\{t \in (0, \tau] : w > \varepsilon\}$, then we estimate J_1, J_2, J_3.

$$
\begin{aligned}
J_1 &= \int_{Q_{\varepsilon,\tau}} w_t F_\varepsilon(w)\,dx\,dt \\
&= \int_\Omega \Big(\int_0^{t_0} w_t F_\varepsilon(w)dt + \int_{t_0}^{\tau} w_t F_\varepsilon(w)dt\Big)dx \\
&\geq \int_\Omega \int_\varepsilon^{w(x,\tau)} F_\varepsilon(s)\,ds\,dx \\
&\geq \int_{\Omega_\delta} \int_\varepsilon^{w(x,\tau)} F_\varepsilon(s)\,ds\,dx \\
&\geq \int_{\Omega_\delta} (w - 2\varepsilon) F_\varepsilon(\varepsilon) dx \\
&\geq (\delta - 2\varepsilon) F_\varepsilon(\varepsilon)\mu(\Omega_\delta),
\end{aligned} \tag{4.2}
$$

Let us first consider the case $p^- \geq 2$. By the first inequality of Lemma 4.1, we obtain

$$
\begin{aligned}
J_2 &= \int_{Q_{\varepsilon,\tau}} (v^\sigma + d_0) w^{-\alpha} (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\nabla w\,dx\,dt \\
&\geq \int_{Q_{\varepsilon,\tau}} (v^\sigma + d_0) w^{-\alpha} 2^{-p(x)} |\nabla w|^{p(x)}\,dx\,dt \\
&\geq 2^{-p^+} \int_{Q_{\varepsilon,\tau}} (v^\sigma + d_0) w^{-\alpha} |\nabla w|^{p(x)}\,dx\,dt \geq 0,
\end{aligned} \tag{4.3}
$$

Noting that $\frac{p(x)}{p(x)-1} \geq \frac{p^+}{p^+-1} = \alpha > 1$ and applying Young's inequality, we estimate integrand of J_3 in the following way

$$
\begin{aligned}
&|(u^\sigma - v^\sigma) w^{-\alpha} |\nabla u|^{p(x)-2}\nabla u \nabla w| \\
&= \big|\sigma w \int_0^1 (\theta u + (1-\theta)v)^{\sigma-1} d\theta w^{-\alpha} |\nabla u|^{p(x)-2}\nabla u\nabla w\big| \\
&\leq \frac{C}{W^\alpha}\Big[\frac{v^\sigma + d_0}{C}|\nabla w|^{p(x)} + C_1(\sigma, d_0, K(T), p^\pm)|w|^{p'(x)}|\nabla u|^{p(x)}\Big] \\
&\leq \frac{v^\sigma + d_0}{2^{p^+ +1} w^\alpha}|\nabla w|^{p(x)} + C_1(\sigma, d_0, K(T), p^\pm)|w|^{p'(x)-\alpha}|\nabla u|^{p(x)} \\
&\leq \frac{v^\sigma + d_0}{2^{p^+ +1} w^\alpha}|\nabla w|^{p(x)} + C_1(\sigma, d_0, K(T), p^\pm)|\nabla u|^{p(x)}.
\end{aligned} \tag{4.4}
$$

Substituting (4.4) into J_3, we obtain

$$
J_3 \leq \frac{1}{2} J_2 + C \int_{Q_{\varepsilon,\tau}} |\nabla u|^{p(x)}\,dx\,dt. \tag{4.5}
$$

Next we consider the case $1 < p^- < p(x) < 2$, $p^+ > 2$. According to the second inequality of Lemma 4.1, it is easy to see that the following inequalities hold

$$
\begin{aligned}
J_2 &= \int_{Q_{\varepsilon,\tau}} (v^\sigma + d_0) w^{-\alpha} (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\nabla w\,dx\,dt \\
&\geq (p^- - 1) \int_{Q_{\varepsilon,\tau}} (v^\sigma + d_0) w^{-\alpha} (|\nabla u| + |\nabla v|)^{p(x)-2} |\nabla w|^2\,dx\,dt \geq 0.
\end{aligned} \tag{4.6}
$$

Using Young's inequality and the fact that $1 < \alpha \leq \frac{p^+}{p^+-1} \leq 2$, we evaluate integrand of J_3 as follows:

$$\begin{aligned} &|(u^\sigma - v^\sigma)w^{-\alpha}|\nabla u|^{p(x)-2}\nabla u\nabla w| \\ &= \Big|\sigma w\int_0^1 (\theta u + (1-\theta)v)^{\sigma-1}d\theta w^{-\alpha}|\nabla u|^{p(x)-2}\nabla u\nabla w\Big| \\ &\leq \frac{(v^\sigma + d_0)(p^- - 1)}{2w^\alpha}(|\nabla u| + |\nabla v|)^{p(x)-2}|\nabla w|^2 \\ &\quad + C_1(\sigma, d_0, K(T), p^\pm)|w|^{2-\alpha}(|\nabla u| + |\nabla v|)^{p(x)} \\ &\leq \frac{(v^\sigma + d_0)(p^- - 1)}{2w^\alpha}(|\nabla u| + |\nabla v|)^{p(x)-2}|\nabla w|^2 \\ &\quad + C_1(\sigma, d_0, K(T), p^\pm)(|\nabla u| + |\nabla v|)^{p(x)}. \end{aligned} \tag{4.7}$$

Inserting (4.7) into J_3, we obtain

$$J_3 \leq \frac{1}{2}J_2 + C\int_{Q_{\varepsilon,\tau}} (|\nabla u| + |\nabla v|)^{p(x)}\,dx\,dt.$$

Plugging the estimates (4.2), (4.3), (4.5) and (4.2), (4.6), (4.7) into (4.1) and dropping the nonnegative terms, we arrive at the inequality

$$(\delta - 2\varepsilon)(1 - 2^{1-\alpha})\varepsilon^{1-\alpha}\mu(\Omega_\delta) \leq \tilde{C},$$

with a constant $\tilde{C}$ independent of ε.

Notice that $\lim_{\varepsilon\to 0}(\delta-2\varepsilon)(1-2^{1-\alpha})\varepsilon^{1-\alpha}\mu(\Omega_\delta) = +\infty$, we obtain a contradiction. This means $\mu(\Omega_\delta) = 0$ and $w \leq 0$, a.e. in Q_τ. Thus the proof is complete.

5. Localization property of solutions

In this section, we shall focus on the study of localization of solutions to problem (1.1). The proof is similar to that of [13, Theorem 4.1], we would like to give the detailed treatment, just for the reader's convenience. In fact, by Definition 1.1, it follows easily that

$$\int_Q u_\tau\xi + (u^\sigma + d_0)|\nabla u|^{p(x)-2}\nabla u\nabla\xi = 0, \tag{5.1}$$

with $\tau \in (0, T)$. Let

$$\Psi = \inf\{\operatorname{dist}(x, \operatorname{supp}\ u_0 \cup \partial\Omega)/\lambda, 1\},$$

where $0 < \lambda < 1$, and $F_\varepsilon(\xi)$ is mentioned in Section 4 with $\alpha = \sigma/2$. Taking $\xi = \Psi F_\varepsilon(u)(0 < \varepsilon < 1)$ and substituting it into (5.1), we obtain

$$\begin{aligned} 0 &= \int_{Q_{\varepsilon,\tau}} u_t\Psi F_\varepsilon(u)\,dx\,dt + \int_{Q_{\varepsilon,\tau}} \Psi(u^\sigma + d_0)|\nabla u|^{p(x)-2}\nabla u\nabla F_\varepsilon(u)\,dx\,dt \\ &\quad + \int_{Q_{\varepsilon,\tau}} F_\varepsilon(u)(u^\sigma + d_0)|\nabla u|^{p(x)-2}\nabla u\nabla\Psi\,dx\,dt := I_1 + I_2 + I_3. \end{aligned} \tag{5.2}$$

with $Q_{\varepsilon,\tau} = Q_\tau \cap \{(x,t) \in Q_\tau : u > \varepsilon\}$. Denote $E = \{x \in \{\Psi = 1\} : u(x,\tau) > \delta\}$ with $\delta > 2\varepsilon > 0$, then

$$\begin{aligned} I_1 &= \int_{Q_{\varepsilon,\tau}} u_t \Psi F_\varepsilon(u)\,dx\,dt \ge \int_{\Omega_\varepsilon} \chi_{\operatorname{supp}\Psi} \Psi \int_\varepsilon^u F_\varepsilon(s)\,ds\,dx \\ &\ge \int_{\Omega_\varepsilon} \chi_{\operatorname{supp}\Psi} \Psi (u-\varepsilon) F_\varepsilon(\delta) dx \\ &\ge \big(\delta - \frac{3}{2}\varepsilon\big) F_\varepsilon\big(\frac{3}{2}\varepsilon\big) \operatorname{meas}(E). \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} I_2 &= \int_{Q_{\varepsilon,\tau}} \Psi(u^\sigma + d_0)|\nabla u|^{p(x)-2}\nabla u \nabla \frac{1}{\alpha-1}(-u^{1-\alpha})\,dx\,dt \\ &\ge \int_{Q_{\varepsilon,\tau}} \Psi(u^\sigma + d_0)|\nabla u|^{p(x)} u^{-\alpha}\,dx\,dt \ge 0. \end{aligned} \tag{5.4}$$

Applying Young's inequality with η and choosing $\eta = (\varepsilon^\beta)^{1-p(x)}$, we may estimate that

$$\begin{aligned} |I_3| &= \Big| \int_{Q_{\varepsilon,\tau}} F_\varepsilon(u)(u^\sigma + d_0)|\nabla u|^{p(x)-2}\nabla u \nabla \Psi\,dx\,dt \Big| \\ &\le C \int_{Q_{\varepsilon,\tau}} \varepsilon^{1-\alpha} |\nabla u|^{p(x)-1} |\nabla \Psi|\,dx\,dt \\ &\le C(\sigma, d_0, K(T), p^\pm) \varepsilon^{\beta + \frac{(1-\alpha)p^-}{p^- - 1}} \int_{Q_{\varepsilon,\tau}} |\nabla u|^{p(x)}\,dx\,dt \\ &\quad + \varepsilon^{\beta(1-p^+)} \int_{Q_{\varepsilon,\tau}} |\nabla \Psi|^{p(x)}\,dx\,dt, \end{aligned} \tag{5.5}$$

where $C > 0$ denote the various constants. Choosing $\beta = \frac{\alpha p^- - 1}{p^- - 1} > 0$ and putting (5.3)–(5.5) into (5.2), we deduce

$$\begin{aligned} &\frac{1}{2}[1 - (3/2)^{1-\alpha}] \varepsilon^{2-\alpha-\beta+\frac{(\alpha-1)p^-}{p^- - 1}} \operatorname{meas}(E) \\ &\le (\delta - 3\varepsilon/2)[1 - (3/2)^{1-\alpha}] \varepsilon^{1-\alpha-\beta+\frac{(\alpha-1)p^-}{p^- - 1}} \operatorname{meas}(E) \\ &\le \widetilde{C}\Big(1 + \varepsilon^{\frac{(\alpha-1)p^-}{p^- - 1} - \beta p^+}\Big), \end{aligned} \tag{5.6}$$

with the positive constant $\widetilde{C}$ independent of ε. Noticing that

$$2 < \sigma < \frac{2(p^+ - p^-)}{p^-(p^+ - 1)} < \frac{2p^+}{p^+ - 1},$$

we have

$$1 < \alpha = \frac{\sigma}{2} < \frac{(p^+ - p^-)}{p^-(p^+ - 1)}, \quad 1 - \beta + \frac{(\alpha-1)p^-}{p^- - 1} = 0; \tag{5.7}$$

$$\frac{(\alpha-1)p^-}{p^- - 1} - \beta p^+ = \frac{(p^+ - p^-) - \alpha p^-(p^+ - 1)}{p^- - 1} > 0. \tag{5.8}$$

Assume that there exists the constant $\tau_0 \in (0,T)$ such that $\operatorname{meas}(E) \ne 0$. Thus, (5.6)–(5.8) yield a contradiction. Hence, we have

$$\operatorname{meas}\{x \in \{\Psi = 1\} : u(x,\tau) > \delta\} = 0, \tag{5.9}$$

for all $\delta \in (0,1)$ and a.e. $\tau \in (0,T)$. Then Theorem 1.4 follows from (5.9) and the arbitrariness of λ.

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